Numerical techniques for approximating the solution of matrix ODE on the general linear group

Nicoletta Del Buono

Joint work with : Luciano Lopez
Outline

- The matrix ODE we deal with
- Theoretical results
- Examples
- Numerical tools:
  - Substituting approach
  - Solution via Riccati equation
  - SVD approach
- Rectangular Case
- Numerical examples
Consider the matrix differential equation

\[ \dot{Y}(t) = Y(t)^{-T} F(Y(t), Y(t)^{-T}) \]

\[ Y(0) = Y_0 \in GL(n) \]

- \( F \) is a continuous matrix function, globally Lipschitz on a subdomain of \( GL(n) \)
- the solution \( Y(t) \) exists and is unique in a neighborhood \( ]-\tau, \tau[ \) of the origin 0
The structure of $GL(n)$

- Two maximal connected and disjoint open subsets comprising $GL(n)$

\[
GL^+(n) = \{ M \in \mathbb{R}^{n\times n} \mid \det(M) > 0 \}
\]

\[
GL^-(n) = \{ M \in \mathbb{R}^{n\times n} \mid \det(M) < 0 \}
\]

Variety of Singular $n\times n$ matrices
Theoretical results

- The existence of the solution $Y(t)$ for all $t$ is not guaranteed \textit{a priori} and the presence of a finite escape time behavior is not precluded.

- The value of the escape point depends on the function $F$
  - If the escape point $\tau$ is finite then $Y(t)$ approaches a singular matrix as $t \rightarrow \tau$
  - If $\tau < \infty$ then $Y(t)$ exists for all $t > 0$
Theoretical results

Example: \( F \) constant function with \( \text{trace}(F) = 0 \)

\[
\dot{Y} = Y^{-T} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad Y(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\]

solution

\[
Y(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+t} & -\sqrt{1+t} \\ \sqrt{1-t} & \sqrt{1-t} \end{bmatrix}
\]

Existence interval \((-1,1)\)
Escape point 1

Nicoletta Del Buono
Theoretical results

- Relationship between the singular values of the solution $Y(t)$, the initial condition $Y(0)$ and the symmetric matrix function:

$$E(t) = \int_{0}^{t} [F^T(Y(s), Y^{-T}(s)) + F(Y(s), Y^{-T}(s))] ds$$

$$\sigma_{\min}(t) \geq \sigma_{\min}^0 + \lambda_{\min}(E(t))$$

- Smallest Singular Value of $Y(t)$
- Smallest Singular Value of $Y(0)$
- Smallest Eigenvalue of $E(t)$
If the matrix function $F$ maps all matrices into the Lie algebra of skew-symmetric matrices, then $Y(t)$ belongs to the orthogonal manifold (whenever $Y(0)$ is orthogonal).

If $\text{diag}(F) = 0$ for all nonsingular matrices, then $\text{diag}(Y(t)^TY(t)) = \text{diag}(Y(0)^TY(0))$. 

Nicoletta Del Buono
Examples

- **Control Theory**
  - Optimal system assignment via Output Feedback Control
  - Balanced Matrix Factorizations
  - Balanced realizations (Isodynamical flows)

- **Multivariate Data Analysis**
  - Weighted Oblique Procrustes problem

- **Inverse Eigenvalue Problem**
  - Pole placement or eigenvalue assignment problem via output feedback
  - Prescribed Entries Inverse Eigenvalue Problem

*Nicoletta Del Buono*
Output Feedback Control of linear system

Consider the linear dynamical system defined by the triple \((A, B, C) \in \mathbb{P}^{n \times n} \times \mathbb{P}^{n \times m} \times \mathbb{P}^{p \times n}\):

\[
\begin{align*}
x(t) &= Ax(t) + Bu(t) \\
y(t) &=Cx(t)
\end{align*}
\]

The process of “feeding back” the output or the state variables in a dynamical system configuration through the input channels.

Output Feedback: \(u(t)\) is replaced by \(u(t) = Ky(t) + v(t)\), where \(K \in \mathbb{P}^{m \times p}\) is the feedback gain matrix.
Examples in Control Theory

- **Output Feedback Control of linear system**
  - The feedback system is

\[
\begin{align*}
\dot{x}(t) &= (A + BK C)x(t) + Bv(t) \\
y(t) &= Cx(t)
\end{align*}
\]

- **Optimal system assignment**
  - Given a **target system** described by the triple \((F,G,H)\in \mathbb{P}^{n\times n}\times \mathbb{P}^{n\times m}\times \mathbb{P}^{p\times n}\) find an **optimal feedback transformation** of \((A,B,C)\) which results the best approximation of \((F,G,H)\).

Nicoletta Del Buono
Examples in Control Theory

- The set $GL(n) \times \mathcal{P}^{m \times p}$ of feedback transformation is a Lie group under the operation
  
  $$(T_1, K_1) \circ (T_2, K_2) = (T_1 T_2, K_1 + K_2)$$

- We can consider action on the output feedback group and orbits, particularly:

  $$\Phi(A, B, C) = \{(T(A + BKC)T^{-1}, TB, CT^{-1} \mid (T, K) \in GL(n) \times \mathcal{P}^{m \times p}\}$$

- The distance function

  $$\Phi = \|T(A + BKC)T^{-1} - F\|^2 + \|TB - G\|^2 + \|CT^{-1} - H\|^2$$

Nicoletta Del Buono
Examples in Control Theory

- The gradient flow of this distance function with respect to a specific Riemannian metric on \( \Phi(A,B,C) \) can be written as:

\[
\dot{T} = T^{-T} f(T, T^{-T}, K)
\]

\[
\dot{K} = -B^T T^T (T (A + BKC) T^{-1} - F) T^{-T} C^T
\]
Examples in Control Theory

- **Balanced matrix factorizations**
  - General matrix factorization problem:
    Given a matrix $H \in \mathbb{P}^{k \times l}$ find two $X \in \mathbb{P}^{k \times n}$ and $Y \in \mathbb{P}^{n \times l}$ such that $H = XY$
  - balanced factorization $X^TX = YY^T$
  - diagonal balanced factorization $X^TX = YY^T = D$

- Balanced and diagonal balanced factorization can be characterized as critical points of cost functions defined on the orbit

$$O(X,Y) = \{(XT^{-1}, TY) \in \mathbb{P}^{k \times n} \times \mathbb{P}^{n \times l} | T \in GL(n)\}$$

*Nicoletta Del Buono*
Examples in Control Theory

- The cost functions are respectively:

\[
\Phi : \mathbb{O} (X, Y) \rightarrow \mathbb{P} \quad \Phi(XT^{-1}, TY) = \| XT^{-1} \|_2^2 + \| TY \|_2^2
\]

\[
\Phi_N : \mathbb{O} (X, Y) \rightarrow \mathbb{P} \quad \Phi_N(XT^{-1}, TY) = \text{tr}(NTT^{-1}XX^{-1} + NTYY^TT^T)
\]

- Applying a gradient flow techniques differential systems on $GL(n)$ can be constructed:

\[
T = T^{-T} (XTX(T^TX)^{-1} - TTY^YT^T) \quad T(0) = T_0
\]

\[
\dot{T} = T^{-T} (XTXT^{-1}NT^{-T} - TTY^YT^T) \quad T(0) = T_0
\]
Examples in Control Theory

- **Balanced realizations in linear system theory**

- Consider the linear dynamical system defined by the triple $(A,B,C) \in \mathbb{P}^{n \times n} \times \mathbb{P}^{n \times m} \times \mathbb{P}^{p \times n}$

- \[
x(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t)
\]

- Gramians:
  
  \[
  W_C = \int_0^\infty e^{At}BB^T e^{A^Tt} \, dt \\
  W_O = \int_0^\infty e^{A^Tt}C^TCe^{At} \, dt
  \]

- $(A,B,C)$ is a **balanced realization** if $W_C = W_O$

- $(A,B,C)$ is a **diagonal balanced realization** if $W_C = W_O = D$

Nicoletta Del Buono
Examples in Control Theory

- Any $T \in GL(n)$ changes a realization by
  $$(A, B, C) \rightarrow (TAT^{-1}, TB, CT^{-1})$$
- and the Gramians via
  $$W_C \rightarrow T W_C T^{-1} \quad W_0 \rightarrow T^{-T}W_0 T^{-1}$$
- Balanced and diagonal balanced realizations have been proved to be critical points of costs functions defined on the orbit

$$O(A, B, C) = \{(TAT^{-1}, TB, CT^{-1}) \in P^{n \times n} \times P^{n \times m} \times P^{k \times n} \mid T \in GL(n)\}$$
Examples in Control Theory

- The cost functions are respectively:

\[ \Phi : O(A, B, C) \to P \quad \Phi(T) = tr(TW_C T^{-1} + T^{-T}W_O T^{-1}) \]

\[ \Phi_N : O(A, B, C) \to P \quad \Phi_N(T) = tr(NTW_C T^{-1} + NT^{-T}W_O T^{-1}) \]

- All balancing transformation \( T \in GL(n) \) for a given asymptotically stable system \((A, B, C)\) can be obtained solving the gradient flow

\[ \dot{T} = T^{-T} (W_O (T^T T)^{-1} - T^T TW_C) \quad T(0) = T_0 \]

\[ \dot{T} = T^{-T} (W_O T^{-1} NT^{-T} - T^T NTW_C) \quad T(0) = T_0 \]
Examples in Multivariate Data Analysis

- **Weighted oblique Procrustes problem (WObPP)**
  - **Manifold of the oblique rotation matrices**
    \[ OB(n) = \{ X \in P^{n \times n} \mid \det(X) \neq 0, \text{diag}(X^T X) = I \} \]

- Given \( A, B, C \) fixed matrices with conformal dimensions
  - Minimize \( \| AXC - B \| \) subject to \( X \in OB(n) \)
    - Problem in factor analysis known as a “rotation to factor-structure matrix”
  - Minimize \( \| AX^TC - B \| \) subject to \( X \in OB(n) \)
    - Problem of finding an approximation to a “factor-pattern” matrix
The solution of the WObPP problem can be obtained solving a **descent matrix ODE**:

\[
\frac{dX}{dt} = -\pi_{OB(n)}(\nabla) = -X^{-T} \text{off} \left( X^T \nabla \right)
\]

being \( \nabla \) the gradient of the function to be minimize with respect to the chosen metric

(N. Trendafilov FGCS 2003)
Examples in Inverse Eigenvalue Problem and control theory

- **Pole placement or eigenvalue assignment via output feedback:**
  - Given a linear system described by the triple \((A, B, C)\) and a self-conjugate set of complex points \(\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\).
  - Find a feedback gain matrix \(K\) such that \(A + BKC\) has eigenvalues \(\lambda_i\).

- Denoted by \(\Lambda\) a fixed matrix with eigenvalues \(\lambda_i\), the pole placement task is equivalent to find a matrix \(T \in \text{GL}(n)\) and \(K \in \text{P}^{m \times p}\) minimizing the distance \(\mu_1\)

\[
\|\Lambda - T(A + BKC)T^{-1}\|
\]
Examples in Inverse Eigenvalue Problem and control theory

Using a gradient flow techniques the solution can be obtained solving

\[ T = T^{-T} \left[ ( A + BKC )^T, T^T ( \Lambda - ( A + BKC )) T^{-T} \right] \]

\[ K = -B^T T^T ( T ( A + BKC ) T^{-1} - F ) T^{-T} C^T \]
Examples in Inverse Eigenvalue Problem

- **Matrix completion with prescribed eigenvalues**
- **PEIEP (prescribed entries inverse eigenvalue problem)**:

Given
- \( \Lambda = \{ (i_v, j_v) \mid v = 1, \ldots, m \} \) \( m \) pairs of integers \( 1 \leq i_v < j_v \leq n \)
- \( a = \{ a_1, \ldots, a_m \} \subseteq P \)
- \( \{ \lambda_1, \ldots, \lambda_n \} \subseteq X \) closed under conjugation

Find a matrix \( X \in P^{n \times n} \) such that \( \sigma(X) = \{ \lambda_1, \ldots, \lambda_n \} \)
and \( x_{i_v, j_v} = a_v \quad v = 1, \ldots, m \)
Examples in Inverse Eigenvalue Problem

- Let $\Lambda$ a matrix with eigenvalues $\lambda_i$ and denoting
  $$M(\Lambda) = \{VAV^{-1} \mid V \in GL(n)\}$$
  the orbit of matrices isospectral to $\Lambda$ under the action group of $GL(n)$ and
  $$\Sigma(\Lambda,a) = \{X = [x_{ij}] \in \mathbb{P}^{n \times n} \mid x_{i,vJ} = a_v \quad v = 1, \ldots, m\}$$

- Solving the PEIEP is to find intersection of the two geometric entities $M(\Lambda)$ and $\Sigma(\Lambda,a)$
Examples in Inverse Eigenvalue Problem

- Minimize for each given $X \in M(\Lambda)$ the distance between $X$ and $\Sigma(\Lambda,a)$

$$\min_{V \in M(\Lambda)} \frac{1}{2} < V\Lambda V^{-1} - P(V\Lambda V^{-1}), V\Lambda V^{-1} - P(V\Lambda V^{-1}) >$$

Projection on $\Sigma(\Lambda,a)$

- Using a descent flow approach we get

$$\frac{dV}{dt} = \kappa(V\Lambda V^{-1})V^{-T} \quad \text{with} \quad \kappa(X) = [X^T, X - P(X)]$$

(M.T. Chu et al. FGCS 2003)
Consider our system:

\[
\dot{Y}(t) = Y(t)^{-T} F(Y(t), Y(t)^{-T}) \\
Y(0) = Y_0 \in GL(n)
\]

Setting \( Z = Y^{-T} \) from \( Y^T Z = I \) we get

\[
Y^T Z + Y^T \dot{Z} = 0 \iff \dot{Z} = -Y^{-T} Y^T Z
\]

\[
\begin{align*}
\dot{Y} &= ZF(Y, Z) = H(Y, Z), & Y(0) &= Y_0 \\
\dot{Z} &= -ZF^T(Y, Z)Z^T Z = -ZH^T(Y, Z)Z, & Z(0) &= Y_0^{-T}
\end{align*}
\]
Substituting Approach

- **Advantages:**
  - No direct use of the inverse of $Y(t)$ (computational advantages)

- **Drawbacks:**
  - Solution of a new matrix ODE with double dimension with respect to the original system;
  - High stiffness (when $Y(t)$ tends to a singular matrix or the Lipschitz constant of $H$ is large);
  - The presence of an additional structure of the solution matrix $Y(t)$ is not considered need of ad hoc numerical scheme.
Solution via Riccati equation

- When the matrix function $F$ does not depend explicitly on $Y^{-T}$, i.e.:
  
  $\dot{Y}(t) = Y(t)^{-T} F(Y(t))$
  
  $Y(0) = Y_0 \in GL(n)$

- It could be convenient to work with the implicit equation
  
  $Y(t) \dot{Y}(t) = F(Y(t))$
  
  $Y(0) = Y_0 \in GL(n)$
Solution via Riccati equation

- Applying the second order Gauss Legendre method, we get:

\[ Y_{n+1}^T Y_{n+1} + Y_n^T Y_{n+1} - Y_{n+1}^T Y_n - Y_n^T Y_n - 2hF\left(\frac{Y_n + Y_{n+1}}{2}\right) = 0 \]

- The previous equation can be iteratively solved starting from an initial approximation \( Y_{n+1}^{(0)} \)

(avoiding the nonlinearity of \( F \))

\[ Y_{n+1}^T Y_{n+1} + Y_n^T Y_{n+1} - Y_{n+1}^T Y_n - Y_n^T Y_n - 2hF\left(\frac{Y_n + Y_{n+1}^{(0)}}{2}\right) = 0 \]
Solution via Riccati Equation

- The latter equation is the prototype of an Algebraic Riccati equation, in fact setting

\[ A = Y_n \quad \text{and} \quad C = Y_n^T Y_n + 2hF \left( \frac{Y_n + Y_n^{(0)}}{2} \right) \]

- we get

\[ R(X) = X^T X + A^T X - X^T A + C = 0 \]
Solution via Algebraic Riccati equation

- Numerical methods to solve Algebraic Riccati equation are based on fixed point or Newton iteration:
  - **Picard iteration:**
    \[ A^T X_{k+1} - X_{k+1}^T A = -C - X_k^T X_k \]
  - **Newton method:**
    \[ R : \mathbb{P}^{n\times n} \to \mathbb{P}^{n\times n} \]
    - its Frechét derivative is: \( R'_X(H) = H^T (X - A) + (X + A)^T H \)
    - the Newton iteration starts from \( X_0 \) and solves \( R(X) = 0 \) via \( X_{k+1} = X_k + D_k \) being \( D_k \) the solution of Sylvester equation
    \[ R'_X(D_k) = -R(X_k) \iff (X_k + A)^T D_k + D_k^T (X_k - A) = -R(X_k) \]
Solution via Riccati equation

- Solving Riccati equation implies the numerical treatment of the Sylvester equation
  \[ AX + X^T B = X \]
  with \( A, B, X \) given \( n \times n \) matrices

**Existence:** there exists a solution \( X \) of the Sylvester equation iff

\[
\begin{bmatrix}
X & A \\
B & O
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
O & A \\
B & O
\end{bmatrix}
\]

are equivalent
To obtain conditions for uniqueness of solution and for constructing it, we reformulate the Sylvester equation as a $n^2 \times n^2$ linear system:

$$(I \otimes A) \text{vec}(X) + (B^T \otimes I) \text{vec}(X^T) = \text{vec}(X)$$

$$\text{vec}(X^T) = P(n,n) \text{vec}(X)$$

$$P(n,n) = \sum_{i=1}^{n} \sum_{j=1}^{n} E_{ij} \otimes E_{ij}^T$$

$$\left[ (I \otimes A) + (B^T \otimes I) P(n,n) \right] \text{vec}(X) = \text{vec}(X)$$
Solution via Riccati equation

\[ M = \begin{bmatrix}
A + e_1 b_1^T & e_2 b_1^T & \cdots & e_n b_1^T \\
e_1 b_2^T & A + e_2 b_2^T & \cdots & e_n b_2^T \\
\vdots & \vdots & \ddots & \vdots \\
e_1 b_n^T & e_2 b_n^T & \cdots & A + e_n b_n^T
\end{bmatrix} \]

being \( b_i \) the columns of the matrix \( B \)

**Uniqueness:** there exists a unique solution \( X \) of the Sylvester equation \( AX + X^T B = X \) if the matrix \( M \) is non-singular (\( \text{rank } (M) = n^2 \))
Solution via Riccati equation

- Considering the linear equation derived from:
  - **Picard iteration:** $A = A^T$ and $B = A \Rightarrow M$ is singular
  - **Newton iteration:** $A = X_k + A^T$ and $B = X_k - A \Rightarrow M$ is non-singular $\Rightarrow$ unique solution!

- Newton method converges in a reasonable number of iterations

- Numerical solution of Sylvester equation:
  - Direct methods (QR, Gaussian Elimination);
  - Iterative algorithms;
  - Generalize Conjugate Residual method.

*Nicoletta Del Buono*
To avoid the inverse matrix computations and to control the singularities of the matrix solution $Y(t)$ we can adopt a continuous Singular Value Decomposition approach.

The continuous SVD of $Y(t)$ is a continuous factorization

$$Y(t) = U(t) \Sigma(t) V^T(t)$$

- $U(t), V(t)$ orthogonal matrices ($U^T U = I_n$ and $V^T V = I_n$)
- $\Sigma(t)$ diagonal matrix with diagonal elements the singular values $\sigma_i(t)$ of $Y(t)$

The motion of $Y(t)$ is now described by the variables $U(t), \Sigma(t), V(t)$ giving more information on the flow.

Nicoletta Del Buono
Suppose that the solution $Y(t)$ possesses distinct and nonzero singular values $\sigma_i(t)$, for $i=1,\ldots,n$ and $t$ in $[0, \tau)$ then there exists a continuous SVD of $Y(t)$ and the factors $U(t)$, $\Sigma(t)$, $V(t)$ of such a decomposition satisfy the following ODEs:

- \[ \Sigma = \Sigma^{-1} V^T F(Y, Y^T) V - H \Sigma + \Sigma K, \quad \Sigma(0) = \Sigma_0 \]
- \[ U = U H, \quad U(0) = U_0 \]
- \[ V = V K, \quad V(0) = V_0 \]
Singular Value Decomposition

- The differential equations for the singular values are

\[ \dot{\sigma}_i = \frac{1}{\sigma_i} \left( V^T F (Y, Y^{-T}) V \right)_{ii}, \quad i = 1, \ldots, n \]

- The elements of the skew-symmetric matrices $H, K$ are

\[
H_{ij} = \frac{1}{\sigma_i \sigma_j (\sigma_j^2 - \sigma_i^2)} \left[ \sigma_j^2 \left( V^T F V \right)_{ij} + \sigma_i^2 \left( V^T F V \right)_{ji} \right]
\]

\[
K_{ij} = \frac{1}{(\sigma_j^2 - \sigma_i^2)} \left[ \left( V^T F V \right)_{ij} + \left( V^T F V \right)_{ji} \right]
\]
Singular Value Decomposition

- Numerical solution of:
  - a diagonal system in $\sigma_i$ (information on the conditioning of the matrix solution $Y(t)$)
  - two linear systems in $H_{ij} K_{ij}$
  - two orthogonal systems in $U$ and $V$
    - our aim is to preserve the non-singular behavior of the numerical solution $\rightarrow$ explicit integration of the systems in $U$ and $V$ (orthogonality preserved up to the order of the method)

- Drawback distinct singular values
  - Block Continuous SVD

Nicoletta Del Buono
Some of the previous results can be extended to differential problems on the manifold

\[ GL(m, n) = \{ Y \in P^{m \times n} \mid \text{rank}(Y) = n \}, \quad n \leq m \]

Differential systems on \( GL(m,n) \) have the following form:

\[ \dot{Y} = G(Y), \quad Y(0) = Y_0 \in GL(n, p) \]

with \( G \) belonging to the tangent space of \( GL(m,n) \):

\[ G(Y) = Y \left( Y^T Y \right)^{-1} F_1(Y) + \left[ I_n - Y \left( Y^T Y \right)^{-1} Y^T \right] F_2(Y) \]
Rectangular Case: numerical treatment

- **Continuous SVD (economy)**

\[ Y(t) = U_1(t)\Sigma_1(t)V^T(t) \]

- Differentiating we obtain the differential systems satisfied by the three factors:
Rectangular Case: numerical treatment

- \( \sigma_i = \frac{1}{\sigma_i} \left( V^T F_1(Y)V \right)_{ij} \) \( i = 1, \ldots, n \)
- \( V = VK, \quad V(0) = V_0 \)
- \( U_1 = U_1H + (I_n - U_1U_1^T)F_2(Y)\Sigma_1^{-1}, \quad U(0) = U_0 \)

Differential System on the Stiefel manifold

\[
H_{ij} = \frac{1}{\sigma_i \sigma_j (\sigma_j^2 - \sigma_i^2)} \left[ \sigma_j^2 \left( V^T F_1(Y)V \right)_{ij} + \sigma_i^2 \left( V^T F_1(Y)V \right)_{ji} \right]
\]

\[
K_{ij} = \frac{1}{(\sigma_j^2 - \sigma_i^2)} \left[ \left( V^T F_1(Y)V \right)_{ij} + \left( V^T F_1(Y)V \right)_{ji} \right]
\]
Rectangular Case: numerical treatment

- **Substituting approach:**
  \[ \dot{Y} = Y(Y^TY)^{-1} F_1(Y) + \left[ I_n - Y(Y^TY)^{-1} Y^T \right] F_2(Y) \]

- **Setting** \( Z = (Y^TY)^{-1} \) we obtain
  \[
  \begin{align*}
  \dot{Y} &= YZF_1(Y) + \left[ I - YZY^T \right] F_2(Y) \\
  \dot{Z} &= -Z \left[ F_1(Y) + F_1^T(Y) \right] Z
  \end{align*}
  \]
First example:
\[
\dot{Y} = Y^{-T} \begin{bmatrix} 0 & -\frac{\delta}{2} \\ -\frac{\delta}{2} & 0 \end{bmatrix}
\]
\[
Y(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\]

With solution existing in \((-1/\delta, 1/\delta)\)

\[
Y(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+\delta t} & -\sqrt{1+\delta t} \\ \sqrt{1-\delta t} & \sqrt{1-\delta t} \end{bmatrix}
\]

We solve the problem with \(\delta = 1/2\)
Numerical Illustrations

- Behaviour of the global error on [0 2)
Numerical Illustrations

- Second example
  
  \[
  Y = Y^{-T} \begin{bmatrix}
  -\sin(t)\cos(t) & \cos(t) \\
  -t\sin(t) & t
  \end{bmatrix} \quad Y(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
  \]

- with solution
  
  \[
  Y(t) = \begin{bmatrix} \cos(t) & t \\ 0 & 1 \end{bmatrix}
  \]

- periodically singular (for each \( \tau_k = k \pi/2 \))
Nicoletta Del Buono

Numerical Illustrations

- Semilog plot of the global error on $(\pi/4, \pi/2)$
Conclusions

- We have considered a particular ODEs on $GL(n)$ often occurring in applications
- Several problems modeled by such ODEs
- Different numerical approaches avoiding the direct use of matrix inversion and detection of singular behavior

**Future works:**
- Improving the validation of the proposed approaches by tackling numerical tests on real examples

*Nicoletta Del Buono*