Lecture notes

Summary: Constantin and Fefferman's regularity result

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1 Introduction

We consider the incompressible Navier–Stokes equations in $\mathbb{R}^3$, assuming suitable decay of the solution at infinity. Our goal is to provide the *essential* arguments underlying the conditional regularity result of Constantin and Fefferman [1] from 1993. The main theorem they prove can be stated as follows.

Theorem 1 (Constantin and Fefferman, 1993) *Suppose there exists constants Ω and ρ such that*

$$|\sin\phi| \leq \frac{|\boldsymbol{y}|}{\rho},$$

holds whenever $|\boldsymbol{\omega}(\boldsymbol{x},t)| > \Omega$ and $|\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y},t)| > \Omega$, for $0 \leqslant t \leqslant T$ for any $T > 0$. Here $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{x},t)$ is the vorticity field and ϕ is the angle between the vorticity vectors $\boldsymbol{\omega}(\boldsymbol{x},t)$ and $\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y},t)$. Then the solution to the initial value problem of the Navier–Stokes equation is strong and hence smooth on the time interval $[0,T]$.

Our proof is brief. We will list the caveats thus induced at the end.

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2 Proof

2.1 Enstrophy evolution

We start by writing the incompressible Navier–Stokes equations in the form

$$\partial_t \boldsymbol{u} + \boldsymbol{\omega} \wedge \boldsymbol{u} = \nu\, \Delta \boldsymbol{u} - \nabla\big(p + \tfrac{1}{2}|\boldsymbol{u}|^2\big),$$
$$\nabla \cdot \boldsymbol{u} = 0,$$

where $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}$ is the vorticity. To achieve this, we substitute the identity

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \tfrac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big) - \boldsymbol{u} \wedge (\nabla \wedge \boldsymbol{u})$$

into the standard formulation of the Navier–Stokes equations. Taking the curl of the Navier–Stokes equations and using for divergence free fields $\boldsymbol{u}$ with $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}$ we have $\nabla \wedge (\boldsymbol{\omega} \wedge \boldsymbol{u}) = \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$, we obtain the following evolution equation for the vorticity,

$$\partial_t \boldsymbol{\omega} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu\, \Delta \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}.$$

If R is the antisymmetric part of $\nabla \boldsymbol{u}$ and D is the symmetric part of $\nabla \boldsymbol{u}$ which is also called the deformation matrix, then we observe that $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = D\boldsymbol{\omega}$ as $R\boldsymbol{\omega} \equiv \boldsymbol{0}$. Hence the evolution of the vorticity is equivalent to the form

$$\partial_t \boldsymbol{\omega} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu\, \Delta \boldsymbol{\omega} + D\boldsymbol{\omega}.$$

Consider the L^2-inner product of this evolution equation with the vorticity itself. This generates the equation for the evolution of the enstrophy

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\omega}\|_{L^2}^2 + \nu\|\nabla \boldsymbol{\omega}\|_{L^2}^2 = \int \boldsymbol{\omega} \cdot (D\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{x}.$$

Here we have used that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\boldsymbol{\omega}(\boldsymbol{x},t)|^2\,\mathrm{d}\boldsymbol{x} = 2\int \boldsymbol{\omega}(\boldsymbol{x},t) \cdot \frac{\partial}{\partial t}\boldsymbol{\omega}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x},$$

and that

$$\int \boldsymbol{\omega} \cdot \Delta \boldsymbol{\omega}\,\mathrm{d}\boldsymbol{x} = \int \Delta\big(\tfrac{1}{2}|\boldsymbol{\omega}|^2\big)\,\mathrm{d}\boldsymbol{x} - \int |\nabla \boldsymbol{\omega}(\boldsymbol{x},t)|^2\,\mathrm{d}\boldsymbol{x}.$$

We implicitly assume suitable decay for the vorticity at infinity so that the boundary integral term above (the first term on the right) is zero.

Remark 1 The main idea in Constantin and Fefferman's paper is to try to be more subtle about estimating the vorticity stretching term.

Remark 2 Since $\boldsymbol{u}$ is divergence-free the following quantities are equivalent:

$$\|\nabla \boldsymbol{u}\|_{L^2}^2 = \|\boldsymbol{\omega}\|_{L^2}^2 = 2\int \mathrm{tr}\big(D^2\big)\,\mathrm{d}\boldsymbol{x}.$$

2.2 **Biot–Savart Law**

Since $\nabla\cdot\boldsymbol{u}=0$, there exists a vector potential $\boldsymbol{\psi}$ such that $\boldsymbol{u}=\nabla\wedge\boldsymbol{\psi}$. Hence we see that $\boldsymbol{\omega}=\nabla\wedge\nabla\wedge\boldsymbol{\psi}\equiv-\Delta\boldsymbol{\psi}$ and thus $\boldsymbol{u}=-\nabla\wedge(\Delta^{-1}\boldsymbol{\omega})$. This is the Biot–Savart Law from potential theory. More explicitly, we have

$$\boldsymbol{u}(\boldsymbol{x})=\frac{1}{4\pi}\int\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\wedge\nabla\left(\frac{1}{|\boldsymbol{y}|}\right)\mathrm{d}\boldsymbol{y}.$$

In the integrand we can freely swap $\boldsymbol{x}+\boldsymbol{y}$ and $\boldsymbol{y}$.

2.3 **Deformation matrix**

Taking the gradient with respect to $\boldsymbol{x}$ of the Biot–Savart Law we see that

$$\begin{aligned}\nabla\boldsymbol{u}(\boldsymbol{x})&=\frac{1}{4\pi}\int\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\wedge\nabla\nabla\left(\frac{1}{|\boldsymbol{y}|}\right)\mathrm{d}\boldsymbol{y}\\&=\frac{1}{4\pi}\int\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\wedge\big(3\,\hat{\boldsymbol{y}}\otimes\hat{\boldsymbol{y}}-I\big)\frac{1}{|\boldsymbol{y}|^3}\,\mathrm{d}\boldsymbol{y},\end{aligned}$$

where we have used that by direct computation

$$\nabla\nabla\left(\frac{1}{|\boldsymbol{y}|}\right)=\big(3\,\hat{\boldsymbol{y}}\otimes\hat{\boldsymbol{y}}-I\big)\frac{1}{|\boldsymbol{y}|^3}.$$

Here I is the 3×3 identity matrix, we set $\hat{\boldsymbol{y}}\coloneqq\boldsymbol{y}/|\boldsymbol{y}|$ and $\hat{\boldsymbol{y}}\otimes\hat{\boldsymbol{y}}$ denotes the 3×3 matrix with (i,j)th entries given by y_iy_j.

Using that for any 3×3 matrix A, the quantity $\boldsymbol{\omega}\wedge A$ denotes the 3×3 matrix whose columns are the vector cross product of $\boldsymbol{\omega}$ with the corresponding column of A, we compute $\boldsymbol{\omega}\wedge I$ as follows. Each of the columns of I is the unit vector $\boldsymbol{e}_i$ with 1 at position i and zero elsewhere. Direct computation of the cross product reveals

$$\begin{aligned}\boldsymbol{\omega}\wedge\boldsymbol{e}_1&=\omega_3\boldsymbol{e}_2-\omega_2\boldsymbol{e}_3,\\\boldsymbol{\omega}\wedge\boldsymbol{e}_2&=\omega_1\boldsymbol{e}_3-\omega_3\boldsymbol{e}_1,\\\boldsymbol{\omega}\wedge\boldsymbol{e}_3&=\omega_2\boldsymbol{e}_1-\omega_1\boldsymbol{e}_2.\end{aligned}$$

Hence $\boldsymbol{\omega}\wedge I\equiv R$ the rotation matrix or antisymmetric part of $\nabla\boldsymbol{u}$. Since $R^{\mathrm{T}}=-R$, we have $R+R^{\mathrm{T}}=O$.

When we compute D, the symmetric part of $\nabla\boldsymbol{u}$, since $\boldsymbol{\omega}\wedge I+(\boldsymbol{\omega}\wedge I)^{\mathrm{T}}=O$, the term involving the identity is zero, and we are left with computing the sum of the terms

$$\boldsymbol{\omega}\wedge(\hat{\boldsymbol{y}}\otimes\hat{\boldsymbol{y}})=(\boldsymbol{\omega}\wedge\hat{\boldsymbol{y}})\otimes\hat{\boldsymbol{y}}\qquad\text{and}\qquad\big(\boldsymbol{\omega}\wedge(\hat{\boldsymbol{y}}\otimes\hat{\boldsymbol{y}})\big)^{\mathrm{T}}=\hat{\boldsymbol{y}}\otimes(\boldsymbol{\omega}\wedge\hat{\boldsymbol{y}}).$$

Hence we deduce

$$D(\boldsymbol{x})=\frac{3}{8\pi}\int\big(\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\wedge\hat{\boldsymbol{y}}\otimes\hat{\boldsymbol{y}}+\hat{\boldsymbol{y}}\otimes\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\wedge\hat{\boldsymbol{y}}\big)\frac{1}{|\boldsymbol{y}|^3}\,\mathrm{d}\boldsymbol{y}.$$

2.4 **Vorticity stretching**

Now consider the vorticity stretching term. If we set $\hat{\boldsymbol{\omega}} := \boldsymbol{\omega}/|\boldsymbol{\omega}|$, then we see

$$\big(\hat{\boldsymbol{\omega}}\cdot(D\hat{\boldsymbol{\omega}})\big)(\boldsymbol{x}) = \frac{3}{4\pi}\int\Big(\hat{\boldsymbol{\omega}}(\boldsymbol{x})\cdot\big(\hat{\boldsymbol{\omega}}(\boldsymbol{x}+\boldsymbol{y})\wedge\hat{\boldsymbol{y}}\big)\Big)\big(\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{\omega}}(\boldsymbol{x})\big)\big|\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\big|\frac{1}{|\boldsymbol{y}|^3}\,\mathrm{d}\boldsymbol{y}.$$

Note that the integrand contains the triple scalar product given by

$$\hat{\boldsymbol{\omega}}(\boldsymbol{x})\cdot\big(\hat{\boldsymbol{\omega}}(\boldsymbol{x}+\boldsymbol{y})\wedge\hat{\boldsymbol{y}}\big).$$

This is invariant to a cyclic rotation of the vectors therein and is equivalent to

$$\hat{\boldsymbol{y}}\cdot\big(\hat{\boldsymbol{\omega}}(\boldsymbol{x})\wedge\hat{\boldsymbol{\omega}}(\boldsymbol{x}+\boldsymbol{y})\big).$$

Importantly though, in magnitude it has an upper bound given by

$$\Big|\hat{\boldsymbol{y}}\cdot\big(\hat{\boldsymbol{\omega}}(\boldsymbol{x})\wedge\hat{\boldsymbol{\omega}}(\boldsymbol{x}+\boldsymbol{y})\big)\Big| \leqslant |\sin\phi|,$$

where ϕ is the angle between $\hat{\boldsymbol{\omega}}(\boldsymbol{x})$ and $\hat{\boldsymbol{\omega}}(\boldsymbol{x}+\boldsymbol{y})$. Hence we immediately see

$$\int\big|\boldsymbol{\omega}(\boldsymbol{x})\big|^2\,\big(\hat{\boldsymbol{\omega}}\cdot(D\hat{\boldsymbol{\omega}})\big)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \leqslant \frac{3}{4\pi}\int\big|\boldsymbol{\omega}(\boldsymbol{x})\big|^2\int|\sin\phi|\,\frac{\big|\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\big|}{|\boldsymbol{y}|^3}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.$$

2.5 **Estimating vorticity stretching**

Using the assumption on $|\sin\phi|$ stated, we see the vorticity stretching term can be bounded as follows

$$\int\big|\boldsymbol{\omega}(\boldsymbol{x})\big|^2\,\big(\hat{\boldsymbol{\omega}}\cdot(D\hat{\boldsymbol{\omega}})\big)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \leqslant \frac{3}{4\pi\rho}\int\big|\boldsymbol{\omega}(\boldsymbol{x})\big|^2\int\big|\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\big|\frac{1}{|\boldsymbol{y}|^2}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.$$

To bound the term on the righthand side we can use the Hardy–Littlewood–Sobolev inequality as follows (see for example Khotyakov [2]).

Theorem 2 (Hardy–Littlewood–Sobolev inequality [2]) *For every* $0<\lambda<d$ *and for every* $f\in L^p(\mathbb{R}^d)$ *and* $g\in L^q(\mathbb{R}^d)$ *with* $p,q>1$ *and* $1/p+\lambda/d+1/q=2$ *there exists a sharp constant* $C(d,\lambda,p)$ *such that*

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\boldsymbol{x})|\boldsymbol{x}-\boldsymbol{y}|^{-\lambda}g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\right| \leqslant C(d,\lambda,p)\,\|f\|_p\|g\|_q.$$

We apply this result to the double integral above where in our case $d=3$, $\lambda=2$, $f(\boldsymbol{x})=|\boldsymbol{\omega}(\boldsymbol{x})|^2$ and $g(\boldsymbol{y})=|\boldsymbol{\omega}(\boldsymbol{y})|$—with time dependence suppressed.

With $p = 1$ and $q = 3$ we explicitly find that (denoting multiplicative constants by a generic finite constant C)

$$\begin{aligned}\frac{3}{4\pi\rho}\int\big|\boldsymbol{\omega}(\boldsymbol{x})\big|^2\int\big|\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\big|\frac{1}{|\boldsymbol{y}|^2}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}&\\ &\leqslant C\big\||\boldsymbol{\omega}|^2\big\|_{L^1}\|\boldsymbol{\omega}\|_{L^3}\\ &= C\big\|\boldsymbol{\omega}\big\|_{L^2}^2\|\boldsymbol{\omega}\|_{L^3}\\ &\leqslant \|\boldsymbol{\omega}\|_{L^2}^2\|\nabla\boldsymbol{\omega}\|_{L^2}^{1/2}\|\boldsymbol{\omega}\|_{L^2}^{1/2}\\ &= C\|\nabla\boldsymbol{\omega}\|_{L^2}^{1/2}\|\boldsymbol{\omega}\|_{L^2}^{5/2}\\ &\leqslant \frac{\nu}{8}\|\nabla\boldsymbol{\omega}\|_{L^2}^2+\frac{C}{\nu^{1/3}}\|\boldsymbol{\omega}\|_{L^2}^{10/3}\\ &= \frac{\nu}{8}\|\nabla\boldsymbol{\omega}\|_{L^2}^2+\frac{C}{\nu^{1/3}}\big(\|\boldsymbol{\omega}\|_{L^2}^2\big)^{2/3}\|\boldsymbol{\omega}\|_{L^2}^2.\end{aligned}$$

where we have used the Sobolev–Gagliardo–Nirenberg and Young inequalities.

2.6 **Bounded enstrophy**

From the energy estimate for $\|\boldsymbol{u}\|_{L^2}^2$ it is known that $\|\boldsymbol{\omega}\|_{L^2}^2$ is integrable on $[0,T]$. Thus the coefficient of $\|\boldsymbol{\omega}\|_{L^2}^2$ in the last inequality is integrable in time. Hence if we combine the inequality above with our result in Section 2.1, we can integrate the resulting inequality to show that $\|\boldsymbol{\omega}\|_{L^2}^2$ is uniformly bounded on $[0,T]$ for any $T > 0$. A standard result in Navier–Stokes analysis is if the quantity $\|\boldsymbol{\omega}\|_{L^2}^2$ is known to be bounded on any time interval $[0,T]$ for any $T > 0$, then the solution to the Navier–Stokes equation is smooth (on $\mathbb{R}^3$) on that time interval.

3 Caveats

Here is the list of results that we glossed over in our proof of the Constantin and Fefferman conditional regularity result above:

1. Constantin and Fefferman establish, in a very succint proof on page 782, that $\|\boldsymbol{\omega}\|_{L_1}$ is bounded on any finite time interval.
2. More rigorously we should actually consider an approximate system to the Navier–Stokes equations for which we know global regularity—say the approximate system is a perturbation by a parameter ϵ away. We would carry through analogous estimates for the approximate system to those above, proving bounds uniform in ϵ. Then passing to a subsequence if necessary, we would take the limit as $\epsilon \to 0$. In the case of Constantin and Fefferman, they constructed their approximate system by mollifying the advecting velocity, i.e. by replacing the term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ by $\boldsymbol{u}_\epsilon\cdot\nabla\boldsymbol{u}$, where $\boldsymbol{u}_\epsilon$ is a smoother (mollified) velocity field.

References

1. Constantin, P. and Fefferman, C. 1993 *Direction of vorticity and the problem of global regularity for the Navier–Stokes equations*, Indiana University Mathematics Journal **42**(3), pp. 775–789.
2. Khotyakov, M. 2011 *Two proofs of the sharp Hardy–Littlewood–Sobolev inequality*, Bachelor Thesis, Mathematics Department, LMU Munich.