Differential Equations and Linear Algebra
Lecture Notes

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Contents

CHAPTER 1

Linear second order ODEs

1.1. Newton's second law

We shall begin by stating Newton's fundamental kinematic law relating the force, mass and acceleration of an object whose position is $y(t)$ at time t.

> *Newton's second law* states that the force F applied to an object is equal to its mass m times its acceleration $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$, i.e.
> $$F = m\frac{\mathrm{d}^2y}{\mathrm{d}t^2}.$$

1.1.1. Example. Find the position/height $y(t)$, at time t, of a body falling freely under gravity (take the convention, that we measure positive displacements upwards).

1.1.2. Solution. The equation of motion of a body falling freely under gravity, is, by Newton's second law,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -g\,. \tag{1.1}$$

We can solve equation (1.1) by integrating with respect to t, which yields an expression for the velocity of the body,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -gt + v_0\,,$$

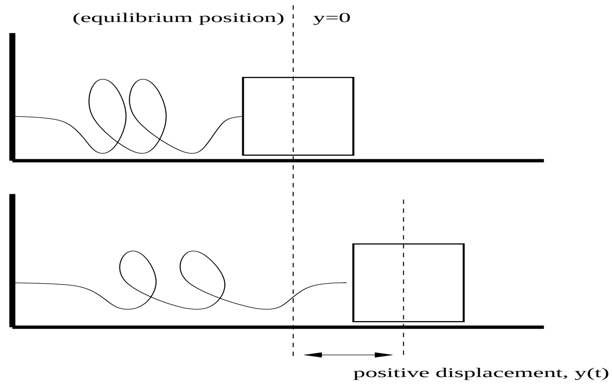


FIGURE 1.1. Mass m slides freely on the horizontal surface, and is attached to a spring, which is fixed to a vertical wall at the other end. We take the convention that positive displacements are measured to the right.

where v_0 is the constant of integration which here also happens to be the initial velocity. Integrating again with respect to t gives

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0 \,,$$

where y_0 is the second constant of integration which also happens to be the initial height of the body.

Equation (1.1) is an example of a second order differential equation (because the highest derivative that appears in the equation is second order):

- the solutions of the equation are a *family of functions with two parameters* (in this case v_0 and y_0);
- choosing values for the two parameters, corresponds to choosing a particular function of the family.

1.2. Springs and Hooke's Law

Consider a mass m Kg on the end of a spring, as in Figure 1.1. With the initial condition that the mass is pulled to one side and then released, what do you expect to happen?

Hooke's law implies that, provided y is not so large as to deform the spring, then the restoring force is

$$F_{\text{spring}} = -ky\,,$$

where the constant $k > 0$ depends on the properties of the spring, for example its stiffness.

1.2.1. Equation of motion. Combining Hooke's law and Newton's second law implies

$$\begin{aligned}
& & m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -ky \\
(\text{assuming } m \neq 0) &\quad\Leftrightarrow & \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -\frac{k}{m}y \\
(\text{setting } \omega = +\sqrt{k/m}) &\quad\Leftrightarrow & \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -\omega^2 y\,. \qquad (1.2)
\end{aligned}$$

Can we guess a solution of (1.2), i.e. a function that satisfies the relation (1.2)? We are essentially asking ourselves: what function, when you differentiate it twice, gives you minus ω^2 times the original function you started with?

The *general solution* to the linear ordinairy differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \omega^2 y = 0\,,$$

is

$$y(t) = C_1 \sin\omega t + C_2 \cos\omega t\,, \qquad (1.3)$$

where C_1 and C_2 are arbitrary constants. This is an oscillatory solution with *frequency of oscillation* ω. The *period* of the oscillations is

$$T = \frac{2\pi}{\omega}\,.$$

Recall that we set $\omega = +\sqrt{k/m}$ and this parameter represents the frequency of oscillations of the mass. How does the general solution change as you vary m and k? Does this match your physical intuition?

What do these solutions really look like? We can re-express the solution (1.3) as follows. Consider the C_1, C_2 plane as shown in Figure 1.2. Hence

$$\begin{aligned}
C_1 &= A\cos\phi\,, \\
C_2 &= A\sin\phi\,.
\end{aligned}$$

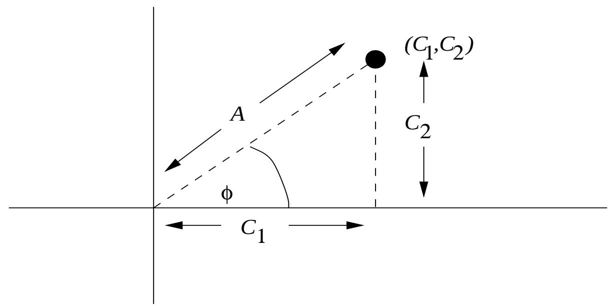


FIGURE 1.2. Relation between (C_1, C_2) and (A, ϕ).

Substituting these expressions for C_1 and C_2 into (1.3) we get

$$\begin{aligned} y(t) &= A\cos\phi\sin\omega t + A\sin\phi\cos\omega t \\ &= A(\cos\phi\sin\omega t + \sin\phi\cos\omega t) \\ &= A\sin(\omega t + \phi)\,. \end{aligned}$$

The general solution (1.3) can *also* be expressed in the form

$$y(t) = A\sin(\omega t + \phi)\,, \tag{1.4}$$

where

$A = +\sqrt{C_1^2 + C_2^2} \geq 0$ is the *amplitude* of oscillation,

$\phi = \arctan(C_2/C_1)$ is the *phase angle*, with $-\pi < \phi \leq \pi$.

1.2.2. Example (initial value problem). Solve the differential equation for the spring,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{k}{m}y\,,$$

if the mass were displaced by a distance y_0 and then released. This is an example of an *initial value problem*, where the initial position and the initial velocity are used to determine the solution.

1.2.3. Solution. We have already seen that the position of the mass at time t is given by

$$y(t) = C_1\sin\omega t + C_2\cos\omega t\,, \tag{1.5}$$

with $\omega = +\sqrt{k/m}$, for some constants C_1 and C_2. The initial position is y_0, i.e. $y(0) = y_0$. Substituting this information into (1.5), we see that

$$\begin{aligned} && y(0) &= C_1 \sin(\omega \cdot 0) + C_2 \cos(\omega \cdot 0) \\ \Leftrightarrow && y_0 &= C_1 \cdot 0 + C_2 \cdot 1 \\ \Leftrightarrow && y_0 &= C_2 \,. \end{aligned}$$

The initial velocity is zero, i.e. $y'(0) = 0$. Differentiating (1.5) and substituting this information into the resulting expression for $y'(t)$ implies

$$\begin{aligned} && y'(0) &= C_1 \omega \cos(\omega \cdot 0) - C_2 \omega \sin(\omega \cdot 0) \\ \Leftrightarrow && 0 &= C_1 \omega \cdot 1 - C_2 \omega \cdot 0 \\ \Leftrightarrow && 0 &= C_1 \,. \end{aligned}$$

Therefore the solution is $y(t) = y_0 \cos \omega t$. Of course this is an oscillatory solution with frequency of oscillation ω, and in this case, the amplitude of oscillation y_0.

1.2.4. Damped oscillations. Consider a more realistic spring which has *friction*.

> In general, the *frictional force* or *drag* is proportional to the velocity of the mass, i.e.
> $$F_{\text{friction}} = -C \, \frac{\mathrm{d}y}{\mathrm{d}t} \,,$$
> where C is a constant known as the *drag* or *friction* coefficient. The frictional force acts in a direction opposite to that of the motion and so $C > 0$.

Newton's Second Law implies (adding the restoring and frictional forces together)

$$m \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = F_{\text{spring}} + F_{\text{friction}} \,,$$

i.e.

$$m \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -ky - C \frac{\mathrm{d}y}{\mathrm{d}t}.$$

Hence the damped oscillations of a spring are described by the differential equation

$$m \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + C \frac{\mathrm{d}y}{\mathrm{d}t} + ky = 0. \tag{1.6}$$

1.2.5. Remark. We infer the general principles: for elastic solids, stress is proportional to strain (how far you are pulling neighbouring particles apart), whereas for fluids, stress is proportional to the rate of strain (how fast you are pulling neighbouring particles apart). Such fluids are said to be Newtonian fluids, and everyday examples include water and simple oils etc. There are also many non-Newtonian fluids. Some of these retain some solid-like elasticity properties. Examples include solutes of long-chain protein molecules such as saliva.

1.3. General ODEs and their classification

1.3.1. Basic definitions. The basic notions of differential equations and their solutions can be outlined as follows.

Differential Equation (DE). An equation relating two or more variables in terms of derivatives or differentials.

Solution of a Differential Equation. Any functional relation, *not* involving derivatives or integrals of "unknown" functions, which satisfies the differential equation.

General Solution. A description of all the functional relations that satisfy the differential equation.

Ordinary Differential Equation (ODE). A differential equation relating only two variables. A general n^{th} order ODE is often represented by

$$F\left(t, y, \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}t^n}\right) = 0\,, \tag{1.7}$$

where F is some given (known) function.

In equation (1.7), we usually call t the *independent variable*, and y is the *dependent variable*.

1.3.2. Example. Newton's second law implies that, if $y(t)$ is the position, at time t, of a particle of mass m acted upon by a force f, then

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = f\left(t, y, \frac{\mathrm{d}y}{\mathrm{d}t}\right)\,,$$

where the given force f may be a function of t, y and the velocity $\frac{\mathrm{d}y}{\mathrm{d}t}$.

1.3.3. Classification of ODEs. ODEs are classified according to *order*, *linearity* and *homogeneity*.

Order. The *order* of a differential equation is the order of the highest derivative present in the equation.

Linear or nonlinear. A second order ODE is said to be *linear* if it can be written in the form

$$a(t)\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b(t)\frac{\mathrm{d}y}{\mathrm{d}t} + c(t)y = f(t)\,, \tag{1.8}$$

where the *coefficients* $a(t)$, $b(t)$ & $c(t)$ can, in general, be functions of t. An equation that is *not* linear is said to be *nonlinear*. Note that linear ODEs are characterised by two properties:

(1) The dependent variable and all its derivatives are of *first degree*, i.e. the power of each term involving y is 1.
(2) Each coefficient depends on the independent variable t *only*.

Homogeneous or non-homogeneous. The linear differential equation (1.8) is said to be *homogeneous* if $f(t) \equiv 0$, otherwise, if $f(t) \neq 0$, the differential equation is said to be *non-homogeneous*. More generally, an equation is said to be *homogeneous* if $ky(t)$ is a solution whenever $y(t)$ is also a solution, for any constant k, i.e. the equation is invariant under the transformation $y(t) \to ky(t)$.

1.3.4. Example. The differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 5\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^3 - 4y = \mathrm{e}^t\,,$$

is second order because the highest derivative is second order, and nonlinear because the second term on the left-hand side is cubic in y'.

1.3.5. Example (higher order linear ODEs). We can generalize our characterization of a *linear* second order ODE to higher order linear ODEs. We recognize that a linear third order ODE must have the form

$$a_3(t)\frac{\mathrm{d}^3y}{\mathrm{d}t^3} + a_2(t)\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + a_1(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_0(t)y = f(t)\,,$$

for a given set of coefficient functions $a_3(t)$, $a_2(t)$, $a_1(t)$ and $a_0(t)$, and a given inhomogeneity $f(t)$. A linear fourth order ODE must have the form

$$a_4(t)\frac{\mathrm{d}^4y}{\mathrm{d}t^4} + a_3(t)\frac{\mathrm{d}^3y}{\mathrm{d}t^3} + a_2(t)\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + a_1(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_0(t)y = f(t)\,,$$

while a general n^{th} order linear ODE must have the form

$$a_n(t)\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}(t)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_2(t)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a_1(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_0(t)y = f(t)\,.$$

1.3.6. Example (scalar higher order ODE as a system of first order ODEs). Any n^{th} order ODE (linear or nonlinear) can always we written as a system of n first order ODEs. For example, if for the ODE

$$F\left(t, y, \frac{\mathrm{d}y}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}t^n}\right) = 0\,, \tag{1.9}$$

we identify new variables for the derivative terms of each order, then (1.9) is equivalent to the system of n first order ODEs in n variables

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= y_1\,,\\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= y_2\,,\\ &\vdots\\ \frac{\mathrm{d}y_{n-2}}{\mathrm{d}t} &= y_{n-1}\,,\\ F\left(t, y, y_1, y_2, \ldots, y_{n-1}, \frac{\mathrm{d}y_{n-1}}{\mathrm{d}t}\right) &= 0\,. \end{aligned}$$

1.4. Exercises

1.1. The following differential equations represent oscillating springs.

(1) $y'' + 4y = 0, \quad y(0) = 5, \quad y'(0) = 0,$

(2) $4y'' + y = 0, \quad y(0) = 10, \quad y'(0) = 0,$

(3) $y'' + 6y = 0, \quad y(0) = 4, \quad y'(0) = 0,$

(4) $6y'' + y = 0, \quad y(0) = 20, \quad y'(0) = 0.$

Which differential equation represents

(a): the spring oscillating most quickly (with the shortest period)?

(b): the spring oscillating with the largest amplitude?

(c): the spring oscillating most slowly (with the longest period)?

(d): the spring oscillating with the largest maximum velocity?

1.2. (Pendulum.) A mass is suspended from the end of a light rod of length, l, the other end of which is attached to a fixed pivot so that the rod can swing freely in a vertical plane. Let $\theta(t)$ be the displacement angle (in radians) at time, t, of the rod to the vertical. Note that the arclength, $y(t)$, of the mass is given by $y = \ell\theta$. Using Newton's second law and the tangential component (to its natural motion) of the weight of the pendulum, the differential equation governing the motion of the mass is (g is the acceleration due to gravity)

$$\theta'' + \frac{g}{\ell}\sin\theta = 0\,.$$

Explain why, if we assume the pendulum bob only performs small oscillations about the equilibrium vertical position, i.e. so that $|\theta(t)| \ll 1$, then the equation governing the motion of the mass is, to a good approximation,

$$\theta'' + \frac{g}{\ell}\theta = 0\,.$$

Suppose the pendulum bob is pulled to one side and released. Solve this initial value problem explicitly and explain how you might have predicted the nature of the solution. How does the solution behave for different values of ℓ? Does this match your physical intuition?

CHAPTER 2

Homogeneous linear ODEs

2.1. The Principle of Superposition

Principle of Superposition for linear homogeneous differential equations. Consider the linear, second order, homogeneous, ordinary differential equation

$$a(t)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b(t)\frac{\mathrm{d}y}{\mathrm{d}t} + c(t)y = 0\,, \tag{2.1}$$

where $a(t)$, $b(t)$ and $c(t)$ are known functions.

(1) If $y_1(t)$ and $y_2(t)$ satisfy (2.1), then for any two constants C_1 and C_2,

$$y(t) = C_1 y_1(t) + C_2 y_2(t) \tag{2.2}$$

is a solution also.

(2) If $y_1(t)$ is not a constant multiple of $y_2(t)$, then the *general solution* of (2.1) takes the form (2.2).

2.2. Linear second order constant coefficient homogeneous ODEs

2.2.1. Exponential solutions. We restrict ourselves here to the case when the coefficients a, b and c in (2.1) are constants, i.e.

$$a\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = 0\,, \tag{2.3}$$

Let us try to find a solution to (2.3) of the form

$$y = \mathrm{e}^{\lambda t}\,. \tag{2.4}$$

The reason for choosing the exponential function is that we know that solutions to *linear first order* constant coefficient ODEs always have this form for a specific value of λ that depends on the coefficients. So we will try to look for a solution to a linear second order constant coefficient ODE of the same form, where at the moment we will not specify what λ is—with hindsight we will see that this is a good choice.

Substituting (2.4) into (2.3) implies

$$\begin{aligned} a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy &= a\lambda^2\mathrm{e}^{\lambda t} + b\lambda\mathrm{e}^{\lambda t} + c\mathrm{e}^{\lambda t} \\ &= \mathrm{e}^{\lambda t}(a\lambda^2 + b\lambda + c) \\ \text{which must} \quad &= 0\,. \end{aligned}$$

Since the exponential function is never zero, i.e. $\mathrm{e}^{\lambda t} \neq 0$, then we see that if λ satisfies the *auxiliary equation*:

$$a\lambda^2 + b\lambda + c = 0\,,$$

then (2.4) will be a solution of (2.3). There are three cases we need to consider.

2.2.2. Case I: $b^2 - 4ac > 0$. There are two real and distinct solutions to the auxiliary equation,

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}\,,$$

and so two functions,

$$\mathrm{e}^{\lambda_1 t} \qquad \text{and} \qquad \mathrm{e}^{\lambda_2 t}\,,$$

satisfy the ordinary differential equation (2.3). The Principle of Superposition implies that the general solution is

$$y(t) = C_1\mathrm{e}^{\lambda_1 t} + C_2\mathrm{e}^{\lambda_2 t}\,.$$

2.2.3. Example: $b^2 - 4ac > 0$. Find the general solution to the ODE

$$y'' + 4y' - 5y = 0\,.$$

2.2.4. Solution. Examining the form of this linear second order constant coefficient ODE we see that $a = 1$, $b = 4$ and $c = -5$; and $b^2 - 4ac = 4^2 - 4(1)(-5) = 36 > 0$. We look for a solution of the form $y = \mathrm{e}^{\lambda t}$. Following through the general theory we just outlined we know that for solutions of this form, λ must satisfy the auxiliary equation

$$\lambda^2 + 4\lambda - 5 = 0\,.$$

There are two real distinct solutions (either factorize the quadratic form on the left-hand side and solve, or use the quadratic equation formula)

$$\lambda_1 = -5 \qquad \text{and} \qquad \lambda_2 = 1\,.$$

Hence by the Principle of Superposition the general solution to the ODE is

$$y(t) = C_1 \mathrm{e}^{-5t} + C_2 \mathrm{e}^{t}\,.$$

2.2.5. Case II: $b^2 - 4ac = 0$. In this case there is one real repeated root to the auxiliary equation, namely

$$\lambda_1 = \lambda_2 = -\frac{b}{2a}\,.$$

Hence we have one solution, which is

$$y(t) = \mathrm{e}^{\lambda_1 t} = \mathrm{e}^{-\frac{b}{2a}t}\,.$$

However, we should suspect that there is another independent solution. It's not obvious what that might be, but let's make the educated guess

$$y = t\mathrm{e}^{\lambda_1 t}$$

where λ_1 is the same as above, i.e. $\lambda_1 = -\frac{b}{2a}$. Substituting this guess for the second solution into our second order differential equation,

$$\begin{aligned}
\Rightarrow a\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy &= a\,(\lambda_1^2 t\mathrm{e}^{\lambda_1 t} + 2\lambda_1 \mathrm{e}^{\lambda_1 t}) \\
&\quad + b\,(\mathrm{e}^{\lambda_1 t} + \lambda_1 t\mathrm{e}^{\lambda_t}) \\
&\quad + c\,t\mathrm{e}^{\lambda_1 t} \\
&= \mathrm{e}^{\lambda_1 t}\big(t\,(a\lambda_1^2 + b\lambda_1 + c) + (2a\lambda_1 + b)\big) \\
\text{which in fact} \quad &= 0\,,
\end{aligned}$$

since we note that $a\lambda_1^2 + b\lambda_1 + c = 0$ and $2a\lambda_1 + b = 0$ because $\lambda_1 = -b/2a$. Thus $t\mathrm{e}^{-\frac{b}{2a}t}$ is another solution (which is clearly not a constant multiple of the first solution). The Principle of Superposition implies that the general solution is

$$y = (C_1 + C_2 t)\mathrm{e}^{-\frac{b}{2a}t}\,.$$

2.2.6. Example: $b^2 - 4ac = 0$. Find the general solution to the ODE

$$y'' + 4y' + 4y = 0\,.$$

2.2.7. Solution. In this example $a = 1$, $b = 4$ and $c = 4$; and $b^2 - 4ac = 4^2 - 4(1)(4) = 0$. Again we look for a solution of the form $y = \mathrm{e}^{\lambda t}$. For solutions of this form λ must satisfy the auxiliary equation

$$\lambda^2 + 4\lambda + 4 = 0\,,$$

which has one (repeated) solution

$$\lambda_1 = \lambda_2 = -2\,.$$

We know from the general theory just above that in this case there is in fact another solution of the form $t\mathrm{e}^{\lambda_1 t}$. Hence by the Principle of Superposition the general solution to the ODE is

$$y(t) = (C_1 + C_2 t)\mathrm{e}^{-2t}\,.$$

2.2.8. Case III: $b^2 - 4ac < 0$. In this case, there are two complex roots to the auxiliary equation, namely

$$\lambda_1 = p + \mathrm{i}q\,, \tag{2.5a}$$

$$\lambda_2 = p - \mathrm{i}q\,, \tag{2.5b}$$

where

$$p = -\frac{b}{2a} \qquad \text{and} \qquad q = \frac{+\sqrt{|b^2 - 4ac|}}{2a}\,.$$

Hence the Principle of Superposition implies that the general solution takes the form

$$\begin{aligned}
y(t) &= A_1\mathrm{e}^{\lambda_1 t} + A_2\mathrm{e}^{\lambda_2 t}\\
&= A_1\mathrm{e}^{(p+\mathrm{i}q)t} + A_2\mathrm{e}^{(p-\mathrm{i}q)t}\\
&= A_1\mathrm{e}^{pt+\mathrm{i}qt} + A_2\mathrm{e}^{pt-\mathrm{i}qt}\\
&= A_1\mathrm{e}^{pt}\mathrm{e}^{\mathrm{i}qt} + A_2\mathrm{e}^{pt}\mathrm{e}^{-\mathrm{i}qt}\\
&= \mathrm{e}^{pt}\left(A_1\mathrm{e}^{\mathrm{i}qt} + A_2\mathrm{e}^{-\mathrm{i}qt}\right)\\
&= \mathrm{e}^{pt}\left(A_1\big(\cos qt + \mathrm{i}\sin qt\big) + A_2\big(\cos qt - \mathrm{i}\sin qt\big)\right)\\
&= \mathrm{e}^{pt}\left(\big(A_1 + A_2\big)\cos qt + \mathrm{i}\big(A_1 - A_2\big)\sin qt\big)\right)\,,
\end{aligned} \tag{2.6}$$

where

(1) we have used Euler's formula

$$\mathrm{e}^{\mathrm{i}z} \equiv \cos z + \mathrm{i}\sin z\,,$$

first with $z = qt$ and then secondly with $z = -qt$, i.e. we have used that

$$\mathrm{e}^{\mathrm{i}qt} \equiv \cos qt + \mathrm{i}\sin qt \tag{2.7a}$$

and

$$\mathrm{e}^{-\mathrm{i}qt} \equiv \cos qt - \mathrm{i}\sin qt \tag{2.7b}$$

since $\cos(-qt) \equiv \cos qt$ and $\sin(-qt) \equiv -\sin qt$;

(2) and A_1 and A_2 are arbitrary (and in general complex) constants—at this stage that means we appear to have a total of four constants because A_1 and A_2 both have real and imaginary parts. However we expect the solution $y(t)$ to be real—the coefficients are real and we will pose real initial data.

The solution $y(t)$ in (2.6) will be real if and only if

$$\begin{aligned} A_1 + A_2 &= C_1 \,, \\ \mathrm{i}(A_1 - A_2) &= C_2 \,, \end{aligned}$$

where C_1 and C_2 are *real* constants—in terms of the initial conditions note that $C_1 = y_0$ and $C_2 = (v_0 - py_0)/q$ where y_0 and v_0 are the initial position and "velocity" data, respectively. Hence the *general solution* in this case has the form

$$y(t) = \mathrm{e}^{pt}(C_1 \cos qt + C_2 \sin qt) \,.$$

2.2.9. Example: $b^2 - 4ac < 0$**.** Find the general solution to the ODE

$$2y'' + 2y' + y = 0 \,.$$

2.2.10. Solution. In this case $a = 2$, $b = 2$ and $c = 1$; and $b^2 - 4ac = 2^2 - 4(2)(1) = -4 < 0$. Again we look for a solution of the form $y = \mathrm{e}^{\lambda t}$. For solutions of this form λ must satisfy the auxiliary equation

$$2\lambda^2 + 2\lambda + 1 = 0 \,.$$

The quadratic equation formula is the quickest way to find the solutions of this equation in this case

$$\begin{aligned} \lambda &= \frac{-2 \pm \sqrt{2^2 - 4(2)(1)}}{2(2)} \\ &= \frac{-2 \pm \sqrt{-4}}{4} \\ &= \frac{-2 \pm \sqrt{(-1)(4)}}{4} \\ &= \frac{-2 \pm \sqrt{-1}\sqrt{4}}{4} \\ &= \frac{-2 \pm 2\mathrm{i}}{4} \\ &= \underbrace{-\tfrac{1}{2}}_{p} \pm \underbrace{\tfrac{1}{2}}_{q}\,\mathrm{i} \,. \end{aligned}$$

In other words there are two solutions

$$\lambda_1 = -\tfrac{1}{2} + \tfrac{1}{2}\mathrm{i} \qquad \text{and} \qquad \lambda_2 = -\tfrac{1}{2} + \tfrac{1}{2}\mathrm{i} \,,$$

Case	Roots of auxiliary equation	General solution
$b^2 - 4ac > 0$	$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$	$y = C_1 \mathrm{e}^{\lambda_1 t} + C_2 \mathrm{e}^{\lambda_2 t}$
$b^2 - 4ac = 0$	$\lambda_{1,2} = -\frac{b}{2a}$	$y = \big(C_1 + C_2 t\big)\mathrm{e}^{\lambda_1 t}$
$b^2 - 4ac < 0$	$\lambda_{1,2} = p \pm \mathrm{i}q$ $p = -\frac{b}{2a}$, $q = \frac{+\sqrt{\lvert b^2-4ac\rvert}}{2a}$	$y = \mathrm{e}^{pt}\big(C_1 \cos qt + C_2 \sin qt\big)$

TABLE 2.1. *Solutions to the linear second order, constant coefficient, homogeneous ODE $ay'' + by' + cy = 0$.*

and we can easily identify $p = -\frac{1}{2}$ as the real part of each solution and $q = \frac{1}{2}$ as the absolute value of the imaginary part of each solution.

We know from the general theory just above and the Principle of Superposition that the general solution to the ODE is

$$y(t) = \mathrm{e}^{-\frac{1}{2}t}\Big(C_1 \cos\big(\tfrac{1}{2}t\big) + C_2 \sin\big(\tfrac{1}{2}t\big)\Big)\,.$$

2.3. Practical example: damped springs

2.3.1. Parameters. For the case of the damped spring note that in terms of the physical parameters $a = m > 0$, $b = C > 0$ and $c = k > 0$. Hence

$$b^2 - 4ac = C^2 - 4mk\,.$$

2.3.2. Overdamping: $C^2 - 4mk > 0$. Since the physical parameters m, k and C are all positive, we have that

$$\sqrt{C^2 - 4mk} < |C|\,,$$

and so λ_1 and λ_2 are both negative. Thus for large times ($t \to +\infty$) the solution $y(t) \to 0$ exponentially fast. For example, the mass might be immersed in thick oil. Two possible solutions, starting from two different initial conditions, are shown in Figure 2.1(a). Whatever initial conditions you choose, there is at most one oscillation. At some point, for example past the vertical dotted line on the right, for all practical purposes the spring is in the equilibrium position.

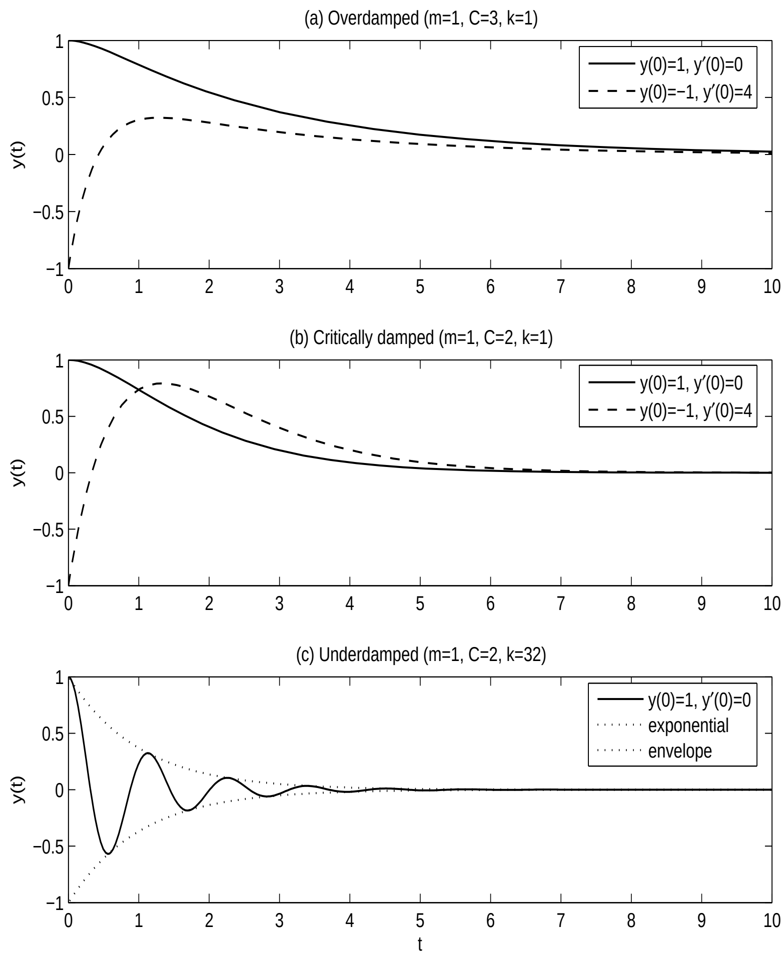


FIGURE 2.1. Overdamping, critical damping and underdamping for a simple mass–spring system. We used the specific values for m, C and k shown. In (a) and (b) we plotted the two solutions corresponding to the two distinct sets of initial conditions shown.

2.3.3. Critical damping: $C^2-4mk=0$. In appearance (see Figure 2.1(b)) the solutions for the critically damped case look very much like those in Figure 2.1(a) for the overdamped case.

2.3.4. Underdamping: $C^2-4mk<0$. Since for the spring

$$p=-\frac{b}{2a}=-\frac{C}{2m}<0\,,$$

the mass will oscillate about the equilibrium position with the amplitude of the oscillations decaying exponentially in time; in fact the solution oscillates between the *exponential envelopes* which are the two dashed curves $A\mathrm{e}^{pt}$ and $-A\mathrm{e}^{pt}$, where $A = +\sqrt{C_1^2 + C_2^2}$—see Figure 2.1(c). In this case, for example, the mass might be immersed in light oil or air.

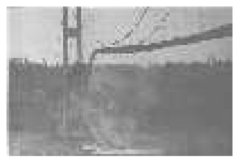
2.4. Exercises

2.1. Find the general solution to the following differential equations:
(a) $y'' + y' - 6y = 0$;
(b) $y'' + 8y' + 16y = 0$;
(c) $y'' + 2y' + 5y = 0$;
(d) $y'' - 3y' + y = 0$.

2.2. For each of the following initial value problems, find the solution, and describe its behaviour:
(a) $5y'' - 3y' - 2y = 0$, with $y(0) = -1$, $y'(0) = 1$;
(b) $y'' + 6y' + 9y = 0$, with $y(0) = 1$, $y'(0) = 2$;
(c) $y'' + 5y' + 8y = 0$, with $y(0) = 1$, $y'(0) = -2$.

CHAPTER 3

Non-homogeneous linear ODEs



3.1. Example applications

3.1.1. Forced spring systems. What happens if our spring system (damped or undamped) is forced externally? For example, consider the following initial value problem for a forced harmonic oscillator (which models a mass on the end of a spring which is forced externally)

$$\begin{aligned} m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + C\frac{\mathrm{d}y}{\mathrm{d}t} + ky &= f(t)\,, \\ y(0) &= 0\,, \\ y'(0) &= 0\,. \end{aligned}$$

Here $y(t)$ is the displacement of the mass, m, from equilibrium at time t. The external forcing $f(t)$ could be oscillatory, say

$$f(t) = A\sin\omega t\,,$$

where A and ω are given positive constants. We will see in this chapter how solutions to such problems can behave quite dramatically when the frequency of the external force ω matches that of the natural oscillations $\omega_0 = +\sqrt{k/m}$ of the undamped ($C \equiv 0$) system—undamped resonance! We will also discuss the phenomenon of resonance in the presence of damping ($C > 0$).

3.1.2. Electrical circuits. Consider a simple loop series circuit which has a resistor with resistance R, a capacitor of capacitance C, an inductor of inductance ℓ (all positive constants) and a battery which provides an impressed voltage $V(t)$. The total charge $Q(t)$ in such a circuit is modelled by the ODE

$$\ell\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{C}Q = V(t)\,. \tag{3.1}$$

'Feedback-squeals' in electric circuits at concerts are an example of resonance effects in such equations.

3.2. Linear operators

3.2.1. Concept. Consider the general non-homogeneous second order linear ODE

$$a(t)\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b(t)\frac{\mathrm{d}y}{\mathrm{d}t} + c(t)y = f(t)\,. \tag{3.2}$$

We can abbreviate the ODE (3.2) to

$$Ly(t) = f(t)\,, \tag{3.3}$$

where L is the *differential operator*

$$L = a(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + b(t)\frac{\mathrm{d}}{\mathrm{d}t} + c(t)\,. \tag{3.4}$$

We can re-interpret our general linear second order ODE as follows. When we operate on a function $y(t)$ by the differential operator L, we generate a new function of t, i.e.

$$Ly(t) = a(t)y''(t) + b(t)y'(t) + c(t)y(t)\,.$$

To solve (3.3), we want the most general expression, y as a function of t, which is such that L operated on y gives $f(t)$.

Definition (Linear operator). An operator L is said to be *linear* if

$$L\big(\alpha y_1 + \beta y_2\big) = \alpha L y_1 + \beta L y_2\,, \tag{3.5}$$

for every y_1 and y_2, and all constants α and β.

3.2.2. Example. The operator L in (3.4) is linear. To show this is true we must demonstrate that the left-hand side in (3.5) equals the right-hand side. Using the properties for differential operators we already know well,

$$\begin{aligned}
L\big(\alpha y_1 + \beta y_2\big) = &\left(a(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + b(t)\frac{\mathrm{d}}{\mathrm{d}t} + c(t)\right)\big(\alpha y_1 + \beta y_2\big) \\
&= a(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(\alpha y_1 + \beta y_2\big) + b(t)\frac{\mathrm{d}}{\mathrm{d}t}\big(\alpha y_1 + \beta y_2\big) + c(t)\big(\alpha y_1 + \beta y_2\big) \\
&= a(t)\left(\alpha\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}^2 y_2}{\mathrm{d}t^2}\right) + b(t)\left(\alpha\frac{\mathrm{d}y_1}{\mathrm{d}t} + \beta\frac{\mathrm{d}y_2}{\mathrm{d}t}\right) \\
&\quad + c(t)\big(\alpha y_1 + \beta y_2\big) \\
&= \alpha\left(a(t)\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + b(t)\frac{\mathrm{d}y_1}{\mathrm{d}t} + c(t)y_1\right) \\
&\quad + \beta\left(a(t)\frac{\mathrm{d}^2 y_2}{\mathrm{d}t^2} + b(t)\frac{\mathrm{d}y_2}{\mathrm{d}t} + c(t)y_2\right) \\
&= \alpha L y_1 + \beta L y_2\,.
\end{aligned}$$

3.3. Solving non-homogeneous linear ODEs

Consider the non-homogeneous linear second order ODE (3.2), which written in abbreviated form is

$$Ly = f\,. \tag{3.6}$$

To solve this problem we first consider the solution to the associated homogeneous ODE (called the *Complementary Function*):

$$Ly_{\mathrm{CF}} = 0\,. \tag{3.7}$$

Since this ODE (3.7) is linear, second order and homogeneous, we can *always* find an expression for the solution—in the constant coefficient case the solution has one of the forms given in Table 2.1. Now suppose that we can find a particular solution—often called the *particular integral (PI)*—of (3.6), i.e. some function, y_{PI}, that satisfies (3.6):

$$Ly_{\mathrm{PI}} = f\,. \tag{3.8}$$

Then the complete, *general solution* of (3.6) is

$$y = y_{\mathrm{CF}} + y_{\mathrm{PI}}\,. \tag{3.9}$$

This must be the general solution because it contains two arbitrary constants (in the y_{CF} part) and satisfies the ODE, since, using that L is a linear operator (i.e. using the property (3.5)),

$$L(y_{\mathrm{CF}} + y_{\mathrm{PI}}) = \underbrace{Ly_{\mathrm{CF}}}_{=0} + \underbrace{Ly_{\mathrm{PI}}}_{=f} = f\,.$$

Hence to summarize, to solve a non-homogeneous equation like (3.6) proceed as follows.

Step 1: Find the complementary function. i.e. find the *general solution* to the corresponding homogeneous equation

$$Ly_{\mathrm{CF}} = 0 \,.$$

Step 2: Find the particular integral. i.e. find *any* solution of

$$Ly_{\mathrm{PI}} = f \,.$$

Step 3: Combine. The *general solution* to (3.6) is

$$y = y_{\mathrm{CF}} + y_{\mathrm{PI}} \,.$$

3.4. Method of undetermined coefficients

We now need to know how to obtain a particular integral y_{PI}. For special cases of the inhomogeneity $f(t)$ we use the *method of undetermined coefficients*, though there is a more general method called the *method of variation of parameters*—see for example Kreyszig [**8**]. In the *method of undetermined coefficients* we make an initial assumption about the form of the particular integral y_{PI} depending on the form of the inhomogeneity f, but with the coefficients left unspecified. We substitute our guess for y_{PI} into the linear ODE, $Ly = f$, and attempt to determine the coefficients so that y_{PI} satisfies the equation.

3.4.1. Example. Find the general solution of the linear ODE

$$y'' - 3y' - 4y = 3\mathrm{e}^{2t} \,.$$

3.4.2. Solution.
Step 1: Find the complementary function. Looking for a solution of the form $\mathrm{e}^{\lambda t}$, the auxiliary equation is $\lambda^2 - 3\lambda - 4 = 0$ which has two real distinct roots $\lambda_1 = 4$ and $\lambda_2 = -1$, hence from Table 2.1, we have

$$y_{\mathrm{CF}}(t) = C_1\mathrm{e}^{4t} + C_2\mathrm{e}^{-t} \,.$$

Step 2: Find the particular integral. Assume that the particular integral has the form (using Table 3.1)

$$y_{\mathrm{PI}}(t) = A\mathrm{e}^{2t} \,,$$

where the coefficient A is yet to be determined. Substituting this form for y_{PI} into the ODE, we get

$$\begin{aligned} & (4A - 6A - 4A)\mathrm{e}^{2t} = 3\mathrm{e}^{2t} \\ \Leftrightarrow \qquad & -6A\mathrm{e}^{2t} = 3\mathrm{e}^{2t} \,. \end{aligned}$$

Hence A must be $-\frac{1}{2}$ and a particular solution is

$$y_{\mathrm{PI}}(t) = -\tfrac{1}{2}\mathrm{e}^{2t} \,.$$

Inhomogeneity $f(t)$	Try $y_{\mathrm{PI}}(t)$
$\mathrm{e}^{\alpha t}$	$A\mathrm{e}^{\alpha t}$
$\sin(\alpha t)$	$A\sin(\alpha t) + B\cos(\alpha t)$
$\cos(\alpha t)$	$A\sin(\alpha t) + B\cos(\alpha t)$
$b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$	$A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n$
$\mathrm{e}^{\alpha t}\sin(\beta t)$	$A\mathrm{e}^{\alpha t}\sin(\beta t) + B\mathrm{e}^{\alpha t}\cos(\beta t)$
$\mathrm{e}^{\alpha t}\cos(\beta t)$	$A\mathrm{e}^{\alpha t}\sin(\beta t) + B\mathrm{e}^{\alpha t}\cos(\beta t)$

TABLE 3.1. *Method of undetermined coefficients.* When the inhomogeneity $f(t)$ has the form (or is any constant multiplied by this form) shown in the left-hand column, then you should try a $y_{\mathrm{PI}}(t)$ of the form shown in the right-hand column. We can also make the obvious extensions for combinations of the inhomogeneities $f(t)$ shown.

Step 3: Combine. Hence the general solution to the differential equation is

$$y(t) = \underbrace{C_1\mathrm{e}^{4t} + C_2\mathrm{e}^{-t}}_{y_{\mathrm{CF}}} \underbrace{-\tfrac{1}{2}\mathrm{e}^{2t}}_{y_{\mathrm{PI}}}\,.$$

3.4.3. Example. Find the general solution of the linear ODE

$$y'' - 3y' - 4y = 2\sin t\,.$$

3.4.4. Solution.

Step 1: Find the complementary function. In this case, the complementary function is clearly the same as in the last example—the corresponding homogeneous equation is the same—hence

$$y_{\mathrm{CF}}(t) = C_1\mathrm{e}^{4t} + C_2\mathrm{e}^{-t}\,.$$

Step 2: Find the particular integral. Assume that y_{PI} has the form (using Table 3.1)

$$y_{\mathrm{PI}}(t) = A\sin t + B\cos t\,,$$

where the coefficients A and B are yet to be determined. Substituting this form for y_{PI} into the ODE implies

$$\begin{aligned}
&(-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t\\
\Leftrightarrow\quad &(-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t = 2\sin t\,.
\end{aligned}$$

Equating coefficients of $\sin t$ and also $\cos t$, we see that

$$-5A + 3B = 2 \quad \text{and} \quad -5B - 3A = 0\,.$$

Hence $A = -\frac{5}{17}$ and $B = \frac{3}{17}$ and so

$$y_{\mathrm{PI}}(t) = -\tfrac{5}{17}\sin t + \tfrac{3}{17}\cos t\,.$$

Step 3: Combine. Thus the general solution is

$$y(t) = \underbrace{C_1\mathrm{e}^{4t} + C_2\mathrm{e}^{-t}}_{y_{\mathrm{CF}}}\underbrace{-\tfrac{5}{17}\sin t + \tfrac{3}{17}\cos t}_{y_{\mathrm{PI}}}\,.$$

3.5. Initial and boundary value problems

Initial value problems (IVPs): given values for the solution, $y(t_0) = y_0$, and its derivative, $y'(t_0) = v_0$, at a given time $t = t_0$, are used to determine the solution.

Boundary value problems (BVPs): given values for the solution, $y(t_0) = y_0$ and $y(t_1) = y_1$, at two distinct times $t = t_0$ and $t = t_1$, are used to determine the solution.

In either case the two pieces of information given (either the initial or boundary data) are used to determine the specific values for the arbitrary constants in the general solution which generate the specific solution satisfying that (initial or boundary) data.

3.5.1. Example (initial value problem). Find the solution to the initial value problem

$$\begin{aligned}
y'' - 3y' - 4y &= 2\sin t\,,\\
y(0) &= 1\,,\\
y'(0) &= 2\,.
\end{aligned}$$

3.5.2. Solution. To solve an initial value problem we start by finding the complementary function and the particular integral; combining them together to get the general solution. We know from the last example above that the general solution is

$$y(t) = C_1 \mathrm{e}^{4t} + C_2 \mathrm{e}^{-t} - \tfrac{5}{17}\sin t + \tfrac{3}{17}\cos t\,. \tag{3.10}$$

Only once we have established the general solution to the full non-homogeneous problem do we start using the initial conditions to determine the constants C_1 and C_2.

First we use that we know $y(0) = 1$; this tells us that the solution function $y(t)$ at the time $t = 0$ has the value 1. Substituting this information into (3.10) gives

$$\begin{aligned} & C_1 \mathrm{e}^{4\cdot 0} + C_2 \mathrm{e}^{-0} - \tfrac{5}{17}\sin 0 + \tfrac{3}{17}\cos 0 = 1 \\ \Leftrightarrow \qquad & C_1 + C_2 = \tfrac{14}{17}\,. \end{aligned} \tag{3.11}$$

Secondly we need to use that $y'(0) = 2$. It is important to interpret this information correctly. This means that the derivative of the solution function, evaluated at $t = 0$ is equal to 2. Hence we first need to differentiate (3.10) giving

$$y'(t) = 4C_1 \mathrm{e}^{4t} - C_2 \mathrm{e}^{-t} - \tfrac{5}{17}\cos t - \tfrac{3}{17}\sin t\,. \tag{3.12}$$

Now we use that $y'(0) = 2$; substituting this information into (3.12) gives

$$\begin{aligned} & 4C_1 \mathrm{e}^{4\cdot 0} - C_2 \mathrm{e}^{-0} - \tfrac{5}{17}\cos 0 - \tfrac{3}{17}\sin 0 = 2 \\ \Leftrightarrow \qquad & 4C_1 - C_2 = \tfrac{39}{17}\,. \end{aligned} \tag{3.13}$$

Equations (3.11) and (3.13) are a pair of linear simultaneous equations for C_1 and C_2. Solving this pair of simultaneous equations we see that

$$C_1 = \tfrac{53}{85} \qquad \text{and} \qquad C_2 = \tfrac{1}{5}\,.$$

Hence the solution to the initial value problem is

$$y(t) = \tfrac{53}{85}\mathrm{e}^{4t} + \tfrac{1}{5}\mathrm{e}^{-t} - \tfrac{5}{17}\sin t + \tfrac{3}{17}\cos t\,.$$

3.5.3. Example (boundary value problem). Find the solution to the boundary value problem

$$\begin{aligned} y'' - 3y' - 4y &= 2\sin t\,, \\ y(0) &= 1\,, \\ y\big(\tfrac{\pi}{2}\big) &= 0\,. \end{aligned}$$

3.5.4. Solution. To solve such a boundary value problem we initially proceed as before to find the complementary function and the particular integral; combining them together to get the general solution. We already know from the last example that the general solution in this case is

$$y(t) = C_1 \mathrm{e}^{4t} + C_2 \mathrm{e}^{-t} - \tfrac{5}{17}\sin t + \tfrac{3}{17}\cos t\,. \tag{3.14}$$

As with initial value problems, only once we have established the general solution to the full non-homogeneous problem do we start using the boundary conditions to determine the constants C_1 and C_2.

First we use that we know $y(0) = 1$; and substitute this information into (3.14) giving

$$\begin{aligned} & C_1 \mathrm{e}^{4\cdot 0} + C_2 \mathrm{e}^{-0} - \tfrac{5}{17}\sin 0 + \tfrac{3}{17}\cos 0 = 1 \\ \Leftrightarrow \qquad & C_1 + C_2 = \tfrac{14}{17}\,. \end{aligned} \tag{3.15}$$

Secondly we use that $y\left(\frac{\pi}{2}\right) = 0$; substituting this information into (3.14) gives

$$\begin{aligned} & C_1 \mathrm{e}^{4\cdot\frac{\pi}{2}} + C_2 \mathrm{e}^{-\frac{\pi}{2}} - \tfrac{5}{17}\sin\tfrac{\pi}{2} + \tfrac{3}{17}\cos\tfrac{\pi}{2} = 0 \\ \Leftrightarrow \qquad & C_1 \mathrm{e}^{2\pi} + C_2 \mathrm{e}^{-\frac{\pi}{2}} = \tfrac{5}{17}\,. \end{aligned} \tag{3.16}$$

We see that (3.15) and (3.16) are a pair of linear simultaneous equations for C_1 and C_2. Solving this pair of simultaneous equations we soon see that

$$C_1 = \frac{5 - 14\mathrm{e}^{-\frac{\pi}{2}}}{17(\mathrm{e}^{2\pi} - \mathrm{e}^{-\frac{\pi}{2}})} \qquad \text{and} \qquad C_2 = \frac{14\mathrm{e}^{2\pi} - 5}{17(\mathrm{e}^{2\pi} - \mathrm{e}^{-\frac{\pi}{2}})}\,.$$

Hence the solution to the boundary value problem is

$$y(t) = \left(\frac{5 - 14\mathrm{e}^{-\frac{\pi}{2}}}{17(\mathrm{e}^{2\pi} - \mathrm{e}^{-\frac{\pi}{2}})}\right)\mathrm{e}^{4t} + \left(\frac{14\mathrm{e}^{2\pi} - 5}{17(\mathrm{e}^{2\pi} - \mathrm{e}^{-\frac{\pi}{2}})}\right)\mathrm{e}^{-t} - \tfrac{5}{17}\sin t + \tfrac{3}{17}\cos t\,.$$

3.6. Degenerate inhomogeneities

3.6.1. Example. Find the general solution of the degenerate linear ODE

$$y'' + 4y = 3\cos 2t\,.$$

3.6.2. Solution.

Step 1: Find the complementary function. First we solve the corresponding homogeneous equation

$$y''_{\mathrm{CF}} + 4y_{\mathrm{CF}} = 0\,, \tag{3.17}$$

to find the complementary function. Two solutions to this equation are $\sin 2t$ and $\cos 2t$, and so the complementary function is

$$y_{\mathrm{CF}}(t) = C_1 \sin 2t + C_2 \cos 2t\,,$$

where C_1 and C_2 are arbitrary constants.

Step 2: Find the particular integral. Using Table 3.1, assume that y_{PI} has the form

$$y_{\mathrm{PI}}(t) = A \sin 2t + B \cos 2t\,,$$

where the coefficients A & B are yet to be determined. Substituting this form for y_{PI} into the ODE implies

$$\begin{aligned} & (-4A \sin 2t - 4B \cos 2t) + 4(A \sin 2t + B \cos 2t) = 3 \cos 2t \\ \Leftrightarrow \qquad & (4B - 4B) \sin 2t + (4A - 4A) \cos 2t = 3 \cos 2t\,. \end{aligned}$$

Since the left-hand side is zero, there is no choice of A and B that satisfies this equation. Hence for some reason we made a poor initial choice for our particular solution $y_{\mathrm{PI}}(t)$. This becomes apparent when we recall the solutions to the homogeneous equation (3.17) are $\sin 2t$ and $\cos 2t$. These are solutions to the homogeneous equation and cannot possibly be solutions to the non-homogeneous case we are considering. We must therefore try a slightly different choice for $y_{\mathrm{PI}}(t)$, for example,

$$y_{\mathrm{PI}}(t) = At \cos 2t + Bt \sin 2t\,.$$

Substituting this form for y_{PI} into the ODE and cancelling terms implies

$$-4A \sin 2t + 4B \cos 2t = 3 \cos 2t$$

Therefore, equating coefficients of $\sin 2t$ and $\cos 2t$, we see that $A = 0$ and $B = \frac{3}{4}$ and so

$$y_{\mathrm{PI}}(t) = \tfrac{3}{4} t \sin 2t\,.$$

Step 3: Combine. Hence the general solution is

$$y(t) = \underbrace{C_1 \sin 2t + C_2 \cos 2t}_{y_{\mathrm{CF}}} + \underbrace{\tfrac{3}{4} t \sin 2t}_{y_{\mathrm{PI}}}\,.$$

Occasionally such a modification, will be insufficient to remove all duplications of the solutions of the homogeneous equation, in which case it is necessary to multiply by t a second time. For a second order equation though, it will never be necessary to carry the process further than two modifications.

3.6.3. Example. Find the general solution of the degenerate linear ODE $y'' - 2y' - 3y = 3\mathrm{e}^{3t}$.

3.6.4. Solution. First we focus on finding the complementary function, and since in this case the auxiliary equation is $\lambda^2 - 2\lambda - 3 = 0$, which has two real distinct solutions $\lambda_1 = -1$ and $\lambda_2 = 3$, we see that

$$y_{\mathrm{CF}}(t) = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{3t} .$$

Now when we try to look for a particular integral, we see that Table 3.1 tells us that we should try

$$y_{\mathrm{PI}}(t) = A\mathrm{e}^{3t} ,$$

as our guess. However we see that this form for y_{PI} is already a part of the complementary function and so cannot be a particular integral. Hence we try the standard modification is such circumstances and that is to change our guess for the particular integral to

$$y_{\mathrm{PI}}(t) = At\mathrm{e}^{3t} .$$

If we substitute this into the non-homogeneous differential equation we get

$$\begin{aligned} & 6A\mathrm{e}^{3t} + 9At\mathrm{e}^{3t} - 2(A\mathrm{e}^{3t} + 3At\mathrm{e}^{3t}) - 3At\mathrm{e}^{3t} = 3\mathrm{e}^{3t} \\ \Leftrightarrow \qquad & 6A\mathrm{e}^{3t} - 2A\mathrm{e}^{3t} = 3\mathrm{e}^{3t} . \end{aligned}$$

Hence $A = \frac{3}{4}$ and so the general solution to the full non-homogeneous differential equation is

$$y(t) = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{3t} + \tfrac{3}{4} t\mathrm{e}^{3t} .$$

3.6.5. Example. Find the general solution of the degenerate linear ODE $y'' - 6y' + 9y = 3\mathrm{e}^{3t}$.

3.6.6. Solution. First we focus on finding the complementary function. In this case the auxiliary equation is $\lambda^2 - 6\lambda + 9 = 0$, which has the repeated solution $\lambda_1 = \lambda_2 = 3$. Hence

$$y_{\mathrm{CF}}(t) = (C_1 + C_2 t)\mathrm{e}^{3t} .$$

Now forewarned by the last example, we see that we should *not* try the form for the particular integral

$$y_{\mathrm{PI}}(t) = A\mathrm{e}^{3t} ,$$

that Table 3.1 tells us that we should try, but rather we should modify our guess to

$$y_{\mathrm{PI}}(t) = At\mathrm{e}^{3t} .$$

However, this is also is already a part of the complementary function and so cannot be a particular integral. Hence we need to modify this guess also. We try the standard modification as before, and change our guess for the particular integral to

$$y_{\mathrm{PI}}(t) = At^2\mathrm{e}^{3t} .$$

If we substitute this into the non-homogeneous differential equation we get

$$\begin{aligned} & 2A\mathrm{e}^{3t} + 12At\mathrm{e}^{3t} + 9At^2\mathrm{e}^{3t} - 6(2At\mathrm{e}^{3t} + 3At^2\mathrm{e}^{3t}) + 9At^2\mathrm{e}^{3t} = 3\mathrm{e}^{3t} \\ \Leftrightarrow \quad & 2A\mathrm{e}^{3t} = 3\mathrm{e}^{3t}\,. \end{aligned}$$

Hence $A = \frac{3}{2}$ and so the general solution to the full non-homogeneous differential equation is

$$y(t) = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{3t} + \tfrac{3}{2}t^2\mathrm{e}^{3t}\,.$$

3.7. Resonance

3.7.1. Spring with oscillatory external forcing. Consider the following initial value problem for a forced harmonic oscillator, which for example, models a mass on the end of a spring which is forced externally,

$$\begin{aligned} y'' + \omega_0^2\, y &= \frac{1}{m} f(t)\,, \\ y(0) &= 0\,, \\ y'(0) &= 0\,. \end{aligned}$$

Here $y(t)$ is the displacement of the mass m from equilibrium at time t, and

$$\omega_0 = +\sqrt{k/m}$$

is a positive constant representing the natural frequency of oscillation when no forcing is present. Suppose

$$f(t) = A\sin\omega t$$

is the external oscillatory forcing, where A and ω are also positive constants.

Assume for the moment that $\omega \neq \omega_0$. We proceed by first finding a solution to the corresponding homogeneous problem,

$$\begin{aligned} & y_{\mathrm{CF}}'' + \omega_0^2\, y_{\mathrm{CF}} = 0 \\ \Rightarrow \quad & y_{\mathrm{CF}}(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t\,, \end{aligned} \tag{3.18}$$

where C_1 and C_2 are arbitrary constants.

Next we look for a particular integral, using Table 3.1 we should try

$$y_{\mathrm{PI}}(t) = D_1\cos\omega t + D_2\sin\omega t\,, \tag{3.19}$$

where D_1 and D_2 are the constants to be determined. Substituting this trial particular integral into the full non-homogeneous solution we get,

$$-\omega^2(D_1\cos\omega t + D_2\sin\omega t) + \omega_0^2(D_1\cos\omega t + D_2\sin\omega t) = \tfrac{1}{m}A\sin\omega t\,.$$

Equating coefficients of $\cos\omega t$ and $\sin\omega t$, we see that

$$-\omega^2 D_1 + \omega_0^2 D_1 = 0 \quad \text{and} \quad -\omega^2 D_2 + \omega_0^2 D_2 = \tfrac{1}{m}A\,.$$

Hence, provided $\omega \neq \omega_0$, then $D_1 = 0$ and $D_2 = A/m(\omega_0^2 - \omega^2)$, so that

$$y_{\mathrm{PI}}(t) = \frac{A}{m(\omega_0^2 - \omega^2)} \sin \omega t \,.$$

Hence the general solution is

$$y(t) = \underbrace{C_1 \cos \omega_0 t + C_2 \sin \omega_0 t}_{y_{\mathrm{CF}}} + \underbrace{\frac{A}{m(\omega_0^2 - \omega^2)} \sin \omega t}_{y_{\mathrm{PI}}} \,.$$

Now using the initial conditions,

$$y(0) = 0 \quad \Rightarrow \quad C_1 = 0 \,,$$

while,

$$y'(0) = 0 \quad \Rightarrow \quad \omega_0 C_2 + \omega \cdot \frac{A}{m(\omega_0^2 - \omega^2)} = 0 \quad \Leftrightarrow \quad C_2 = -\frac{A\omega}{\omega_0 m(\omega_0^2 - \omega^2)} \,.$$

Thus finally the solution to the initial value problem is given by

$$y(t) = \frac{A\omega}{m(\omega^2 - \omega_0^2)} \cdot \Bigg(\underbrace{\frac{1}{\omega_0} \sin \omega_0 t}_{\text{natural oscillation}} - \underbrace{\frac{1}{\omega} \sin \omega t}_{\text{forced oscillation}} \Bigg) \,, \tag{3.20}$$

where the first oscillatory term represents the natural oscillations, and the second, the forced mode of vibration.

3.7.2. Undamped resonance. What happens when $\omega \to \omega_0$? If we naively take the limit $\omega \to \omega_0$ in (3.20) we see that the two oscillatory terms combine to give zero, but also, the denominator in the multiplicative term $\frac{A\omega}{m(\omega^2 - \omega_0^2)}$ also goes to zero. This implies we should be much more careful about taking this limit. Let's rewrite this problematic ratio as follows:

$$\begin{aligned} \frac{\frac{1}{\omega_0} \sin \omega_0 t - \frac{1}{\omega} \sin \omega t}{\omega^2 - \omega_0^2} &= \frac{\frac{1}{\omega_0} \sin \omega_0 t - \frac{1}{\omega} \sin \omega t}{(\omega + \omega_0)(\omega - \omega_0)} \\ &= \frac{-1}{(\omega + \omega_0)} \cdot \underbrace{\frac{\frac{1}{\omega} \sin \omega t - \frac{1}{\omega_0} \sin \omega_0 t}{\omega - \omega_0}}_{\text{careful limit}} \,. \end{aligned} \tag{3.21}$$

Now the limit issue is isolated to the term on the right–hand side shown. The idea now is to imagine the function

$$F(\omega) = \frac{1}{\omega} \sin \omega t$$

as a function of ω, with time t as a constant parameter and to determine the limit

$$\lim_{\omega \to \omega_0} \frac{F(\omega) - F(\omega_0)}{\omega - \omega_0} \,,$$

which in fact, by definition, is nothing other than $F'(\omega_0)$! i.e. it is the derivative of $F(\omega)$ with respect to ω evaluated at ω_0.

Hence since $F(\omega) = \frac{1}{\omega}\sin\omega t$, we have that

$$\lim_{\omega\to\omega_0} \frac{\frac{1}{\omega}\sin\omega t - \frac{1}{\omega_0}\sin\omega_0 t}{\omega - \omega_0} = F'(\omega_0) = -\frac{1}{\omega_0^2}\sin\omega_0 t + \frac{1}{\omega_0} t\cos\omega_0 t\,.$$

Thus if we take the limit $\omega \to \omega_0$ in (3.21) we see that

$$\lim_{\omega\to\omega_0} \frac{-1}{(\omega+\omega_0)} \cdot \frac{\frac{1}{\omega}\sin\omega t - \frac{1}{\omega_0}\sin\omega_0 t}{\omega - \omega_0} = \frac{1}{2\omega_0^2} \cdot \left(\frac{1}{\omega_0}\sin\omega_0 t - t\cos\omega_0 t\right)\,.$$

Hence the solution to the initial value problem when $\omega = \omega_0$ is

$$y(t) = \frac{A}{2m\omega_0} \cdot \Bigg(\underbrace{\frac{1}{\omega_0}\sin\omega_0 t}_{\text{natural oscillation}} - \underbrace{t\cos\omega_0 t}_{\text{resonant term}} \Bigg)\,. \tag{3.22}$$

The important aspect to notice is that when $\omega = \omega_0$, the second term '$t\cos\omega_0 t$' grows without bound (the amplitude of these oscillations grows like t) and this is the *signature of undamped resonance*.

3.7.3. Damped resonance. Now suppose we introduce damping into our simple spring system so that the coefficient of friction $C > 0$. The equation of motion for the mass on the end of a spring which is forced externally, now becomes

$$y'' + \frac{C}{m}y' + \omega_0^2\, y = \frac{1}{m}f(t)\,.$$

By analogy with the undamped case, we have set

$$\omega_0 = +\sqrt{k/m}\,.$$

However in the scenario here with damping, this no longer simply represents the natural frequency of oscillation when no forcing is present. This is because in the overdamped, critically damped or underdamped cases the complementary function is always exponentially decaying in time. We call this part of the solution the *transient* solution—it will be significant initially, but it decays to zero exponentially fast—see Section 2.3.

We will still suppose that

$$f(t) = A\sin\omega t$$

is the external oscillatory forcing, where A and ω are also positive constants. The contribution to the solution from the particular integral which arises from the external forcing, cannot generate unbounded resonant behaviour for any bounded driving oscillatory force. To see this we look for a particular solution of the form

$$y_{\mathrm{PI}}(t) = D_1\cos\omega t + D_2\sin\omega t\,, \tag{3.23}$$

where D_1 and D_2 are the constants to be determined. Substituting this into the full non-homogeneous solution, equating coefficients of $\cos\omega t$ and $\sin\omega t$, and then solving the resulting pair of linear simultaneous equations for D_1 and D_2 we find that the particular integral is

$$y_{\mathrm{PI}} = \underbrace{-\frac{C\omega A/m^2}{(C\omega/m)^2 + (\omega_0^2 - \omega^2)^2}}_{D_1} \cos\omega t + \underbrace{\frac{(\omega_0^2 - \omega^2)A/m}{(C\omega/m)^2 + (\omega_0^2 - \omega^2)^2}}_{D_2} \sin\omega t\,. \tag{3.24}$$

The general solution is

$$y(t) = \underbrace{y_{\mathrm{CF}}}_{\text{decaying transient}} + \underbrace{y_{\mathrm{PI}}}_{\text{large time solution}}\,.$$

Since the complementary function part of the solution is exponentially decaying, the long term dynamics of the system is governed by the particular integral part of the solution and so for large times ($t \gg 1$)

$$y(t) \approx y_{\mathrm{PI}}(t)\,, \tag{3.25}$$

with y_{PI} given by (3.24). A second consequence of this 'large time' assumption is that the initial conditions are no longer relevant.

Let us now examine the amplitude of the solution (3.25) at large times, or more precisely, the square of the amplitude of the solution

$$H(\omega) \equiv D_1^2 + D_2^2 = \frac{A^2/m^2}{(C\omega/m)^2 + (\omega_0^2 - \omega^2)^2}\,.$$

For what value of ω is this amplitude a maximum? This will be given by the value of ω for which the denominator of $H(\omega)$ is a minimum (since the numerator in $H(\omega)$ is independent of ω). Hence consider

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\big((C\omega/m)^2 + (\omega_0^2 - \omega^2)^2\big) = 2C^2\omega/m^2 - 2(\omega_0^2 - \omega^2)\cdot 2\omega\,.$$

The right-hand side is zero when $\omega = 0$ or when

$$\omega = \omega_* \equiv \sqrt{\omega_0^2 - C^2/2m^2}\,.$$

In fact, $\omega = 0$ is a local maximum for the denominator of $H(\omega)$ and thus a local minimum for $H(\omega)$ itself. Hence we can discard this case (it corresponds to zero forcing afterall). However $\omega = \omega_*$ is a local minimum for the denominator of $H(\omega)$ and hence a local maximum for $H(\omega)$ itself. This means that the amplitude of the oscillations of the solution is largest when $\omega = \omega_*$ and is the resonant frequency!

The frequency ω_* is also known as the *practical resonance* frequency—see Figure 3.7.3. This is covered in detail in many engineering books, for example Kreyszig [**8**] p. 113.

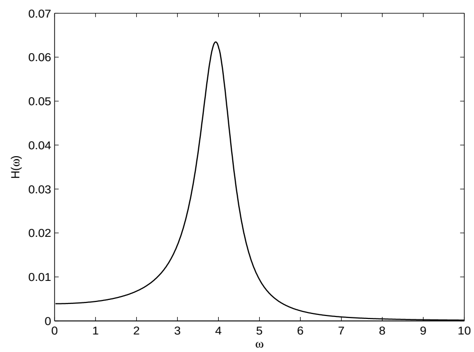


FIGURE 3.1. The square of the amplitude of the large time solution for a damped spring vs the frequency of the external forcing. The amplitude and hence the solution is always bounded but has a maximum at the practical resonance frequency $\omega = \omega_*$ (where as an example $\omega_* = 4$ above).

3.8. Equidimensional equations

A linear differential equation of the form
$$at^2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + bt \frac{\mathrm{d}y}{\mathrm{d}t} + cy = f(t)\,, \tag{3.26}$$
with a, b & c all constant, is an example of an *equidimensional* or *Cauchy–Euler* equation of second order.

To solve equations of this form, introduce a new independent variable
$$z = \log t \quad \Leftrightarrow \quad t = \mathrm{e}^z.$$
Then the chain rule implies
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}z} \cdot \frac{1}{t}\,, \qquad \text{i.e.} \qquad t\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}z}\,. \tag{3.27}$$
Further,
$$\begin{aligned}
\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}z} \cdot \frac{1}{t}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right) \cdot \frac{1}{t} - \frac{\mathrm{d}y}{\mathrm{d}z} \cdot \frac{1}{t^2} \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right) \cdot \frac{\mathrm{d}z}{\mathrm{d}t} \cdot \frac{1}{t} - \frac{\mathrm{d}y}{\mathrm{d}z} \cdot \frac{1}{t^2} \\
&= \left(\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}\right) \cdot \frac{1}{t^2}\,,
\end{aligned}$$

i.e.

$$t^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=\frac{\mathrm{d}^2y}{\mathrm{d}z^2}-\frac{\mathrm{d}y}{\mathrm{d}z}. \tag{3.28}$$

If we substitute (3.27) and (3.28) into the equidimensional equation (3.26), we get

$$a\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+(b-a)\frac{\mathrm{d}y}{\mathrm{d}z}+cy=f(\mathrm{e}^z)\,. \tag{3.29}$$

Now to solve (3.29), we can use the techniques we have learned to solve constant coefficient linear second order ODEs. Once you have solved (3.29), remember to substitute back that $z=\log t$.

3.8.1. Remark. That such equations are called *equidimensional* refers to the fact that they are characterized by the property that the linear operator on the left-hand side

$$L\equiv at^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}+bt\frac{\mathrm{d}}{\mathrm{d}t}+c$$

is invariant under the transformation $t\to k$.

3.8.2. Example (Black–Scholes). In 1973 Black & Scholes derived the partial differential equation

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0\,,$$

where r and σ are constants representing the risk–free interest and volatility of the underlying traded index, respectively. Here S is the price (which varies stochastically) of the underlying traded index and V is the value of a financial option on that index. Myron Scholes won the *Nobel Prize* in 1997 for his work involving this partial differential equation!

Notice that it is an equidimensional equation with respect to S, and is solved using the same change of variables we performed above. (This equation is also known—in different guises—as the Fokker-Planck or backward Kolmogorov equation.)

3.9. Exercises

3.1. Find the general solution to the non-homogeneous differential equations:

(a) $y''+4y=\sin 3t$;
(b) $4y''+7y'-2y=1+2t^2$;
(c) $y''+y'+y=3+5\mathrm{e}^{2t}$;
(d) $y''+8y'+16y=50\sin 2t+8\cos 4t$;
(e) $y''+2y'-8y=t^2\mathrm{e}^{3t}$.

3.2. For each of the following initial value problems, find the solution:

(a) $y''-5y'+6y=\cos 3t$, with $y(0)=0$, $y'(0)=5$;
(b) $y''+4y'+4y=\mathrm{e}^{-2t}$, with $y(0)=-3$, $y'(0)=2$.

3.3. Find the general solution to the non-homogeneous differential equation

$$y'' + 4y = \mathrm{e}^{-t} + \sin 2t\,.$$

How does the solution behave?

3.4. Consider the simple loop series electrical circuit mentioned in the introduction to this chapter. Describe how the charge $Q(t)$ behaves for all $t > 0$, when $L = 1$ Henrys, $R = 2$ Ohms, $C = 1/5$ Farads, $Q(0) = Q_0$, $Q'(0) = 0$, and the impressed voltage is
(a) $V(t) = \mathrm{e}^{-t} \sin 3t$,
(b) $V(t) = \mathrm{e}^{-t} \cos 2t$.

3.5. Find the general solution of the following equidimensional ODEs:
(a) $t^2 y'' - 2ty' + 2y = (\ln(t))^2 - \ln(t^2)$;
(b) $t^3 y''' + 2ty' - 2y = t^2 \ln(t) + 3t$.

3.6. (Resonance and damping.) How does damping effect the phenomenon of resonance? For example, suppose that for our frictionally damped spring system, we apply an external sinusoidal force (we might think here of a wine glass, with such a force induced by a pressure wave such as sound), i.e. suppose the equation of motion for the mass on the end of the spring system is, $my'' + Cy' + ky = f(t)$. Take the mass $m = 1$ Kg, stiffness $k = 2$ Kg/s^2, coefficient of friction $C = 2$ Kg/s and the external forcing as $f(t) = \mathrm{e}^{-t} \sin(t)$ Newtons. Assuming that the mass starts at rest at the origin, describe the subsequent behaviour of the mass for all $t > 0$.

Summary: solving linear constant coefficient second order IVPs

In general, to solve the linear second order non-homogeneous constant coefficient ordinary differential initial value problem,

$$a\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = f(t)\,, \tag{3.30a}$$

$$y(0) = \alpha\,, \tag{3.30b}$$

$$y'(0) = \beta\,, \tag{3.30c}$$

where a, b and c are given intrinsic constants, and α and β are given initial data, proceed as follows.

Step 1: Find the complementary function. i.e. find the general solution to the associated homogeneous ODE

$$Ly_{\mathrm{CF}} = 0\,. \tag{3.31}$$

To achieve this, try to find a solution to (3.31) of the form $y_{\mathrm{CF}} = \mathrm{e}^{\lambda t}$. This generates the *auxiliary equation* $a\lambda^2 + b\lambda + c = 0$. Then pick the solution given in Table 2.1 depending on whether $b^2 - 4ac$ is positive, zero or negative. This solution always has the form (where C_1 and C_2 are arbitrary constants)

$$y_{\mathrm{CF}}(t) = C_1y_1(t) + C_2y_2(t)\,.$$

Step 2: Find the particular integral. i.e. find any solution y_{PI} of the full non-homogeneous equation (3.30)

$$Ly_{\mathrm{PI}} = f\,,$$

using the method of undetermined coefficients (see Table 3.1).

Step 3: Combine. The *general solution* of (3.30) is

$$y(t) = y_{\mathrm{CF}} + y_{\mathrm{PI}} \tag{3.32}$$

$$\Rightarrow \qquad y(t) = C_1y_1(t) + C_2y_2(t) + y_{\mathrm{PI}}(t)\,. \tag{3.33}$$

Step 4: Use the initial conditions to determine the arbitrary constants. Using the first initial condition, and then differentiating the general solution (3.33) and substituting in the second initial condition we get, respectively,

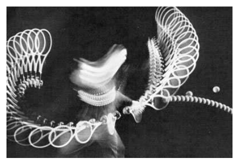
$$C_1y_1(0) + C_2y_2(0) + y_{\mathrm{PI}}(0) = \alpha\,, \tag{3.34a}$$

$$C_1y_1'(0) + C_2y_2'(0) + y_{\mathrm{PI}}'(0) = \beta\,. \tag{3.34b}$$

Now solve the simultaneous equations (3.34) for C_1 and C_2 and substitute these values into (3.33).

CHAPTER 4

Laplace transforms



4.1. Introduction

4.1.1. Example. Consider a damped spring system which consists of a mass which slides on a horizontal surface, and is attached to a spring, which is fixed to a vertical wall at the other end (see Figure 1.1 in Chapter 1). Suppose that the mass, initially at rest in the equilibrium position, is given a sharp hammer blow at time $t_0 > 0$, so that the equation of motion and initial conditions for the mass are

$$\begin{aligned} y'' + 3y' + 2y &= \delta(t - t_0)\,, \\ y(0) &= 0\,, \\ y'(0) &= 0\,. \end{aligned}$$

Here the external forcing function is $f(t) = \delta(t - t_0)$. The Dirac delta function, $\delta(t - t_0)$, is supposed to represent the action of a force acting instantaneously at the time t_0 and imparting a unit impulse (momentum) to the mass. The method of Laplace transforms is ideally suited to dealing with such situations and can be used to determine the solution to such initial value problems very conveniently.

4.1.2. Definition (Laplace transform). Formally:

Suppose the function $f(t)$ is defined for all $t \geq 0$. The *Laplace transform* of $f(t)$ is defined, as a function of the variable s by the integral,

$$\overline{f}(s) = \mathcal{L}\{f(t)\} \equiv \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t\,.$$

$\overline{f}(s)$ is defined for those values of s for which the right-hand integral is finite.

4.1.3. Example. For any $s > 0$, $\mathcal{L}\{1\} = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t = \left[-\frac{\mathrm{e}^{-st}}{s}\right]_0^\infty = \frac{1}{s}$.

4.1.4. Example. For any $s > a$,
$$\mathcal{L}\{\mathrm{e}^{at}\} = \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{at}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-(s-a)t}\,\mathrm{d}t = \frac{1}{s-a}\,.$$

4.1.5. Example. For any $s > 0$,
$$\mathcal{L}\{\sin at\} = \int_0^\infty \mathrm{e}^{-st}\sin at\,\mathrm{d}t = \frac{a}{s^2+a^2}\,.$$

4.1.6. Example. For any $s > 0$,
$$\mathcal{L}\{\cos at\} = \int_0^\infty \mathrm{e}^{-st}\cos at\,\mathrm{d}t = \frac{s}{s^2+a^2}\,.$$

4.1.7. Example (Derivative theorem). Formally, using the definition of the Laplace transform and then integrating by parts
$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty \mathrm{e}^{-st}f'(t)\,\mathrm{d}t \\ &= \big[\mathrm{e}^{-st}f(t)\big]_0^\infty + s\int_0^\infty \mathrm{e}^{-st}f(t)\,\mathrm{d}t \\ &= -f(0) + s\overline{f}(s)\,.\end{aligned}$$

4.1.8. Example (Derivative theorem application). We know that the Laplace transform of $f(t) = \sin at$ is
$$\overline{f}(s) = \frac{a}{s^2+a^2}\,.$$
Hence to find the Laplace transform of $g(t) = \cos at$, we note that
$$g(t) = \frac{1}{a}f'(t) \quad \Rightarrow \quad \overline{g}(s) = \frac{1}{a}\cdot\frac{sa}{s^2+a^2} - \frac{1}{a}\sin 0 = \frac{s}{s^2+a^2}\,.$$

4.1.9. Example (Shift theorem). Using the Shift Theorem,
$$\mathcal{L}\{\mathrm{e}^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}\,.$$
i.e. the function is *shifted* in the transform space when it is multiplied by e^{at} in the non–transform space.

4.2. Properties of Laplace transforms

4.2.1. Basic properties. The essential properties can be summarized as follows.

Suppose $f(t)$ and $g(t)$ are any two functions with Laplace transforms $\overline{f}(s)$ and $\overline{g}(s)$, respectively, and that a and b are any two constants.

- $\mathcal{L}$ is a *linear integral operator.*

$$\mathcal{L}\{af(t)+bg(t)\} = a\overline{f}(s) + b\overline{g}(s)\,.$$

- *Derivative theorem.*

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= s\overline{f}(s) - f(0)\,,\\ \mathcal{L}\{f''(t)\} &= s^2\overline{f}(s) - sf(0) - f'(0)\,,\\ \mathcal{L}\{f^{(n)}(t)\} &= s^n\overline{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)\,.\end{aligned}$$

- *Shift theorem.*

$$\mathcal{L}\{\mathrm{e}^{-at}f(t)\} = \overline{f}(s+a)\,.$$

- *Second shift theorem.* If

$$f(t) = \begin{cases} g(t-a), & t \geq a,\\ 0, & t < a,\end{cases}$$

  then

$$\mathcal{L}\{f(t)\} = \mathrm{e}^{-sa}\overline{g}(s)\,.$$

- *Convolution theorem.* If we define the convolution product of two functions to be

$$f(t) * g(t) \equiv \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)\,\mathrm{d}\tau \equiv \int_{-\infty}^{+\infty} f(t-\tau)g(\tau)\,\mathrm{d}\tau\,,$$

  then

$$\mathcal{L}\{f(t) * g(t)\} = \overline{f}(s)\cdot\overline{g}(s)\,.$$

4.2.2. Example (Second shift theorem). Consider

$$f(t) = \begin{cases} g(t-a), & t \geq a,\\ 0, & t < a,\end{cases}$$

with $a > 0$ constant. i.e. $f(t)$ is the function g shifted to the right along the real axis by a distance a. Setting $u = t - a$ we see that

$$\begin{aligned}\overline{f}(s) &= \int_0^\infty \mathrm{e}^{-st} g(t-a)\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-sa}\mathrm{e}^{-su} g(u)\mathrm{d}u \\ &= \mathrm{e}^{-sa}\int_0^\infty \mathrm{e}^{-su} g(u)\mathrm{d}u \\ &= \mathrm{e}^{-sa}\overline{g}(s)\,.\end{aligned}$$

4.3. Solving linear constant coefficients ODEs via Laplace transforms

4.3.1. Example. Find the solution to the following initial value problem:

$$\begin{aligned} y'' + 5y' + 6y &= 1\,, \\ y(0) &= 0\,, \\ y'(0) &= 0\,. \end{aligned}$$

4.3.2. Example. Find the solution to the following initial value problem:

$$\begin{aligned} y'' + 4y' + 8y &= 1\,, \\ y(0) &= 0\,, \\ y'(0) &= 0\,. \end{aligned}$$

4.3.3. Solution. Take the Laplace transform of both sides of the ODE, we have that

$$\begin{aligned} & & \mathcal{L}\{y''(t) + 4y'(t) + 8y(t)\} &= \mathcal{L}\{1\} \\ &\Leftrightarrow & \mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y'(t)\} + 8\mathcal{L}\{y(t)\} &= \mathcal{L}\{1\} \\ &\Leftrightarrow & s^2\overline{y}(s) - sy(0) - y'(0) + 4\big(s\overline{y}(s) - y(0)\big) + 8\overline{y}(s) &= \frac{1}{s} \\ &\Leftrightarrow & (s^2 + 4s + 8)\overline{y}(s) - (s+4)y(0) - y'(0) &= \frac{1}{s} \\ &\Leftrightarrow & (s^2 + 4s + 8)\overline{y}(s) &= \frac{1}{s}\,, \end{aligned}$$

where in the last step we have used that $y(0) = y'(0) = 0$ for this problem. Now look at this last equation—notice that by taking the Laplace transform of the differential equation for $y(t)$, we have converted it to an algebraic equation for $\overline{y}(s)$. This linear algebraic equation can be easily solved:

$$\overline{y}(s) = \frac{1}{s(s^2+4s+8)}\,.$$

Hence we now know what the Laplace transform of the solution of the differential equation (plus initial conditions) looks like. The question now is, knowing $\overline{y}(s)$, can we figure out what $y(t)$ is?

We use partial fractions to split up $\overline{y}(s)$ as follows, i.e. we seek to write $\overline{y}(s)$ in the form

$$\frac{1}{s(s^2+4s+8)} = \frac{A}{s} + \frac{Bs+C}{(s^2+4s+8)}\,, \tag{4.1}$$

(the idea is to try to split up $\overline{y}(s)$ into simpler parts we can handle).

The question is, can we find constants A, B and C such that this last expression is true for all $s \neq 0$? Multiply both sides of the equation by the denominator on the left-hand side; this gives

$$\begin{aligned} && 1 &= A(s^2+4s+8) + (Bs+C)s \\ \Leftrightarrow && 1 &= (A+B)s^2 + (4A+C)s + 8A\,. \end{aligned}$$

We want this to hold for all $s \neq 0$. Hence equating powers of s we see that

$$\begin{aligned} s^0 &: \;\Rightarrow\quad 1 = 8A \quad\Rightarrow\quad A = 1/8\,,\\ s^1 &: \;\Rightarrow\quad 0 = 4A + C \quad\Rightarrow\quad C = -1/2\,,\\ s^2 &: \;\Rightarrow\quad 0 = A + B \quad\Rightarrow\quad B = -1/8\,. \end{aligned}$$

Hence

$$\overline{y}(s) = \frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s+4}{(s^2+4s+8)}\,.$$

Completing the square for the denominator in the second term we see that

$$\begin{aligned} && \overline{y}(s) &= \frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s+4}{\big((s+2)^2+4\big)} \\ \Leftrightarrow && \overline{y}(s) &= \frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s+2}{(s+2)^2+4} - \frac{1}{8}\cdot\frac{2}{(s+2)^2+4} \\ \Leftrightarrow && y(t) &= \frac{1}{8} - \frac{1}{8}\mathrm{e}^{-2t}\cos 2t - \frac{1}{8}\mathrm{e}^{-2t}\sin 2t\,, \end{aligned}$$

using the table of Laplace transforms in the last step.

4.3.4. Example. Solve the following initial value problem using the method of Laplace transforms

$$\begin{aligned} y'' + 4y' + 4y &= 6\mathrm{e}^{-2t}\,,\\ y(0) &= -2\,,\\ y'(0) &= 8\,. \end{aligned}$$

4.3.5. Solution. Taking the Laplace transform of both sides of the ODE, we get

$$
\begin{aligned}
&& \mathcal{L}\{y''(t)+4y'(t)+4y(t)\} &= \mathcal{L}\{6\mathrm{e}^{-2t}\}\\
\Leftrightarrow && \mathcal{L}\{y''(t)\}+4\mathcal{L}\{y'(t)\}+4\mathcal{L}\{y(t)\} &= \mathcal{L}\{6\mathrm{e}^{-2t}\}\\
\Leftrightarrow && s^2\overline{y}(s)-sy(0)-y'(0)+4\big(s\overline{y}(s)-y(0)\big)+4\overline{y}(s) &= \frac{6}{s+2}\\
\Leftrightarrow && (s^2+4s+4)\overline{y}(s)-(s+4)y(0)-y'(0) &= \frac{6}{s+2}\\
\Leftrightarrow && (s^2+4s+4)\overline{y}(s) &= \frac{6}{s+2}-2s\\
\Leftrightarrow && (s+2)^2\overline{y}(s) &= \frac{6}{s+2}-2s\,.
\end{aligned}
$$

We now solve this equation for $\overline{y}(s)$, and after simplifying our expression for $\overline{y}(s)$, use the table of Laplace transforms to find the original solution $y(t)$:

$$
\begin{aligned}
&& \overline{y}(s) &= \frac{6}{(s+2)^3}-\frac{2s}{(s+2)^2}\\
\Leftrightarrow && \overline{y}(s) &= \frac{6}{(s+2)^3}-\frac{2(s+2-2)}{(s+2)^2}\\
\Leftrightarrow && \overline{y}(s) &= \frac{6}{(s+2)^3}-\frac{2}{s+2}+\frac{4}{(s+2)^2}\\
\Leftrightarrow && \overline{y}(s) &= 6\mathcal{L}\{\mathrm{e}^{-2t}\tfrac{1}{2}t^2\}-2\mathcal{L}\{\mathrm{e}^{-2t}\}+4\mathcal{L}\{\mathrm{e}^{-2t}t\}\\
\Leftrightarrow && y(t) &= 6\mathrm{e}^{-2t}\tfrac{1}{2}t^2-2\mathrm{e}^{-2t}+4\mathrm{e}^{-2t}t\\
\Leftrightarrow && y(t) &= (3t^2+4t-2)\mathrm{e}^{-2t}\,.
\end{aligned}
$$

4.4. Impulses and Dirac's delta function

4.4.1. Impulse. Laplace transforms are particularly useful when we wish to solve a differential equation which models a mechanical or electrical system and which involves an impulsive force or current. For example, if a mechanical system is given a blow by a hammer. In mechanics, the *impulse* $I(t)$ of a force $f(t)$ which acts over a given time interval, say $t_0 \leq t \leq t_1$, is defined to be

$$I(t)=\int_{t_0}^{t_1} f(t)\,\mathrm{d}t.$$

It represents the total momentum imparted to the system over the time interval $t_0 \leq t \leq t_1$ by $f(t)$. For an electrical circuit the analogous quantity is obtained by replacing $f(t)$ by the electromotive force (applied voltage) $V(t)$.

4.4.2. Dirac's delta function. Let's suppose we apply a force

$$f_\epsilon(t) = \begin{cases} 1/\epsilon, & \text{if } t_0 \le t \le t_0 + \epsilon\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

i.e. we apply a constant force $1/\epsilon$ over the time interval $t_0 \le t \le t_0 + \epsilon$, where $\epsilon \ll 1$ is a small parameter and $t_0 > 0$. Then the total impulse, for all $t \ge 0$, corresponding to this force is

$$I_\epsilon(t) = \int_0^\infty f_\epsilon(t)\,\mathrm{d}t = \int_{t_0}^{t_0+\epsilon} \tfrac{1}{\epsilon}\,\mathrm{d}t = 1\,.$$

Note that $I_\epsilon(t)$ represents the area under the graph of $f_\epsilon(t)$, and this last result shows that this area is independent of ϵ. Hence if we take the limit as $\epsilon \to 0$, then

$$f_\epsilon(t) \to \delta(t - t_0)\,,$$

where $\delta(t - t_0)$ is called the *Dirac delta function*. It has the property that

$$\int_0^\infty \delta(t - t_0)\,\mathrm{d}t = 1\,,$$

and that it is zero everywhere except at $t = t_0$, where it is undefined. In fact it is not really a function at all, but is an example of a *generalized function* or *(singular) distribution*. Of interest to us here is that it represents an impulse, of magnitude 1, acting over an infinitesimally short time interval, exactly as a hammer hitting a mass and imparting some momentum to it (via an impulse).

4.4.3. The Laplace transform of the Dirac delta function. This is particularly simple and means that the method of Laplace transforms is suited to problems involving delta function impulses/forces. First, let's consider the Laplace transform of $f_\epsilon(t)$:

$$\mathcal{L}\{f_\epsilon(t)\} = \int_0^\infty f_\epsilon(t)\,\mathrm{e}^{-st}\,\mathrm{d}t = \int_{t_0}^{t_0+\epsilon} \tfrac{1}{\epsilon}\,\mathrm{e}^{-st}\,\mathrm{d}t = \mathrm{e}^{-st_0} \cdot \frac{1 - \mathrm{e}^{-\epsilon s}}{\epsilon s}\,.$$

Now taking the limit $\epsilon \to 0$ in this last expression, we get

$$\mathcal{L}\{\delta(t - t_0)\} = \mathrm{e}^{-st_0}\,.$$

Hence the Dirac delta function is much more easily handled in the Laplace transform space, where it is represented by an ordinary exponential function, as opposed to its generalized function guise in the non-transform space. Note also that if we take the limit $t_0 \to 0$ we get that $\mathcal{L}\{\delta(t)\} = 1$.

4.4.4. Example. Consider the damped spring system shown in Figure 1.1. Suppose that the mass, initially at rest in the equilibrium position, is given a sharp hammer blow at time $t_0 > 0$, so that the equation of motion and initial conditions for the mass are

$$\begin{aligned} y'' + 3y' + 2y &= \delta(t - t_0)\,, \\ y(0) &= 0\,, \\ y'(0) &= 0\,. \end{aligned}$$

Use the Laplace transform to determine the solution to this initial value problem and sketch the behaviour of the solution for all $t \geq 0$.

4.4.5. Solution. Taking the Laplace transform of both sides of the ODE, we get

$$\begin{aligned} & \mathcal{L}\{y''(t) + 3y'(t) + 2y(t)\} = \mathcal{L}\{\delta(t - t_0)\} \\ \Leftrightarrow \quad & \mathcal{L}\{y''(t)\} + 3\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} = \mathcal{L}\{\delta(t - t_0)\} \\ \Leftrightarrow \quad & s^2\overline{y}(s) - sy(0) - y'(0) + 3\big(s\overline{y}(s) - y(0)\big) + 2\overline{y}(s) = \mathrm{e}^{-t_0 s} \\ \Leftrightarrow \quad & (s^2 + 3s + 2)\overline{y}(s) - (s + 3)y(0) - y'(0) = \mathrm{e}^{-t_0 s} \\ \Leftrightarrow \quad & (s^2 + 3s + 2)\overline{y}(s) = \mathrm{e}^{-t_0 s} \\ \Leftrightarrow \quad & (s + 1)(s + 2)\overline{y}(s) = \mathrm{e}^{-t_0 s}\,. \end{aligned}$$

We now solve this equation for $\overline{y}(s)$,

$$\overline{y}(s) = \frac{\mathrm{e}^{-t_0 s}}{(s + 1)(s + 2)}\,.$$

We now try to simplify our expression for $\overline{y}(s)$ as far as possible. Using partial fractions, can we write

$$\begin{aligned} & \frac{1}{(s + 1)(s + 2)} \equiv \frac{A}{(s + 1)} + \frac{B}{(s + 2)} \\ \Leftrightarrow \quad & 1 \equiv A(s + 2) + B(s + 1)\,, \end{aligned}$$

for some constants A and B? Using the 'cover-up method' if we set

$$\begin{aligned} s = -2: \quad & \Rightarrow \quad B = -1\,, \\ s = -1: \quad & \Rightarrow \quad A = 1\,. \end{aligned}$$

Hence

$$\begin{aligned} & \overline{y}(s) = \mathrm{e}^{-t_0 s}\left(\frac{1}{(s + 1)} - \frac{1}{(s + 2)}\right) \\ \Leftrightarrow \quad & \overline{y}(s) = \mathrm{e}^{-t_0 s} \cdot \mathcal{L}\{\mathrm{e}^{-t} - \mathrm{e}^{-2t}\}\,. \end{aligned}$$

Using the table of Laplace transforms to find the original solution,

$$y(t) = \begin{cases} \mathrm{e}^{-(t - t_0)} - \mathrm{e}^{-2(t - t_0)}\,, & \text{if } t > t_0\,, \\ 0\,, & \text{if } t < t_0\,. \end{cases}$$

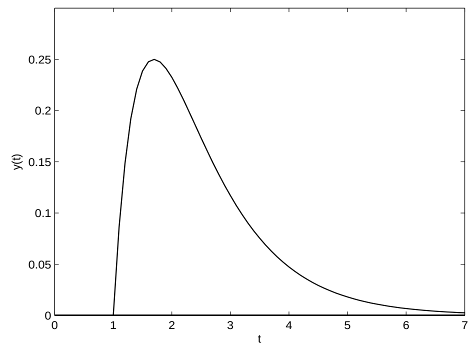


FIGURE 4.1. The mass sits in the equilibrium position $y = 0$ until it is hit by the hammer at $t = t_0$ (as an example we took $t_0 = 1$ above). Note that the values for the mass, coefficient of friction and spring stiffness mean that we are in the overdamped case.

How do we interpret this solution? Recall that the mass starts in the equilibrium position $y = 0$ with zero velocity and no force acts on it until the time $t = t_0$. Hence we expect the mass to sit in its equilibrium position until it is given a hammer blow at $t = t_0$ which imparts a unit impluse of momentum to it. Since its mass is $m = 1$, the hammer blow is equivalent to giving the mass one unit of velocity at $t = t_0$ and the mass starts off from $y = 0$ with that velocity. The solution thereafter is equivalent to the mass starting from the origin with a velocity of 1 (and no subsequent force)—see Figure 4.1.

4.4.6. The big picture. We will see in Chapter 7 that we can re-express any scalar higher order linear constant coefficient ODE (linear or nonlinear) as a system of first order ODEs of the form

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(t)\,.$$

Here $\boldsymbol{y}$ is the unknown vector $n \times 1$ solution, A is a constant $n \times n$ matrix and $\boldsymbol{f}$ is a vector $n \times 1$ external force (we will know how to interpret this equation more readily once we have completed the material in Chapter 5). The initial data can be expressed as

$$\boldsymbol{y}(0) = \boldsymbol{y}_0\,.$$

A nice way to visualize the process of solving linear constant coefficient ODE initial value problems via Laplace transform is as follows.

$$
\begin{tikzcd}
\begin{array}{r} \boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(t) \\ \boldsymbol{y}'(0) = \boldsymbol{y}_0 \end{array} \arrow[r, "\text{Laplace transform}"] \arrow[d, "\text{Solve}"'] & s\overline{\boldsymbol{y}}(s) - \boldsymbol{y}_0 = A\overline{\boldsymbol{y}}(s) + \overline{\boldsymbol{f}}(s) \arrow[d, "\text{Solve}"] \\
\boldsymbol{y}(t) & \overline{\boldsymbol{y}}(s) = (sI - A)^{-1}\big(\boldsymbol{y}_0 + \overline{\boldsymbol{f}}(s)\big) \arrow[l, "\text{inverse Laplace transform}"]
\end{tikzcd}
$$

Hence instead of solving the corresponding system of first order ODEs with initial conditions directly (the arrow down the left-hand side), we solve the system indirectly, by first taking the Laplace transform of the ODEs and initial conditions—the arrow across the top. We must then solve the resulting linear algebraic problem for the Laplace transform of the solution $\overline{\boldsymbol{y}}(s)$. This corresponds to the arrow down the right-hand side. Then finally, to find the actual solution to the original initial value problem $\boldsymbol{y}(t)$, we must take the inverse Laplace transform—the arrow along the bottom.

4.5. Exercises

For all the exercises below use the table of Laplace transforms!

4.1. Find the Laplace transforms of the following functions:

(a) $\sin(2t)\cos(2t)$; (b) $\cosh^2(2t)$; (c) $\cos(at)\sinh(at)$; (d) $t^2\mathrm{e}^{-3t}$.

Hint, you will find the following identities useful:

$$\sin 2\varphi \equiv 2\sin\varphi\cos\varphi\,; \quad \sinh\varphi \equiv \frac{1}{2}\left(\mathrm{e}^{\varphi} - \mathrm{e}^{-\varphi}\right)\,; \quad \cosh\varphi \equiv \frac{1}{2}\left(\mathrm{e}^{\varphi} + \mathrm{e}^{-\varphi}\right)\,.$$

4.2. Find the inverse Laplace transforms of the following functions (you may wish to re-familiarize yourself with partial fractions first):

$$\text{(a)}\ \frac{s}{(s+3)(s+5)}\,; \quad \text{(b)}\ \frac{1}{s(s^2+k^2)}\,; \quad \text{(c)}\ \frac{1}{(s+3)^2}\,.$$

Use Laplace transforms to solve the following initial value problems:

4.3. $y'' + y = t$, $y(0) = 0$, $y'(0) = 2$.

4.4. $y'' + 2y' + y = 3t\mathrm{e}^{-t}$, $y(0) = 4$, $y'(0) = 2$.

4.5. $y'' + 16y = 32t$, $y(0) = 3$, $y'(0) = -2$.

4.6. $y'' - 3y' + 2y = 4$, $y(0) = 1$, $y'(0) = 0$.

4.7. $y'' + 4y' + 4y = 6e^{-2t}$, $y(0) = -2$, $y'(0) = 8$.

4.8. Consider the damped spring system of Chapters 2&3. In particular let's suppose that the mass, initially at rest in the equilibrium position, is given a sharp hammer blow at time $t = 4\pi$, so that the equation of motion for the mass is,

$$y'' + 4y' + 5y = \delta(t - 4\pi)\,, \quad \text{with} \quad y(0) = 0\,,\ y'(0) = 3\,.$$

Use the Laplace transform to determine the solution to this initial value problem and sketch the behaviour of the solution for all $t \geq 0$.

Table of Laplace transforms

$f(t)$	$\int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t$
1	$\dfrac{1}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$af(t)+bg(t)$	$a\overline{f}(s)+b\overline{g}(s)$
$f'(t)$	$s\overline{f}(s)-f(0)$
$f''(t)$	$s^2\overline{f}(s)-sf(0)-f'(0)$
$\mathrm{e}^{-at}f(t)$	$\overline{f}(s+a)$
$f(t)=\begin{cases} g(t-a), & t\geq a, \\ 0, & t<a, \end{cases}$	$\mathrm{e}^{-sa}\overline{g}(s)$
$\delta(t-a)$	e^{-sa}
$f(t)*g(t)$	$\overline{f}(s)\cdot\overline{g}(s)$

TABLE 4.1. *Table of Laplace transforms.*

CHAPTER 5

Linear algebraic equations

5.1. Physical and engineering applications

When modelling many physical and engineering problems, we are often left with a system of algebraic equations for unknown quantities x_1, x_2, x_3, $\ldots$, x_n, say. These unknown quantities may represent components of modes of oscillation in structures for example, or more generally[1]:

Structures: stresses and moments in complicated structures;

Hydraulic networks: hydraulic head at junctions and the rate of flow (discharge) for connecting pipes;

General networks: abstract network problems, global communication systems;

Optimization: Linear programming problems, simplex algorithm, distribution networks, production totals for factories or companies, flight and ticket availability in airline scheduling;

Finite difference schemes: nodal values in the numerical implementation of a finite difference scheme for solving differential equation boundary value problems;

Finite element method: elemental values in the numerical implementation of a finite element method for solving boundary value problems (useful in arbitrary geometrical configurations);

Surveying: error adjustments via least squares method;

Curve fitting: determining the coefficients of the best polynomial approximation;

Circuits: electrical currents in circuits;

Nonlinear cable analysis: bridges, structures.

[1]see `www.nr.com` and `www.ulib.org`

In any of these contexts, the system of algebraic equations that we must solve will in many cases be linear or at least can be well approximated by a linear system of equations. Linear algebraic equations are characterized by the property that no variable is raised to a power other than one or is multiplied by any other variable. The question is: is there a systematic procedure for solving such systems?

5.2. Systems of linear algebraic equations

5.2.1. One equation with one unknown. For example, suppose we are asked to solve

$$2x = 6\,.$$

Clearly the solution is

$$x = 3\,.$$

In general if a and b are any two given real numbers, we might be asked to find the values of x for which

$$ax = b \tag{5.1}$$

is satisfied. There are three cases—we will return to this problem (5.1) time and again.

For the equation (5.1) if

- $a \neq 0$ then we can divide (5.1) by a. Thus $x = b/a$ and this is the *unique solution*, i.e. the only value of x for which (5.1) is satisfied.
- $a = 0$ and $b \neq 0$, then there is *no solution*. There do not exist any real values of x such that $0 \cdot x = b$ ($\neq 0$).
- $a = 0$ and $b = 0$, then any value of x will satisfy $0 \cdot x = 0$. Hence there are *infinitely many solutions*.

5.2.2. A system of two equations with two unknowns. A simple example is the system of two equations for the unknowns x_1 and x_2:

$$3x_1 + 4x_2 = 2\,, \tag{5.2}$$

$$x_1 + 2x_2 = 0\,. \tag{5.3}$$

There are many ways to solve this simple system of equations—we describe one that is easily generalised to much larger systems of linear equations.

Step 1. The first step is to eliminate x_1 from (5.3) by replacing (5.3) by $3\cdot$(5.3)$-$(5.2):

$$3x_1 + 4x_2 = 2 \tag{5.4}$$

$$2x_2 = -2\,. \tag{5.5}$$

Step 2. Equation (5.5) is easy to solve (divide both sides by 2), and gives

$$x_2 = -1\,.$$

Step 3. Substitution of this result back into (5.4) then gives

$$\begin{aligned} && 3x_1 - 4 &= 2 \\ \Leftrightarrow && 3x_1 &= 6 \\ \Leftrightarrow && x_1 &= 2\,. \end{aligned}$$

The solution to equations (5.2) and (5.3) is therefore $x_1 = 2$ and $x_2 = -1$. The process in this last step is known as *back–substitution*.

Note that having found the solution, we can always check it is correct by substituting our solution values into the equations, and showing that the equations are satisfied by these values.

$$\begin{aligned} 3 \cdot (2) + 4 \cdot (-1) &= 2 \\ (2) + 2 \cdot (-1) &= 0. \end{aligned}$$

Remark (algebraic interpretation). What we have just done in the steps above is exactly equivalent to the following: first solve (5.2) for $3x_1$, i.e. $3x_1 = 2 - 4x_2$. Now multiply (5.3) by 3 so that it becomes $3x_1 + 6x_2 = 0$. Now substitute the expression for $3x_1$ in the first equation into the second (thus eliminating x_1) to get $(2 - 4x_2) + 6x_2 = 0 \;\Rightarrow\; 2x_2 = -2$, etc....

Remark (geometric interpretation). The pair of simultaneous equations (5.2) & (5.3) also represent a pair of straight lines in the (x_1, x_2)-plane, rearranging: $x_2 = -\frac{3}{4}x_1 + \frac{2}{3}$, $x_2 = -\frac{1}{2}x_1$. In posing the problem of finding the solution of this pair of simultaneous equations, we are asked to find the values of x_1 and x_2 such that both these constraints (each of these equations represents a constraint on the set of values of x_1 and x_2 in the plane) are satisfied simultaneously. This happens at the intersection of the two lines.

In general, consider the system of two linear equations:

$$a_{11}x_1 + a_{12}x_2 = b_1 \tag{5.6}$$

$$a_{21}x_1 + a_{22}x_2 = b_2. \tag{5.7}$$

Solving these equations gives us

$$(5.6) \;\Rightarrow\; a_{11}x_1 + a_{12}x_2 = b_1\,, \tag{5.8}$$

$$a_{11} \cdot (5.7) - a_{21} \cdot (5.6) \;\Rightarrow\; \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{D}\, x_2 = \underbrace{a_{11}b_2 - a_{21}b_1}_{B}\,. \tag{5.9}$$

For the system (5.6), (5.7) if

- $D \neq 0$, then we may solve (5.9) to get x_2, and then by substituting this value back into (5.8) we determine x_1, i.e. there is a *unique solution* given by

$$x_2 = \frac{B}{D} = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} \quad \text{and} \quad x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}.$$

- $D = 0$ and $B \neq 0$ then there is *no solution*;
- $D = 0$ and $B = 0$, then any value of x_2 will satisfy (5.9) and there is an *infinite number of solutions*.

5.2.3. Remark (determinant). The quantity $D = a_{11}a_{22} - a_{12}a_{21}$ is clearly important, and is called the *determinant* of the system (5.6), (5.7). It is denoted by

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}. \tag{5.10}$$

5.2.4. A system of three equations with three unknowns. A similar procedure can also be used to solve a system of three linear equations for three unknowns x_1, x_2, x_3. For example, suppose we wish to solve

$$2x_1 + 3x_2 - x_3 = 5, \tag{5.11}$$
$$4x_1 + 4x_2 - 3x_3 = 3, \tag{5.12}$$
$$2x_1 - 3x_2 + x_3 = -1. \tag{5.13}$$

Step 1. First we eliminate x_1 from equations (5.12) and (5.13) by subtracting multiples of (5.11). We replace (5.12) by (5.12)$-2\cdot$(5.11) and (5.13) by (5.13)$-$(5.11), to leave the system

$$2x_1 + 3x_2 - x_3 = 5, \tag{5.14}$$
$$-2x_2 - x_3 = -7, \tag{5.15}$$
$$-6x_2 + 2x_3 = -6. \tag{5.16}$$

Step 2. Next we eliminate x_2 from (5.16). We do this by subtracting an appropriate multiple of (5.15). (If we subtract a multiple of (5.14) from (5.16) instead, then in the process of eliminating x_2 from (5.16) we re-introduce x_1!). We therefore replace (5.16) by (5.16)$-3\cdot$(5.15) to leave

$$2x_1 + 3x_2 - x_3 = 5, \tag{5.17}$$
$$-2x_2 - x_3 = -7, \tag{5.18}$$
$$5x_3 = 15. \tag{5.19}$$

Step 3. Now that we have *triangularized* the system, we can use *back substitution* to find the solution: first solve (5.19) to give

$$x_3 = \frac{15}{5} = 3\,.$$

Then substitute this result back into (5.18) to give

$$\begin{aligned} && -2x_2 - 3 &= -7 \\ \Leftrightarrow && 2x_2 &= 4 \\ \Leftrightarrow && x_2 &= 2\,. \end{aligned}$$

Finally we substitute the values of x_2 and x_3 back into (5.17) to give

$$\begin{aligned} && 2x_1 + 6 - 3 &= 5 \\ \Leftrightarrow && 2x_1 &= 2 \\ \Leftrightarrow && x_1 &= 1\,. \end{aligned}$$

So the solution to the system (5.11), (5.12), (5.13), is $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$.

Remark (algebraic interpretation). Effectively, what we did in the steps above was to solve (5.11) for $2x_1$, multiply the result by 2, and substitute the resulting expression for $4x_1$ into (5.12), thus eliminating x_1 from that equation, and then also we have substituted our expression for $2x_1$ from the first equation into (5.13) to eliminate x_1 from that equation also, etc. . . .

Remark (geometric interpretation). An equation of the form $ax + by + cz = d$, where a, b, c and d are given constants, actually represents an infinite plane in three dimensional space. Thus (5.11), (5.12) and (5.13) represent three planes and we are asked to find the values of (x_1, x_2, x_3) such that each of the three equations is satisfied simultaneously—i.e. the point(s) of intersection of the three planes. If no two of the three planes are parallel (beware of the 'toblerone' case) then since two planes intersect in a line, and the third plane must cut that line at a single/unique point, we therefore have a unique solution in this case (where all three planes meet) which we deduce algebraically as above is $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$.

5.3. Gaussian elimination

5.3.1. The idea. When modelling many physical and engineering problems we often need to simultaneously solve a large number of linear equations involving a large number of unknowns—there may be thousands of equations. We therefore need a systematic way of writing down (coding) and solving (processing) large systems of linear equations—preferably on a computer.

The method we used in the last section is known as *Gaussian elimination*, and can be applied to any size of system of equations. Just as importantly though, you may have noticed that to solve the system of linear equations by this method, we were simply performing operations on the coefficients in the equations, and the end result was an equivalent (triangularized) system of equations that was extremely easy to solve by back-substitution. With this in mind, a convenient way to write and analyze such systems is to use *matrices* as follows.

5.3.2. Example. To solve the system

$$\begin{aligned} 2x + 3y - z &= 5, \\ 4x + 4y - 3z &= 3, \\ 2x - 3y + z &= -1, \end{aligned}$$

we begin by writing down the *augmented matrix* of coefficients and right-hand sides:

$$H \equiv \left(\begin{array}{cccc} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ \underbrace{2 \quad -3 \quad 1}_{\text{coeffs on LHS}} & & & \underbrace{-1}_{\text{RHS}} \end{array} \right).$$

Then following the Gaussian elimination procedure, we try to make the terms below the leading diagonal zero. Note that we do not need to write down the equations for x, y & z at this stage, we can simply deal with the numerical coefficients in the augmented matrix H. The *advantage* of this approach using matrices, as we shall see, is that it is very easy to automate the solution process and implement the Gaussian elimination algorithm on a computer.

Step 1: Clear the first column below the diagonal. Replace Row2 by Row2 − 2·Row1:

$$\begin{pmatrix} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 2 & -3 & 1 & -1 \end{pmatrix}$$

And now replace Row3 by Row3 − Row1:

$$\begin{pmatrix} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 2 & -6 \end{pmatrix}$$

Step 2: Clear the second column below the diagonal. Replace Row3 by Row3 − 3·Row2:

$$\begin{pmatrix} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 5 & 15 \end{pmatrix}$$

Step 3: Use back-substitution to find the solution. Now we rewrite the rows of the augmented matrix as equations for x, y & z and proceed to solve the system of equations by *back-substitution* (a process which can also be coded and automated) giving us the solution we had before:

$$\begin{aligned}
&\text{Row } 3 \Rightarrow \quad 5z = 15 && \Rightarrow \quad z = 3;\\
&\text{Row } 2 \Rightarrow \quad -2y - z = -7 && \Rightarrow \quad y = 2;\\
&\text{Row } 1 \Rightarrow \quad 2x - 3y + z = 5 && \Rightarrow \quad x = 1.
\end{aligned}$$

5.3.3. Pivots. Note that in the first two steps we focused on the *pivot position*—the encircled numbers—which we use to eliminate all the terms below that position. The coefficient in the pivot position is called the *pivotal element* or *pivot*. The pivotal element must always be non-zero—if it is ever zero, then the pivotal row/equation is interchanged with an equation below it to produce a non-zero pivot—see the next example. This is always possible for systems with a unique solution. When implementing the Gaussian elimination algorithm on a computer, to minimize rounding errors, a practice called *partial pivoting* is used, whereby we interchange (if necessary) the pivot row with any row below it, to ensure that the pivot has the maximum possible magnitude—we will discuss partial pivoting in much more detail shortly—also see Meyer [**10**].

5.3.4. Example (of simple pivoting). The following system of equations models the currents in the electrical circuit.

$$\begin{aligned}
I_1 - I_2 + I_3 &= 0\,,\\
-I_1 + I_2 - I_3 &= 0\,,\\
10I_2 + 25I_3 &= 90\,,\\
20I_1 + 10I_2 &= 80\,.
\end{aligned}$$

To solve this system we construct the augmented matrix

$$H \equiv \begin{pmatrix} 1 & -1 & 1 & 0\\ -1 & 1 & -1 & 0\\ 0 & 10 & 25 & 90\\ 20 & 10 & 0 & 80 \end{pmatrix}.$$

To begin, the pivot row is the first row (notice also that the first two rows are in fact the same equation).

$$\begin{aligned}
&\text{Row2} \to \text{Row2} + \text{Row1};\\
&\text{Row4} \to \text{Row4} - 20 \cdot \text{Row1};
\end{aligned}
\qquad
\begin{pmatrix} 1 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 10 & 25 & 90\\ 0 & 30 & -20 & 80 \end{pmatrix}$$

Now move Row 2 to the end so that we have a non-zero element in the new pivot position.

$$H \equiv \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that the last equation is redundant as it contains no new information.

$$\text{Row3} \to \text{Row3} - 3 \cdot \text{Row2}; \qquad \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can now use back-substitution to find the solution:

$$\begin{aligned} I_3 &= \frac{-190}{-95} = 2\,; \\ I_2 &= \frac{1}{10}(90 - 25I_3) = 4\,; \\ I_1 &= I_2 - I_3 = 2\,. \end{aligned}$$

Hence the solution is $I_1 = 2$, $I_2 = 4$ and $I_3 = 2$.

5.3.5. Elementary row operations. The operations we carried out above are examples of *elementary row operations (EROs)*.

> There are three types of elementary row operations permitted in Gaussian elimination:
>
> - interchange any two rows—*this is equivalent to swapping the order of any two equations.*
> - multiply any row by any non-zero constant—*this is equivalent to multiplying any given equation by the constant.*
> - add the multiple of one row to another—*this is equivalent to adding the multiple of one equation to another.*

It is important to distinguish EROs from the broader range of operations that may be applied to determinants. For example, operations to columns are *not allowed* when solving a system of equations by EROs (for example swapping columns—any except the last column—in the augmented matrix corresponds to relabelling the variables).

In general, we may have a system of m equations in n unknowns

$$\mathcal{S}\left\{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\,, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\,, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\,. \end{aligned}\right. \tag{5.20}$$

Performing EROs we would get

$$\mathcal{S}'\left\{\begin{aligned} a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n &= b'_1\,, \\ a'_{21}x_1 + a'_{22}x_2 + \cdots + a'_{2n}x_n &= b'_2\,, \\ &\vdots \\ a'_{m1}x_1 + a'_{m2}x_2 + \cdots + a'_{mn}x_n &= b'_m\,. \end{aligned}\right. \tag{5.21}$$

The solution scheme of *Gaussian elimination* described above, exploits the fact that if H' (corresponding to system $\mathcal{S}'$) is derived from the augmented matrix H (corresponding to system $\mathcal{S}$) by a sequence of EROs, then the problems $\mathcal{S}$ and $\mathcal{S}'$ are equivalent.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\text{code}} & H \\ \Big\downarrow & & \Big\downarrow \text{EROs} \\ \mathcal{S}' & \xleftarrow[\text{decode}]{} & H' \end{array}$$

5.3.6. Example. Consider the set of equations

$$\begin{aligned} 2w + \ 4x + \ y + 2z &= 5\,, \\ 4w + 14x - \ y + 6z &= 11\,, \\ w - \ \ x + 5y - \ z &= 9\,, \\ -4w + \ 2x - 6y + \ z &= -2\,. \end{aligned}$$

5.3.7. Solution. We begin by writing down the augmented matrix:

$$H \equiv \begin{pmatrix} 2 & 4 & 1 & 2 & 5 \\ 4 & 14 & -1 & 6 & 11 \\ 1 & -1 & 5 & -1 & 9 \\ -4 & 2 & -6 & 1 & -2 \end{pmatrix}$$

The solution to our original system can be obtained by carrying out elementary row operations on the augmented matrix H (here and below, by

$R2 \to R2 - 2R1$ we mean replace Row 2 by Row 2 minus twice Row 1, and so forth).

$$\begin{aligned} &R2 \to R2 - 2R1; \\ &R3 \to R3 - \tfrac{1}{2}R1; \\ &R4 \to R4 + 2R1; \end{aligned} \qquad \begin{pmatrix} 2 & 4 & 1 & 2 & 5 \\ 0 & 6 & -3 & 2 & 1 \\ 0 & -3 & \frac{9}{2} & -2 & \frac{13}{2} \\ 0 & 10 & -4 & 5 & 8 \end{pmatrix}$$

$$\begin{aligned} &R3 \to R3 + \tfrac{1}{2}R2; \\ &R4 \to R4 - \tfrac{10}{6}R2; \end{aligned} \qquad \begin{pmatrix} 2 & 4 & 1 & 2 & 5 \\ 0 & 6 & -3 & 2 & 1 \\ 0 & 0 & 3 & -1 & 7 \\ 0 & 0 & 1 & \frac{5}{3} & \frac{19}{3} \end{pmatrix}$$

$$R4 \to R4 - \tfrac{1}{3}R3; \qquad \begin{pmatrix} 2 & 4 & 1 & 2 & 5 \\ 0 & 6 & -3 & 2 & 1 \\ 0 & 0 & 3 & -1 & 7 \\ 0 & 0 & 0 & 2 & 4 \end{pmatrix}$$

Now use back-substitution to obtain the solution (starting from the bottom):

$$\begin{aligned} &\text{Row } 4 \Rightarrow \quad 2z = 4 && \Rightarrow \quad z = 2; \\ &\text{Row } 3 \Rightarrow \quad 3y - 2z = 7 && \Rightarrow \quad y = 3; \\ &\text{Row } 2 \Rightarrow \quad 6x - 3y + 2z = 1 && \Rightarrow \quad x = 1; \\ &\text{Row } 1 \Rightarrow \quad 2w + 4x + y + 2z = 5 && \Rightarrow \quad w = -3\,. \end{aligned}$$

5.3.8. Example. Consider the system

$$\begin{aligned} 3x + 2y + \ z &= 3\,, \\ 2x + \ y + \ z &= 0\,, \\ 6x + 2y + 4z &= 6\,. \end{aligned}$$

5.3.9. Solution. We begin by writing down the augmented matrix:

$$H \equiv \begin{pmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{pmatrix}$$

$$\begin{aligned} &R2 \to R2 - \tfrac{2}{3}R1; \\ &R3 \to R3 - 2R1: \end{aligned} \qquad \begin{pmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{pmatrix}$$

$$R3 \to R3 - 6R2: \qquad \begin{pmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{pmatrix}$$

i.e.

$$\begin{aligned} 3x + 2y + \ z &= 3\,, \\ -\tfrac{1}{3}y + \tfrac{1}{3}z &= 2\,, \\ 0 \cdot z &= 12\,. \end{aligned}$$

This system has no solution (there is no solution to the last equation).

5.4. Solution of general rectangular systems

Suppose we are required to solve the system of m linear equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\,, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\,, \\ &\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\,. \end{aligned}$$

The nature of the solution is easily determined if we use elementary row operations (EROs) to reduce the augmented matrix $H = [A\ \boldsymbol{b}']$ to *row-echelon* form—see Figure 5.1.
There are three cases (referring to Figure 5.1 directly):

- a *unique solution* if $r = n$ and $b'_{r+1}, \ldots, b'_m$ are all zero; and the solution can be obtained by back-substitution;
- an *infinite number of solutions* if $r < n$ and $b'_{r+1}, \ldots, b'_m$ are all zero;
- *no solution* if $r < m$ and one of $b'_{r+1}, \ldots, b'_m$ is non-zero.

The system of equations is said to be *consistent* if there is at least one solution (either a unique solution or an infinite number of solutions) and *not consistent* if there is no solution.

5.5. Matrix Equations

5.5.1. Example. The system of equations

$$2x_1 + 3x_2 - \ x_3 = 5\,, \tag{5.22a}$$
$$4x_1 + 4x_2 - 3x_3 = 3\,, \tag{5.22b}$$
$$2x_1 - 3x_2 + \ x_3 = -1\,, \tag{5.22c}$$

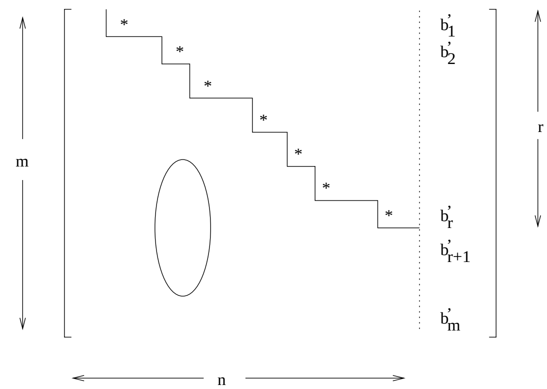


FIGURE 5.1. $[A \;\; \boldsymbol{b}]$ reduced to row-echelon form ($r \leq \min\{m, n\}$); the *'s represent non-zero elements—the other elements above the diagonal line may or may not be zero.

we solved previously may be written more compactly by introducing

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 1 \end{pmatrix} = \text{ the } \textit{matrix of coefficients},$$

with

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \text{ the } \textit{vector of unknowns}$$

and

$$\boldsymbol{b} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} = \text{ the } \textit{vector of right-hand sides.}$$

Then we simply write the system of equations (5.22) as

$$A\boldsymbol{x} = \boldsymbol{b}\,. \tag{5.23}$$

5.5.2. General linear systems. For large systems of equations the notation (5.23) is very compact.

More generally, if we have a system of m equations in n unknowns

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\,, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\,, \\ &\;\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\,, \end{aligned}\right\} \tag{5.24}$$

these may be written in matrix form as

$$A\boldsymbol{x} = \boldsymbol{b}\,, \tag{5.25}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Here A is an $m \times n$ matrix (m = number of rows, n = number of columns), $\boldsymbol{x}$ is an $n \times 1$ matrix (also known as a column vector), $\boldsymbol{b}$ is an $m \times 1$ matrix (again, a column vector). Note that a_{ij} = element of A at the intersection of the i-th row and the j-th column. The dimensions or size, '$m \times n$', of the matrix A, is also known as the *order* of A.

5.5.3. Multiplication of matrices. Multiplication of one matrix by another is a more involved operation than you might expect. To motivate matrix multiplication, recall that we rewrote the system (5.24) in the form

$$A\boldsymbol{x} = \boldsymbol{b}\,,$$

i.e.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

We therefore want the product $A\boldsymbol{x}$ to mean

$$A\boldsymbol{x} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}. \tag{5.26}$$

To make sense of matrix multiplication we need: number of columns of A = number of rows of $\boldsymbol{x}$. Thus

$$\underbrace{\begin{matrix} A \\ (m \times \end{matrix}\underbrace{\begin{matrix} \\ n) \quad (n \end{matrix}}_{\text{equal}}\begin{matrix} \boldsymbol{x} \\ \times 1) \end{matrix}}_{m\times 1 \text{ matrix}} \quad = \quad \begin{matrix} \boldsymbol{b} \\ (m \times 1) \end{matrix}$$

Matrix multiplication of an $m \times n$ matrix A by an $n \times 1$ matrix $\boldsymbol{x}$ is defined by (5.26).

5.6. Linear independence

5.6.1. Definition (linear combination). Given any set of m vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ (with the same number of components in each), a *linear combination* of these vectors is an expression of the form

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_m\boldsymbol{v}_m$$

where c_1, $c_2, \ldots, c_m$ are any scalars.

5.6.2. Definition (linear independence). Now consider the equation

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_m\boldsymbol{v}_m = \boldsymbol{0}. \tag{5.27}$$

Linear independence. The vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\ldots$, $\boldsymbol{v}_m$ are said to be *linearly independent* if the only values for c_1, c_2, $\ldots$, c_m for which (5.27) holds are when c_1, c_2, $\ldots$, c_m are all zero, i.e.

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_m\boldsymbol{v}_m = \boldsymbol{0} \quad \Rightarrow \quad c_1 = c_2 = \cdots = c_m = 0\,.$$

Linear dependence. If (5.27) holds for some set of c_1, c_2, $\ldots$, c_m which are not all zero, then the vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are said to be *linearly dependent*, i.e. we can express at least one of them as a linear combination of the others. For instance, if (5.27) holds with, say $c_1 \neq 0$, we can solve (5.27) for $\boldsymbol{v}_1$:

$$\boldsymbol{v}_1 = k_2\boldsymbol{v}_2 + \cdots + k_m\boldsymbol{v}_m \quad \text{where} \quad k_i = -\frac{c_i}{c_1}.$$

5.6.3. Example. The three vectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are linearly independent because

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 \equiv \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \text{equals the zero vector} \quad \boldsymbol{0} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

only when $c_1 = c_2 = c_3 = 0$.

5.6.4. Example. Suppose

$$\boldsymbol{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

The three vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_4$ are also linearly independent because

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_4\boldsymbol{v}_4 \equiv \begin{pmatrix} c_1 \\ c_2 + c_4 \\ c_4 \end{pmatrix}$$

will equal the zero vector only when $c_1 = c_2 = c_4 = 0$.

5.6.5. Example. Suppose

$$\boldsymbol{v}_5 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

The three vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_5$ are linearly dependent because

$$\boldsymbol{v}_1 + 2\boldsymbol{v}_2 - \boldsymbol{v}_5 = \boldsymbol{0}, \tag{5.28}$$

i.e. there exists three constants (at least one of which is non-zero), namely $c_1 = 1$, $c_2 = 2$ and $c_5 = -1$ such that (5.28) holds. Equivalently we see that any one of the three vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_5$ can be expressed as a linear combination of the other two, for example

$$\boldsymbol{v}_5 = \boldsymbol{v}_1 + 2\boldsymbol{v}_2.$$

5.6.6. Example. The three vectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -6 \\ 42 \\ 24 \\ 54 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}_3 = \begin{pmatrix} 21 \\ -21 \\ 0 \\ -15 \end{pmatrix}$$

are linearly dependent because

$$6\boldsymbol{v}_1 - \frac{1}{2}\boldsymbol{v}_2 - \boldsymbol{v}_3 = \mathbf{0}\,.$$

However note that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent because

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 = \mathbf{0},$$

implies $c_2 = 0$ (from the second components) and then $c_1 = 0$ (from any of the other components).

Although it is easy to verify that $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are linearly dependent given an appropriate linear combination, it is not so easy to determine if a given set if vectors is either linearly dependent or independent when starting from scratch. Next we show a method to help us in this direction.

5.7. Rank of a matrix

5.7.1. Definition (rank). The maximum number of linearly independent row vectors of a matrix $A = [a_{ij}]$ is called the *rank* of A and is denoted by 'rank$[A]$'.

5.7.2. Example. The matrix

$$A = \begin{pmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{pmatrix}$$

has rank 2, since from the previous example above, the first two row vectors are linearly independent, whereas the three row vectors are linearly dependent.

Note further that, $\text{rank}[A] = 0 \Leftrightarrow A = O$. This follows directly from the definition of rank. Another surprising result is that *row-rank equals column-rank*, i.e. the rank of a matrix A equals the maximum number of linearly independent column vectors of A, or equivalently, the matrix A and its transpose A^T, have the same rank[2]. However, the most important result of this section concerns *row-equivalent matrices*—these are matrices that can be obtained from each other by finitely many EROs.

[2]See the end of appendix D for the definition of the transpose of a matrix.

Invariance of rank under EROs. Row-equivalent matrices have the same rank, i.e. EROs do not alter the rank of a matrix A.

Test for linear dependence/independence. A given set of p vectors, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_p$ (with n components each), are linearly independent if the matrix with row vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_p$ have rank p; they are linearly dependent if that rank is less than p.

Hence to determine the rank of a matrix A, we can reduce A to echelon form using Gaussian elimination (this leaves the rank unchanged) and from the echelon form we can recognize the rank directly. Further, the second result above implies that, to test if a given set of vectors is linearly dependent/independent, we combine them to form rows (or columns) of a matrix and reduce that matrix to echelon form using Gaussian elimination. If the rank of the resulting matrix is equal to the number of given vectors then they're linearly independent, otherwise, they're linearly dependent.

5.7.3. Example.

$$A = \begin{pmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{pmatrix}$$

$$\begin{aligned} R2 &\to R2 + 2R1; \\ R3 &\to R3 - 7R1: \end{aligned} \qquad \begin{pmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{pmatrix}$$

$$R3 \to R3 + \frac{1}{2}R2: \qquad \begin{pmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the matrix is now in echelon form. Hence $\operatorname{rank}[A] = 2$.

5.8. Fundamental theorem for linear systems

The nature of the solution to a general system of linear equations can, as an alternative to the possibilities outlined in Section 5.4, be characterized using the notion of *rank* as follows—the fact that EROs do not alter the rank of a matrix establishes the equivalence of both statements.

Fundamental theorem for linear systems. A linear system of m equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\,, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\,, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\,, \end{aligned}$$

in n unknowns x_1, x_2, $\ldots$, x_n *has solutions* if and only if the coefficient matrix A and the augmented matrix $H = [A\ \boldsymbol{b}]$ have the same rank, i.e. if and only if

$$\operatorname{rank}[H] = \operatorname{rank}[A]\,.$$

- If this rank, r, equals n, then the system has a *unique solution.*
- If $r < n$, the system has *infinitely many solutions* (all of which are obtained be determining r suitable unknowns in terms of the remaining $n - r$ unknowns, to which arbitrary values can be assigned).

If solutions exist, they can be obtained by Gaussian elimination.

5.9. Gauss-Jordan method

Another method for solving linear systems of algebraic equations is the *Gauss-Jordan method.* It is a continuation of the Gaussian elimination process. For example, consider the linear system

$$\begin{aligned} 3x - \ \ y + 2z &= 3\,, \\ 2x + \ \ y + \ \ z &= -2\,, \\ x - 3y \qquad &= 5\,. \end{aligned}$$

Start by constructing the augmented matrix:

$$H = \begin{pmatrix} 3 & -1 & 2 & 3 \\ 2 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{pmatrix}.$$

$$\begin{matrix} R2 \to 3R2 - 2R1; \\ R3 \to 3R3 - R1; \end{matrix} \qquad \begin{pmatrix} 3 & -1 & 2 & 3 \\ 0 & 5 & -1 & -12 \\ 0 & -8 & -2 & 12 \end{pmatrix}$$

$$R3 \to 5R3 + 8R2; \qquad \begin{pmatrix} 3 & -1 & 2 & 3 \\ 0 & 5 & -1 & -12 \\ 0 & 0 & -18 & -36 \end{pmatrix}.$$

At this point, with Gaussian elimination, we would back-substitute by first solving the 3rd equation etc. In the Gauss-Jordan method we continue applying EROs to reduce the left-hand submatrix to the 3×3 identity matrix:

$$\begin{array}{ll} R1 \to R1 + \frac{1}{9}R3; \\ R2 \to 2R2 - \frac{1}{9}R3; \end{array} \qquad \begin{pmatrix} 3 & -1 & 0 & -1 \\ 0 & 10 & 0 & -20 \\ 0 & 0 & -18 & -36 \end{pmatrix}$$

$$R1 \to R1 + \tfrac{1}{10}R2; \qquad \begin{pmatrix} 3 & 0 & 0 & -3 \\ 0 & 10 & 0 & -20 \\ 0 & 0 & -18 & -36 \end{pmatrix}$$

$$\begin{array}{l} R1 \to \frac{1}{3}R1; \\ R2 \to \frac{1}{10}R2; \\ R3 \to -\frac{1}{18}R3; \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

The solution to this system is therefore $x = -1$, $y = -2$ and $z = 2$.

Which method should we use? Gaussian elimination or the Gauss-Jordan method? The answer lies in the *efficiency* of the respective methods when solving large systems. Gauss-Jordan requires about 50% more effort than Gaussian elimination and this difference becomes significant when n is large—see Meyer [**10**].

Thus Gauss-Jordan is not recommended for solving linear systems of equations that arise in practical situations, though it does have theoretical advantages—for example for finding the inverse of a matrix.

5.10. Matrix Inversion via EROs

Row reduction methods can be used to find the inverse of a matrix—in particular via the Gauss-Jordan approach. For example, to calculate the inverse of

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix},$$

consider the augmented matrix

$$\left(\underbrace{\begin{matrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{matrix}}_{\text{matrix } A} \quad \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\text{identity}} \right).$$

The plan is to perform row operations on this augmented matrix in such a way as to reduce it to the form

$$\left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\text{identity}} \quad \underbrace{\begin{matrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{matrix}}_{\text{matrix } B} \right).$$

The 3×3 matrix B is then guaranteed to be A^{-1}, the inverse of A.

$$\begin{array}{l} R2 \to R_2 - 2R_1; \\ R3 \to R_3 - R_1; \end{array} \qquad \begin{pmatrix} 1 & 1 & 3 & \vdots & 1 & 0 & 0 \\ 0 & -1 & -5 & \vdots & -2 & 1 & 0 \\ 0 & 2 & 2 & \vdots & -1 & 0 & 1 \end{pmatrix}$$

This pair of row operations clears the elements in column 1, rows 2 and 3.

$$R3 \quad \to R_3 + 2R_2; \qquad \begin{pmatrix} 1 & 1 & 3 & \vdots & 1 & 0 & 0 \\ 0 & -1 & -5 & \vdots & -2 & 1 & 0 \\ 0 & 0 & -8 & \vdots & -5 & 2 & 1 \end{pmatrix}$$

This clears column 2, row 3. Now we try to "clear" the top half:

$$\begin{array}{l} R_2 \to R_2 - \frac{5}{8}R_3; \\ R_1 \to R_1 + \frac{3}{8}R_3; \end{array} \qquad \begin{pmatrix} 1 & 1 & 0 & \vdots & -\frac{7}{8} & \frac{3}{4} & \frac{3}{8} \\ 0 & -1 & 0 & \vdots & \frac{9}{8} & -\frac{1}{4} & -\frac{5}{8} \\ 0 & 0 & -8 & \vdots & -5 & 2 & 1 \end{pmatrix}$$

$$R_1 \quad \to \quad R_1 + R_2; \qquad \begin{pmatrix} 1 & 0 & 0 & \vdots & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & -1 & 0 & \vdots & \frac{9}{8} & -\frac{1}{4} & -\frac{5}{8} \\ 0 & 0 & -8 & \vdots & -5 & 2 & 1 \end{pmatrix}$$

$$\begin{array}{l} R_2 \to -R_2; \\ R_3 \to -\frac{1}{8}R_3; \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 & \vdots & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 0 & \vdots & -\frac{9}{8} & \frac{1}{4} & \frac{5}{8} \\ 0 & 0 & 1 & \vdots & \frac{5}{8} & -\frac{1}{4} & -\frac{1}{8} \end{pmatrix}$$

$$\Rightarrow \qquad A^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{9}{8} & \frac{1}{4} & \frac{5}{8} \\ \frac{5}{8} & -\frac{1}{4} & -\frac{1}{8} \end{pmatrix}.$$

Now check that $AA^{-1} = I$.

In practice we very rarely need to invert matrices. However if we do need to do so on a computer, then when inverting a matrix, say A, it's roughly equivalent, in terms of raw numerical computations, to use the Gauss-Jordan procedure described above as opposed to using Gaussian elimination to perform EROs on augmented matrix $[A \colon I]$ so that the left-hand matrix becomes upper diagonal, and then to use back-substitution on each of the three underlying problems to find each of the three columns of A^{-1}, see Meyer [**10**].

5.11. Exercises

Reduce the augmented matrix to *row echelon form* using elementary row transformations—i.e. implement the method of Gaussian elimination. Hence determine which systems are *consistent* (in which case either state the unique solution if there is one, or classify the set of infinite solutions) or *not consistent* (there is no solution).

5.1.

$$\begin{aligned} x - \ y + 2z &= -2\,, \\ 3x - 2y + 4z &= -5\,, \\ 2y - 3z &= 2\,. \end{aligned}$$

5.2.

$$\begin{aligned} x - 2y + 2z &= -3\,, \\ 2x + \ y - 3z &= 8\,, \\ -x + 3y + 2z &= -5\,. \end{aligned}$$

5.3.

$$\begin{aligned} 3x + \ y + \ z &= 8\,, \\ -x + \ y - 2z &= -5\,, \\ x + \ y + \ z &= 6\,, \\ -2x + 2y - 3z &= -7\,. \end{aligned}$$

5.4.

$$\begin{aligned} 3x - \ y + 2z &= 3\,, \\ 2x + 2y + \ z &= 2\,, \\ x - 3y + \ z &= 4\,. \end{aligned}$$

5.5.

$$\begin{aligned} 3x - 7y + 35z &= 18\,, \\ 5x + 4y - 20z &= -17\,. \end{aligned}$$

5.6.

$$\begin{aligned} -3w + \ x - 2y + 13z &= -3\,, \\ 2w - 3x + \ y - \ 8z &= 2\,, \\ w + 4x + 3y - \ 9z &= 1\,. \end{aligned}$$

5.7. Solve the following systems of equations using Gaussian elimination. For what values of α are these systems consistent?

$$(a)\quad \begin{aligned} x - 2y + 2z &= -3\,, \\ 2x + y - 3z &= 8\,, \\ 9x - 3y - 3z &= \alpha\,. \end{aligned} \qquad\qquad (b)\quad \begin{aligned} 2x + 3y &= 7\,, \\ x - y &= 1\,, \\ \alpha x + 2y &= 8\,. \end{aligned}$$

5.8. Using that the rank of a matrix is invariant to EROs, find the rank of the following matrices.

$$(a)\quad \begin{pmatrix} -4 & 1 & 3 \\ 2 & 2 & 0 \end{pmatrix}, \qquad (b)\quad \begin{pmatrix} 2 & 2 & 1 \\ 1 & -1 & 3 \\ 0 & 0 & 1 \\ 4 & 0 & 7 \end{pmatrix},$$

$$(c)\quad \begin{pmatrix} 7 & -2 & 1 & -2 \\ 0 & 2 & 6 & 3 \\ 7 & 2 & 13 & 4 \\ 7 & 0 & 7 & 1 \end{pmatrix}, \qquad (d)\quad \begin{pmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{pmatrix},$$

$$(e)\quad \begin{pmatrix} 1 & 2 & -1 \\ 4 & 3 & 1 \\ 2 & 0 & -3 \end{pmatrix}, \qquad (f)\quad \begin{pmatrix} -1 & 3 & -2 & 4 \\ -1 & 4 & -3 & 5 \\ -1 & 5 & -4 & 6 \end{pmatrix}.$$

5.9. Calculate the inverse of $A = \begin{pmatrix} 2 & 1 & -4 \\ 0 & -2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$. Check $AA^{-1} = I$.

CHAPTER 6

Linear algebraic eigenvalue problems

6.1. Eigenvalues and eigenvectors

In this chapter we study eigenvalue problems. These arise in many situations, for example: calculating the natural frequencies of oscillation of a vibrating system; finding principal axes of stress and strain; calculating oscillations of an electrical circuit; image processing; data mining (web search engines); etc.

Differential equations: solving arbitrary order linear differential equations analytically;

Vibration analysis: calculating the natural frequencies of oscillation of a vibrating system—bridges, cantilevers;

Principal axes of stress and strain: mechanics;

Dynamic stability: linear stability analysis;

Column buckling: lateral deflections—modes of buckling;

Electrical circuit: oscillations, resonance;

Principle component analysis: extracting the salient features of a mass of data;

Markov Chains: transition matrices;

Data mining: web search engines—analysis of fixed point problems;

Image processing: fixed point problems again;

Quantum mechanics: quantized energy levels.

In fact when we solved the linear second order equations in Chapters 2–3, we were actually solving an eigenvalue problem.

Eigenvalue problems. Given an $n \times n$ matrix A, *the problem* is to find values of λ for which the equation

$$A\boldsymbol{x} = \lambda \boldsymbol{x}\,, \tag{6.1}$$

has a *nontrivial* (nonzero) vector $n \times 1$ solution $\boldsymbol{x}$.
Such values of λ are called *eigenvalues*, and the corresponding vectors $\boldsymbol{x}$, are the *eigenvectors* of the matrix A.

Equation (6.1) may be written as follows (here I is the identity matrix)

$$\begin{aligned} & & A\boldsymbol{x} &= \lambda \boldsymbol{x} \\ &\Leftrightarrow & A\boldsymbol{x} &= \lambda I \boldsymbol{x} \\ &\Leftrightarrow & A\boldsymbol{x} - \lambda I\boldsymbol{x} &= \mathbf{0} \\ &\Leftrightarrow & (A - \lambda I)\boldsymbol{x} &= \mathbf{0}\,. \end{aligned} \tag{6.2}$$

As λ is varied, the entries of the matrix $A - \lambda I$ vary. If $\det(A - \lambda I) \neq 0$ the matrix $A - \lambda I$ can be inverted, and the only solution of (6.2) is

$$\boldsymbol{x} = (A - \lambda I)^{-1}\mathbf{0} = \mathbf{0}\,.$$

Eigenvalue problems. A alternative useful way to re-write the eigenvalue equation (6.1) is in the form

$$(A - \lambda I)\boldsymbol{x} = \mathbf{0}\,. \tag{6.3}$$

Characteristic equation. Nonzero solutions to (6.3) $\boldsymbol{x}$ *will* exist provided λ is such that

$$\det(A - \lambda I) = 0\,. \tag{6.4}$$

This is called the *characteristic equation.*

Characteristic polynomial. If A is an $n \times n$ matrix then $\det(A - \lambda I)$ is an n-th degree polynomial in λ, i.e. it has the form

$$\det(A - \lambda I) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0\,. \tag{6.5}$$

Characteristic roots. The characteristic equation is equivalent to

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0\,, \tag{6.6}$$

and the solutions of this equation $\lambda_1, \ldots, \lambda_n$ are *eigenvalues.*

By the *Fundamental Theorem of Algebra* there are n roots of a polynomial of degree n (including repetitions); hence there are n solutions to the characteristic equation (including repetitions).

Note also that if $\boldsymbol{x}$ is an eigenvector of A with eigenvalue λ, and $c \neq 0$ is a scalar then $c\boldsymbol{x}$ is also an eigenvector of A with eigenvalue λ, since

$$A(c\boldsymbol{x}) = cA\boldsymbol{x} = c\lambda\boldsymbol{x} = \lambda(c\boldsymbol{x})\,.$$

This is simply due to the fact that (6.1) is a *homogeneous* algebraic system of equations.

6.1.1. Example. Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}.$$

6.1.2. Solution. First we solve the characteristic equation.

$$\begin{aligned} & & \det(A - \lambda I) &= 0 \\ &\Leftrightarrow & \det\begin{pmatrix} 3-\lambda & -2 \\ 4 & -3-\lambda \end{pmatrix} &= 0 \\ &\Leftrightarrow & \lambda &= \pm 1\,. \end{aligned}$$

For *each* eigenvalue we must find the corresponding eigenvector. Write $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Case $\lambda = 1$*:*

$$\begin{aligned} A\boldsymbol{x} = \lambda\boldsymbol{x} \quad &\Leftrightarrow \quad \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}, \\ &\Leftrightarrow \quad \begin{matrix} 3x - 2y = x & \Rightarrow & x = y \\ 4x - 3y = y & \Rightarrow & x = y\,. \end{matrix} \end{aligned}$$

So any vector of the form $\alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for any $\alpha \neq 0$.

Case $\lambda = -1$*:*

$$\begin{aligned} A\boldsymbol{x} = \lambda\boldsymbol{x} \quad &\Leftrightarrow \quad \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}, \\ &\Leftrightarrow \quad \begin{matrix} 3x - 2y = -x & \Rightarrow & 2x = y \\ 4x - 3y = -y & \Rightarrow & 2x = y\,. \end{matrix} \end{aligned}$$

Hence any vector of the form $\beta\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for any $\beta \neq 0$.

6.1.3. Example. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}.$$

6.1.4. Solution. We first find the eigenvalues λ from the characteristic equation,

$$
\begin{aligned}
& & \det\begin{pmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix} = 0 \\
\Leftrightarrow & & -(\lambda+2)(\lambda-2)^2 = 0\,,
\end{aligned}
$$

which has the solutions 2 (twice) and -2. For each eigenvalue we must calculate the corresponding eigenvectors.

Case $\lambda = -2$*:*

$$
\begin{aligned}
(A-\lambda I)\boldsymbol{x} = \mathbf{0} \quad & \Leftrightarrow \quad \begin{pmatrix} 2-(-2) & -2 & 2 \\ 1 & 1-(-2) & 1 \\ 1 & 3 & -1-(-2) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
& \Leftrightarrow \quad \begin{pmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

We can solve this system of linear equations by Gaussian elimination. Hence, consider the augmented matrix

$$
H \equiv \begin{pmatrix} 4 & -2 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 3 & 1 & 0 \end{pmatrix}
$$

$$
\begin{aligned}
R2 \to 4R2 - R1; \quad & \\
R3 \to 4R3 - R1; \quad & \begin{pmatrix} 4 & -2 & 2 & 0 \\ 0 & 14 & 2 & 0 \\ 0 & 14 & 2 & 0 \end{pmatrix}
\end{aligned}
$$

$$
R3 \to R3 - R2; \qquad \begin{pmatrix} 4 & -2 & 2 & 0 \\ 0 & 14 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.7}
$$

Note that the last equation is redundant. This is because the linear eigenvalue problem is a homogeneous problem (or you could use the fact that row rank equals column rank so the fact that the last column of the augmented matrix is filled with zeros also tells you that at least one row is redundant). More intuitively and precisely, for our example we know that unless λ is equal to an eigenvalue the solution is $\boldsymbol{x} = 0$. This makes sense because all the right-hand sides in the three equations are zero meaning that they all represent planes that intersect at the origin. And the origin would be the only unique point of intersection between the three planes, and the unique solution unless the three planes exactly line up to intersect in a line in which case there are an infinite number of solutions. The values of λ for which such a 'lining up' occurs, are precisely the eigenvalues!

Back to solving our eigenvalue problem. We are left with the system of equations represented in the augmented matrix (6.7), i.e.

$$\begin{aligned} 4x - 2y + 2z &= 0\,, \\ 14y + 2z &= 0\,. \end{aligned}$$

By back substitution we thus know that

$$y = -\tfrac{1}{7}z \qquad \text{and} \qquad x = -\tfrac{4}{7}z\,.$$

So the eigenvector corresponding to $\lambda = -2$ is

$$z\begin{pmatrix} -4/7 \\ -1/7 \\ 1 \end{pmatrix}$$

for any $z \neq 0$.

Case $\lambda = 2$*:*

$$\begin{aligned} (A - \lambda I)\boldsymbol{x} = \mathbf{0} \quad &\Leftrightarrow \quad \begin{pmatrix} 2-(2) & -2 & 2 \\ 1 & 1-(2) & 1 \\ 1 & 3 & -1-(2) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \quad \begin{pmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

We can solve this system by Gaussian elimination. Hence, consider the augmented matrix

$$H \equiv \begin{pmatrix} 0 & -2 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 3 & -3 & 0 \end{pmatrix}$$

$$R1 \leftrightarrow R2; \qquad \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 1 & 3 & -3 & 0 \end{pmatrix}$$

$$R3 \to R3 - R1; \qquad \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 4 & -4 & 0 \end{pmatrix}$$

$$R3 \to R3 + 2R2; \qquad \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The system of equations represented in this augmented matrix is

$$\begin{aligned} x - y + z &= 0\,, \\ -2y + 2z &= 0\,. \end{aligned}$$

By back substitution we thus know that

$$y = z \qquad \text{and} \qquad x = 0\,.$$

So the eigenvector corresponding to $\lambda = 2$ is

$$z\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

for any $z \neq 0$.

What about the (potentially) third eigenvector? There is not necessarily a third one—see the discussion on multiplicity after the next example.

6.1.5. Example. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

6.1.6. Solution. We first find the eigenvalues λ from the characteristic equation

$$\begin{aligned} & \det\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{pmatrix} = 0 \\ \Leftrightarrow \qquad & (\lambda-5)(\lambda+3)^2 = 0\,, \end{aligned}$$

which has the solutions $\lambda = -3$ (twice) and $\lambda = 5$. For each eigenvalue we must calculate the corresponding eigenvectors.

Case $\lambda = 5$*:*

$$\begin{aligned} (A-\lambda I)\boldsymbol{x} = \mathbf{0} \quad \Leftrightarrow \quad & \begin{pmatrix} -2-(5) & 2 & -3 \\ 2 & 1-(5) & -6 \\ -1 & -2 & 0-(5) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \Leftrightarrow \quad & \begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\,. \end{aligned}$$

We can solve this system by Gaussian elimination. Hence consider the augmented matrix

$$H \equiv \begin{pmatrix} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{pmatrix}$$

$$\begin{matrix} R2 \to 7R2 + 2R1; \\ R3 \to 7R3 - R1; \end{matrix} \qquad \begin{pmatrix} -7 & 2 & -3 & 0 \\ 0 & -24 & -48 & 0 \\ 0 & -16 & -32 & 0 \end{pmatrix}$$

$$R3 \to 3R3 - 2R2; \qquad \begin{pmatrix} -7 & 2 & -3 & 0 \\ 0 & -24 & -48 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The system of equations represented in this augmented matrix is

$$\begin{aligned} -7x + 2y - 3z &= 0\,, \\ -24y - 48z &= 0\,. \end{aligned}$$

By back substitution we thus know that

$$y = -2z \qquad \text{and} \qquad x = -z\,.$$

So the eigenvector corresponding to $\lambda = 5$ is

$$\alpha \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

for any value $z = \alpha$ ($\neq 0$).

Case $\lambda = -3$*:*

$$\begin{aligned} (A - \lambda I)\boldsymbol{x} = \boldsymbol{0} \quad &\Leftrightarrow \quad \begin{pmatrix} -2-(-3) & 2 & -3 \\ 2 & 1-(-3) & -6 \\ -1 & -2 & 0-(-3) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \quad \begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\,. \end{aligned}$$

We can now solve this system of linear equations by Gaussian elimination. Hence consider the augmented matrix

$$H \equiv \begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{pmatrix}$$

$$\begin{matrix} R2 \to R2 - 2R1; \\ R3 \to R3 + R1; \end{matrix} \qquad \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There is only one equation represented by this augmented matrix, namely

$$x + 2y - 3z = 0\,. \tag{6.8}$$

What is happening here is that for this repeated value of $\lambda = -3$, all three planes come together as the single plane represented by this single equation. Any point on this plane represents a nonzero solution to the eigenvalue problem, i.e. there is a two dimensional space of solutions. (An equivalent interpretation is to note that $A - \lambda I$ has rank 1 when $\lambda = -3$. This implies that the basis of solutions corresponding to $\lambda = -3$ consists of two linearly independent vectors.)

By inspection we notice that for any values $\beta \neq 0$ and $\gamma \neq 0$,

$$\beta \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \gamma \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

are both in fact eigenvectors corresponding to $\lambda = -3$. i.e. these two linearly independent eigenvectors correspond to this (repeated) eigenvalue and span the space of solutions represented by (6.8).

Algebraic and geometric multiplicity. The degree of a root (eigenvalue) of the characteristic polynomial of a matrix—the number of times the root is repeated—is called the *algebraic multiplicity* of the eigenvalue.
The number of linearly independent eigenvectors corresponding to an eigenvalue is called the *geometric multiplicity* of the eigenvalue.
It can be shown that for any eigenvalue

$$\text{geometric multiplicity} \ \leq \ \text{algebraic multiplicity}\,.$$

6.2. Diagonalization

6.2.1. Similarity transformations. An $n \times n$ matrix $\tilde{A}$ is said to be *similar* to and $n \times n$ matrix A if

$$\tilde{A} = X^{-1} A X$$

for some (non-singular) $n \times n$ matrix X. This transformation, which gives $\tilde{A}$ from A, is called a *similarity transformation*.

6.2.2. Diagonalization. If an $n \times n$ matrix A has a set of n linearly independent eigenvectors, $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\ldots$, $\boldsymbol{x}_n$, then if we set

$$X = [\boldsymbol{x}_1 \ \boldsymbol{x}_2 \ \ldots \ \boldsymbol{x}_n]$$

(i.e. the matrix whose columns are the eigenvectors of A), we have

$$X^{-1} A X = \Lambda,$$

where Λ is the $n \times n$ diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \ldots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix},$$

i.e. the matrix whose diagonal entries are the eigenvalues of the matrix A and whose all other entries are zero.

6.2.3. Symmetric Matrices. A *real* square matrix A is said to be *symmetric* if transposition leaves it unchanged, i.e. $A^T = A$. If A is a real-symmetric $n \times n$ matrix, then

- its eigenvalues, $\lambda_1, \ldots, \lambda_n$, are all real;
- the corresponding eigenvectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\ldots$, $\boldsymbol{x}_n$ are linearly independent.

Hence in particular, all real-symmetric matrices are diagonalizable.

6.2.4. Example. Find the matrix X that diagonalizes the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

via a similarity transformation.

6.2.5. Solution. The eigenvalues of A are 1, 3 and 6. The corresponding eigenvectors are

$$\boldsymbol{x}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{x}_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Hence

$$X = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

Now we can check, since

$$X^{-1} = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 \\ \frac{1}{5} & \frac{2}{5} & 0 \end{pmatrix},$$

then

$$X^{-1}AX = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 \\ \frac{1}{5} & \frac{2}{5} & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

6.3. Exercises

Find the eigenvalues and corresponding eigenvectors of the following matrices. Also find the matrix X that diagonalizes the given matrix via a similarity transformation. Verify your calculated eigenvalues.

6.1.

$$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}.$$

6.2.
$$\begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}.$$

6.3.
$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -2 & 3 \end{pmatrix}.$$

6.4.
$$\begin{pmatrix} 26 & -2 & 2 \\ 2 & 21 & 4 \\ 4 & 2 & 28 \end{pmatrix}.$$

6.5.
$$\begin{pmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{pmatrix}.$$

6.6.
$$\begin{pmatrix} 5 & -3 & 13 \\ 0 & 4 & 0 \\ -7 & 9 & -15 \end{pmatrix}.$$

6.7.
$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix}.$$

CHAPTER 7

Systems of differential equations

7.1. Linear second order systems

We consider here how to solve linear second order homogeneous systems of equations. We will see later in this chapter that such systems can always be reduced to a system of linear first order equations of double the size. However, as we will see presently, in some practical cases it is convenient to solve such systems directly. Hence in this section we are interested in finding the general solution to a system of equations of the form

$$\frac{\mathrm{d}^2\boldsymbol{y}}{\mathrm{d}t^2} = A\,\boldsymbol{y}\,, \tag{7.1}$$

where A is an $n \times n$ constant matrix and the unknown $n \times 1$ solution vector $\boldsymbol{y}$ consists of the components $y_1, y_2, \ldots, y_n$.

To solve such a system of equations we use the tried and trusted method of looking for solutions of the form

$$\boldsymbol{y}(t) = \boldsymbol{c}\,\mathrm{e}^{\lambda t}$$

where now $\boldsymbol{c}$ is a constant $n \times 1$ vector with components $c_1, c_2, \ldots, c_n$. Substituting this form of the solution into the differential equation system (7.1), we get

$$\begin{aligned}
& & \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigl(\boldsymbol{c}\,\mathrm{e}^{\lambda t}\bigr) &= A\,\boldsymbol{c}\,\mathrm{e}^{\lambda t} \\
\Leftrightarrow & & \lambda^2\boldsymbol{c}\,\mathrm{e}^{\lambda t} &= A\,\boldsymbol{c}\,\mathrm{e}^{\lambda t} \\
\Leftrightarrow & & \lambda^2\boldsymbol{c} &= A\,\boldsymbol{c} \\
\Leftrightarrow & & (A - \lambda^2 I)\,\boldsymbol{c} &= \boldsymbol{0}\,. & (7.2)
\end{aligned}$$

Thus solving the system of differential equations (7.1) reduces to solving the eigenvalue problem (7.2)—though note that it contains λ^2 rather than λ. This is a standard eigenvalue problem however, which we can solve using the techniques we learned in Chapter 6. In this section we will suppose there are n independent eigenvectors $\boldsymbol{c}_1$, $\boldsymbol{c}_2$,...,$\boldsymbol{c}_n$ corresponding to the eigenvalues λ_1^2, λ_2^2,...,λ_n^2.

In typical applications A is real and symmetric and hence has real eigenvalues, however further, it is typically negative definite in our examples below and hence has negative eigenvalues. To avoid getting too technical here we will explore the method of solution and practical interpretations of the solution through the following two illustrative examples.

7.1.1. Example (coupled pendula). Two identical simple pendula oscillate in the plane as shown in Figure 7.1. Both pendula consist of light rods of length $\ell = 10$ and are suspended from the same ceiling a distance $L = 15$ apart, with equal masses $m = 1$ attached to their ends. The angles the pendula make to the downward vertical are θ_1 and θ_2, and they are coupled through the spring shown which has stiffness coefficient $k = 1$. The spring has unstretched length $L = 15$. You may also assume that the acceleration due to gravity $g = 10$.

(a) Assuming that the oscillations of the spring remain small in amplitude, so that $|\theta_1| \ll 1$ and $|\theta_2| \ll 1$, by applying Newton's second law and and Hooke's law, show that the coupled pendula system gives rise to the system of differential equations

$$\frac{\mathrm{d}^2\boldsymbol{y}}{\mathrm{d}t^2} = A\,\boldsymbol{y}\,, \tag{7.3}$$

where

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{y} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

Here A is known as the stiffness matrix and $\boldsymbol{y}$ is the vector of unknown angles for each of the pendula shown in Figure 7.1.

(b) By looking for a solution of the form

$$\boldsymbol{y}(t) = \boldsymbol{c}\,\mathrm{e}^{\lambda t}$$

for a constant vector $\boldsymbol{c}$, show that solving the system of differential equations (7.3) reduces to solving the eigenvalue problem

$$(A - \lambda^2 I)\,\boldsymbol{c} = \boldsymbol{0}\,. \tag{7.4}$$

(c) Solve the eigenvalue problem (7.4) in part (b) above, stating clearly the eigenvalues and associated eigenvectors.

(d) Hence enumerate the possible modes of oscillation of the masses corresponding to each eigenvalue-eigenvector pair.

(e) Finally, write down the general solution of the system of equations (7.3).

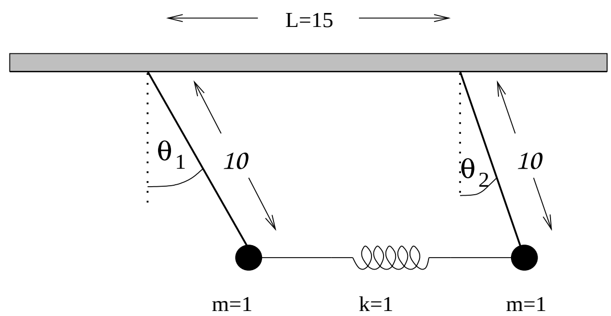


FIGURE 7.1. Simple coupled pendula system.

7.1.2. Example (mass–spring system).

(a) Show that 0, -4 and -16 are the eigenvalues of the matrix

$$A = \begin{pmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{pmatrix}. \tag{7.5}$$

For each eigenvalue, find the corresponding eigenvector.

(b) Three railway cars of mass $m_1 = m_3 = 750$Kg, $m_2 = 500$Kg move along a track and are connected by two buffer springs as shown in Figure 7.2. The springs have stiffness constants $k_1 = k_2 = 3000$Kg/m. Applying Newton's second law and Hooke's law, this mass-spring system gives rise to the differential equation system

$$\frac{\mathrm{d}^2\boldsymbol{y}}{\mathrm{d}t^2} = A\,\boldsymbol{y}\,, \tag{7.6}$$

where A is the matrix given in (7.5) above, and

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

is the vector of unknown position displacements (from equilibrium) for each of the masses shown in Figure 7.2. By looking for a solution of the form

$$\boldsymbol{y}(t) = \boldsymbol{c}\,\mathrm{e}^{\lambda t}$$

for a constant vector $\boldsymbol{c}$, show that solving the system of differential equations (7.6) reduces to solving the eigenvalue problem

$$(A - \lambda^2 I)\,\boldsymbol{c} = \boldsymbol{0}\,. \tag{7.7}$$

(c) We know from part (a) above that the solutions to this eigenvalue problem are $\lambda_1^2 = 0$, $\lambda_2^2 = -4$ and $\lambda_3^2 = -16$. Writing $\lambda = \mathrm{i}\omega$ so that $\lambda^2 = -\omega^2$, deduce the fundamental frequencies of oscillation ω_1, ω_2 and ω_3 of the mechanical system in Figure 7.2.

(d) For each fundamental frequency of oscillation ω_1, ω_2 and ω_3 corresponding to λ_1, λ_2 and λ_3, the eigenvectors you deduced in part (a) above represent the possible modes of oscillation. Use those eigenvectors to enumerate the possible modes of oscillation of the masses corresponding to each eigenvalue–eigenvector pair.

(e) Finally, write down the general solution of the system of equations (7.6).

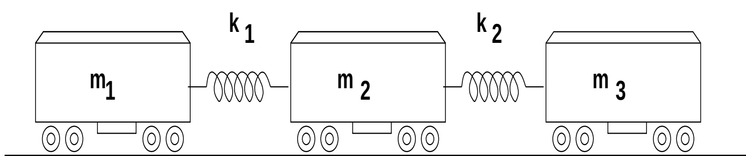


FIGURE 7.2. Simple mass-spring three-particle system.

7.2. Linear second order scalar ODEs

Consider the linear second order constant coefficient homogeneous ODE:

$$ay'' + by' + cy = 0\,.$$

If we set $v = y'$, we can re-write this linear second order ODE as the following coupled system of linear first order ODEs (assuming $a \neq 0$),

$$\begin{aligned} y' &= v\,, \\ v' &= -(c/a)y - (b/a)v\,, \end{aligned}$$

in the form

$$\boldsymbol{y}' = A\boldsymbol{y}\,,$$

where

$$\boldsymbol{y} = \begin{pmatrix} y \\ v \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}\,.$$

To solve this system of linear constant coefficient first order ODEs, we look for a solution of the form

$$\boldsymbol{y} = \boldsymbol{c}\,\mathrm{e}^{\lambda t}$$

where

$$\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \text{ constant vector.}$$

Substituting this ansatz[1] for the solution into the differential equation for $\boldsymbol{y}$ above, we soon see that indeed, this exponential form is a solution, provided

[1]Ansatz is German for “approach”.

we can find a λ for which there exists a non-trivial solution to (after dividing through by the exponential factor $\mathrm{e}^{\lambda t}$)

$$\begin{aligned} && A\boldsymbol{c} &= \lambda \boldsymbol{c}\,, \\ \Leftrightarrow && (A-\lambda I)\boldsymbol{c} &= \boldsymbol{0}\,. \end{aligned} \tag{7.8}$$

To find non-trivial solutions $\boldsymbol{c}$ to this eigenvalue problem, we first solve the characteristic equation

$$\begin{aligned} && \det(A-\lambda I) &= 0 \\ \Leftrightarrow && \det\begin{pmatrix} -\lambda & 1 \\ -c/a & -b/a-\lambda \end{pmatrix} &= 0 \\ \Leftrightarrow && a\lambda^2 + b\lambda + c &= 0\,. \end{aligned}$$

There are two solutions (eigenvalues) λ_1 and λ_2. The corresponding eigenvectors satisfy (from (7.8))

$$\begin{pmatrix} -\lambda & 1 \\ -c/a & -b/a-\lambda \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,,$$

or equivalently the system of equations

$$\begin{aligned} -\lambda c_1 + c_2 &= 0\,, \\ -(c/a)c_1 - (b/a+\lambda)\,c_2 &= 0\,. \end{aligned}$$

The first equation gives

$$c_2 = \lambda c_1\,,$$

and we expect the second equation to give the same (which it does):

$$c_2 = -\frac{c/a}{\lambda + b/a}\,c_1 = -\frac{c\lambda}{a\lambda^2+b\lambda}\,c_1 = \lambda c_1\,.$$

So for $\lambda = \lambda_1$ the corresponding eigenvector is (for arbitrary values of $\alpha \neq 0$)

$$\boldsymbol{c}_1 = \alpha\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}\,.$$

For $\lambda = \lambda_2$ the corresponding eigenvector is (for arbitrary values of $\beta \neq 0$)

$$\boldsymbol{c}_2 = \beta\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}\,,$$

Hence using the Principle of Superposition, we see that the final solution to our original second order ODE can be expressed in the form

$$\boldsymbol{y} = \boldsymbol{c}_1\mathrm{e}^{\lambda_1 t} + \boldsymbol{c}_2\mathrm{e}^{\lambda_2 t} \equiv \alpha\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}\mathrm{e}^{\lambda_1 t} + \beta\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}\mathrm{e}^{\lambda_2 t}\,.$$

How does this solution compare with that given in Table 2.1?

7.3. Higher order linear ODEs

Recall from Chapter 1 that we can re-express any scalar n^{th} order ODE (linear or nonlinear) as a system of n first order ODEs. In particular, any n^{th} order linear ODE of the form (we will restrict ourselves to the homogeneous case for the moment)

$$a_n(t)\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}(t)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_2(t)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a_1(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_0(t)y = 0\,, \quad (7.9)$$

after we identify new variables

$$y_1 = \frac{\mathrm{d}y}{\mathrm{d}t}\,,\ y_2 = \frac{\mathrm{d}y_1}{\mathrm{d}t}\,, \ldots,\ y_{n-1} = \frac{\mathrm{d}y_{n-2}}{\mathrm{d}t}\,,$$

is equivalent to the linear system of n first order ODEs (also set $y_0 \equiv y$)

$$\begin{aligned}
\frac{\mathrm{d}y_0}{\mathrm{d}t} &= y_1\,,\\
\frac{\mathrm{d}y_1}{\mathrm{d}t} &= y_2\,,\\
&\vdots\\
\frac{\mathrm{d}y_{n-2}}{\mathrm{d}t} &= y_{n-1}\,,\\
a_n(t)\frac{\mathrm{d}y_{n-1}}{\mathrm{d}t} &= -a_{n-1}(t)y_{n-1} - \cdots - a_2(t)y_2 - a_1(t)y_1 - a_0(t)y_0\,.
\end{aligned}$$

We can express this system of n first order linear ODEs much more succinctly in matrix form as

$$\boldsymbol{y}' = A(t)\boldsymbol{y}\,, \quad (7.10)$$

where $\boldsymbol{y}$ is the vector of unknowns $y_0, \ldots, y_{n-1}$ and

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & & & \cdots & & \vdots\\ 0 & 0 & 0 & 0 & 0 & 1\\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \cdots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{pmatrix}.$$

7.4. Solution to linear constant coefficient ODE systems

7.4.1. Solution in exponential form. We shall also restrict ourselves to the case when the coefficient matrix A is constant. In the last section we saw how the coefficient matrix could look when we re–write a scalar n^{th} order ODE as a system of n first order ODEs (if we assumed in that case that the coefficients $a_0, \ldots, a_n$ were all constant, then the coefficient matrix A would also constant). Here we are interested in the case when A could be a constant matrix of any form (even with complex entries). Hence in general, our system of linear constant coefficient ODEs has the form

$$\boldsymbol{y}' = A\,\boldsymbol{y}\,. \quad (7.11)$$

If we are given the initial condition $\boldsymbol{y}(0) = \boldsymbol{y}_0$, then the solution to (7.11) is

$$\boldsymbol{y} = \exp(At)\boldsymbol{y}_0\,, \tag{7.12}$$

where we define the exponential function of a matrix, say B, as the sum of the matrix power series

$$\exp(B) = I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \cdots\,. \tag{7.13}$$

We can quite quickly check that '$\exp(At)\boldsymbol{y}_0$' is indeed the solution to the initial value problem associated with (7.11), note that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(At)\boldsymbol{y}_0\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\big)\boldsymbol{y}_0 \\
&= \big(A + A^2t + \frac{1}{2!}A^3t^2 + \cdots\big)\boldsymbol{y}_0 \\
&= A\big(I + At + \frac{1}{2!}A^2t^2 + \cdots\big)\boldsymbol{y}_0 \\
&= A\exp(At)\boldsymbol{y}_0\,.
\end{aligned}$$

7.4.2. Solution via diagonalization. Suppose that the $n \times n$ constant coefficient matrix A has eigenvalues $\lambda_1, \ldots, \lambda_n$ and n linearly independent eigenvectors. Set X to be the matrix whose n columns are the eigenvectors of A. For the initial value problem associated with (7.11), consider the linear transformation

$$\boldsymbol{y}(t) = X\boldsymbol{u}(t)\,.$$

Differentiating this formula, we see that

$$\boldsymbol{y}'(t) = X\boldsymbol{u}'(t)\,,$$

noting X is constant because A is. Substituting these last two formulae into (7.11) we see that we get (here Λ denotes the diagonal matrix with the eigenvalues of A down the diagonal)

$$\begin{aligned}
&& X\boldsymbol{u}' &= AX\boldsymbol{u} \\
\Leftrightarrow && \boldsymbol{u}' &= \underbrace{X^{-1}AX}_{\equiv\Lambda}\boldsymbol{u} \\
\Leftrightarrow && \boldsymbol{u}' &= \Lambda\boldsymbol{u} \\
\Leftrightarrow && u_i' &= \lambda_i u_i \qquad \text{for each } i = 1, \ldots, n \\
\Leftrightarrow && u_i(t) &= \mathrm{e}^{\lambda_i t}u_i(0) \\
\Leftrightarrow && \boldsymbol{u}(t) &= \exp(\Lambda t)\boldsymbol{u}(0) \\
\Leftrightarrow && X\boldsymbol{y}(t) &= \exp(\Lambda t)X\boldsymbol{y}_0 \\
\Leftrightarrow && \boldsymbol{y}(t) &= X^{-1}\exp(\Lambda t)X\boldsymbol{y}_0\,.
\end{aligned}$$

Hence the solution to (7.11) can also be expressed in the form

$$\boldsymbol{y} = X^{-1}\exp(\Lambda t)X\boldsymbol{y}_0\,.$$

Note that, using the uniqueness of solutions to linear ODEs, we have also established that

$$\exp(At) \equiv X^{-1} \exp(\Lambda t) X \,.$$

7.5. Solution to general linear ODE systems

When the coefficient matrix is not constant, then the solution to the initial value problem

$$\boldsymbol{y}' = A(t)\boldsymbol{y}\,, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0\,, \tag{7.14}$$

is given by the Neumann series

$$\boldsymbol{y}(t) = \left(I + \int_0^t A(\tau)\,\mathrm{d}\tau + \int_0^t A(\tau_1) \int_0^{\tau_1} A(\tau_2)\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 + \ldots \right) \boldsymbol{y}_0\,.$$

Check that it is indeed correct by differentiating the series term by term and showing that it satisfies the linear ODE system (7.14). This Neumann series converges provided $\int_0^t \|A(\tau)\|\,\mathrm{d}\tau < \infty$.

7.6. Exercises

7.1. Two particles of equal mass $m = 1$ move in one dimension at the junction of three springs. The springs each have unstretched length $a = 1$ and have spring stiffness constants, k, $3k$ and k (with $k \equiv 1$) respectively—see Figure 7.3. Applying Newton's second law and Hooke's law, this mass-spring system gives rise to the differential equation system

$$\frac{\mathrm{d}^2\boldsymbol{y}}{\mathrm{d}t^2} = A\,\boldsymbol{y}\,, \tag{7.15}$$

where A is the stiffness matrix given by

$$A = \begin{pmatrix} -4 & 3 \\ 3 & -4 \end{pmatrix}\,, \tag{7.16}$$

and

$$\boldsymbol{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is the vector of unknown position displacements for each of the masses shown in Figure 7.3.

Find the eigenvalues and associated eigenvectors of the stiffness matrix A. Hence enumerate the possible modes of vibration of the masses corresponding to each eigenvalue-eigenvector pair. Finally, write down the general solution of the system of equations (7.15).

7.2. Consider the simple model for a tri-atomic shown in Figure 7.4. The molecule consists of three atoms of the same mass m, constrained so that only longitudinal motion is possible. Molecular bonds are modelled by the springs shown, each with stiffness $k = 1$.

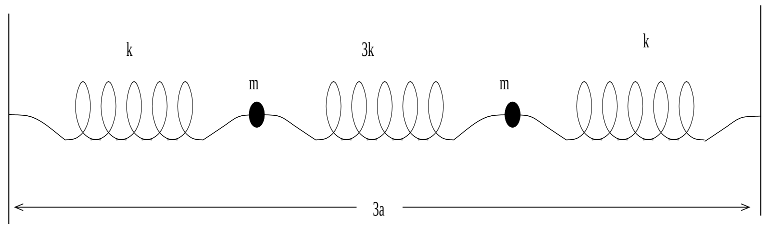


FIGURE 7.3. Simple mass-spring two-particle system.

Applying Newton's second law and Hooke's law, this mass-spring system gives rise to the differential equation system

$$\frac{\mathrm{d}^2\boldsymbol{y}}{\mathrm{d}t^2} = A\,\boldsymbol{y}\,, \tag{7.17}$$

where A is the stiffness matrix given by

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\,,$$

and

$$\boldsymbol{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

is the vector of unknown position displacements for each of the three masses shown in Figure 7.4.

Find the eigenvalues and associated eigenvectors of the stiffness matrix, A, and enumerate the possible modes of vibration of the masses corresponding to each eigenvalue-eigenvector pair. Hence write down the general solution of the system of equations (7.17).

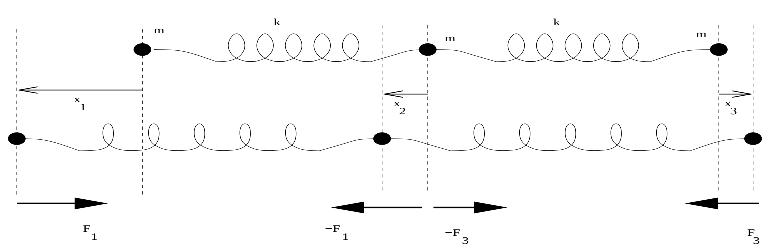


FIGURE 7.4. Simple model for a tri-atomic molecule.

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