Lecture notes

Introductory fluid mechanics

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1 Introduction

The derivation of the equations of motion for an *ideal fluid* by Euler in 1755, and then for a *viscous fluid* by Navier (1822) and Stokes (1845) were a tour-de-force of 18th and 19th century mathematics. These equations have been used to describe and explain so many physical phenomena around us in nature, that currently billions of dollars of research grants in mathematics, science and engineering now revolve around them. They can be used to model the coupled atmospheric and ocean flow used by the meteorological office for weather prediction down to any application in chemical engineering you can think of, say to development of the thrusters on NASA's Apollo programme rockets. The incompressible *Navier–Stokes equations* are given by

$$\begin{aligned}\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} &= \nu\,\nabla^2\boldsymbol{u} - \nabla p + \boldsymbol{f},\\ \nabla\cdot\boldsymbol{u} &= 0,\end{aligned}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is a three dimensional incompressible fluid velocity (indicated by the last equation), $p = p(\boldsymbol{x}, t)$ is the pressure and $\boldsymbol{f}$ is an external force field. The frictional force due to stickiness of a fluid is represented by the term $\nu\,\nabla^2\boldsymbol{u}$. An ideal fluid corresponds to the case $\nu = 0$, when the equations above are known as the *Euler equations* for a homogeneous incompressible ideal fluid. We will derive the Navier–Stokes equations and in the process learn about the subtleties of fluid mechanics and along the way see lots of interesting applications.

2 Fluid flow

2.1 Flow

A material exhibits *flow* if shear forces, however small, lead to a deformation which is unbounded—we could use this as definition of a *fluid*. A *solid* has a fixed shape, or at least a strong limitation on its deformation when force is applied to it. With the category of "fluids", we include liquids and gases. The main distinguishing feature between these two fluids is the notion of compressibility. Gases are usually compressible—as we know from everyday aerosols and air canisters. Liquids are generally incompressible—a feature essential to all modern car braking mechanisms.

Fluids can be further subcatergorized. There are *ideal* or *inviscid* fluids. In such fluids, the *only* internal force present is pressure which acts so that fluid flows from a region of high pressure to one of low pressure. The equations for an ideal fluid have been applied to wing and aircraft design (as a limit of high Reynolds number flow). However fluids can exhibit internal frictional forces which model a "stickiness" property of the fluid which involves energy loss—such fluids are known as *viscous* fluids. Some fluids/material known as "non-Newtonian or complex fluids" exhibit even stranger behaviour, their reaction to deformation may depend on: (i) past history (earlier deformations), for example some paints; (ii) temperature, for example some polymers or glass; (iii) the size of the deformation, for example some plastics or silly putty.

2.2 Continuum hypothesis

For any *real* fluid there are three natural length scales:

1. $L_{\text{molecular}}$, the molecular scale characterized by the mean free path distance of molecules between collisions;
2. L_{fluid}, the medium scale of a fluid parcel, the fluid droplet in the pipe or ocean flow;
3. L_{macro}, the macro-scale which is the scale of the fluid geometry, the scale of the container the fluid is in, whether a beaker or an ocean.

And, of course we have the asymptotic inequalities:

$$L_{\text{molecular}} \ll L_{\text{fluid}} \ll L_{\text{macro}}.$$

We will assume that the properties of an elementary volume/parcel of fluid, however small, are the same as for the fluid as a whole—i.e. we suppose that the properties of the fluid at scale L_{fluid} propagate all the way down and through the molecular scale $L_{\text{molecular}}$. This is the *continuum assumption*. For everyday fluid mechanics engineering, this assumption is extremely accurate (Chorin and Marsden [3, p. 2]).

2.3 Conservation principles

Our derivation of the basic equations underlying the dynamics of fluids is based on three basic principles:

1. *Conservation of mass*, mass is neither created or destroyed;
2. *Newton's 2nd law/balance of momentum*, for a parcel of fluid the rate of change of momentum equals the force applied to it;
3. *Conservation of energy*, energy is neither created nor destroyed.

In turn these principles generate the:

1. *Continuity equation* which governs how the density of the fluid evolves locally and thus indicates compressibility properties of the fluid;
2. *Navier–Stokes equations* of motion for a fluid which indicates how the fluid moves around from regions of high pressure to those of low pressure and under the effects of viscosity;
3. *Equation of state* which indicates the mechanism of energy exchange within the fluid.

3 Trajectories and streamlines

Suppose that our fluid is contained with a region/domain $\mathcal{D} \subseteq \mathbb{R}^d$ where $d = 2$ or 3, and $\boldsymbol{x} = (x, y, z) \in \mathcal{D}$ is a position/point in $\mathcal{D}$. Imagine a small fluid particle or a speck of dust moving in a fluid flow field prescribed by the *velocity field* $\boldsymbol{u}(\boldsymbol{x}, t) = (u, v, w)$. Suppose the position of the particle at time t is recorded by the variables $\big(x(t), y(t), z(t)\big)$. The velocity of the particle at time t at position $\big(x(t), y(t), z(t)\big)$ is

$$\begin{aligned}
\dot{x}(t) &= u\big(x(t), y(t), z(t), t\big),\\
\dot{y}(t) &= v\big(x(t), y(t), z(t), t\big),\\
\dot{z}(t) &= w\big(x(t), y(t), z(t), t\big).
\end{aligned}$$

In shorter vector notation this is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t).$$

The *trajectory* or *particle path* of a fluid particle is the curve traced out by the particle as time progresses. It is the solution to the differential equation above (with suitable initial conditions).

Suppose now for a given fluid flow $\boldsymbol{u}(\boldsymbol{x}, t)$ we fix time t. A *streamline* is an integral curve of $\boldsymbol{u}(\boldsymbol{x}, t)$ for t fixed, i.e. it is a curve $\boldsymbol{x} = \boldsymbol{x}(s)$ parameterized by the variable s, that satisfies the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{x}(s) = \boldsymbol{u}(\boldsymbol{x}(s), t),$$

with t held constant. If the velocity field $\boldsymbol{u}$ is time-independent, i.e. $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ only, or equivalently $\partial_t \boldsymbol{u} = \boldsymbol{0}$, then trajectories and streamlines coincide. Flows for which $\partial_t \boldsymbol{u} = \boldsymbol{0}$ are said to be *stationary*.

Example. Suppose a velocity field $\boldsymbol{u}(\boldsymbol{x}, t) = (u, v, w)$ is given for $t > -1$ by

$$u = \frac{x}{1+t}, \qquad v = \frac{y}{1+\frac{1}{2}t} \qquad \text{and} \qquad w = z.$$

To find the *particle paths or trajectories*, we must solve the system of equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = v \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = w,$$

and then eliminate the time variable t between them. Hence for the particle paths we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{1+t}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{1+\frac{1}{2}t} \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = z.$$

Using the method of separation of variables and integrating in time from t_0 to t, in each of the three equations, we get

$$\ln\left(\frac{x}{x_0}\right) = \ln\left(\frac{1+t}{1+t_0}\right), \qquad \ln\left(\frac{y}{y_0}\right) = 2\ln\left(\frac{1+\frac{1}{2}t}{1+\frac{1}{2}t_0}\right) \qquad \text{and} \qquad \ln\left(\frac{z}{z_0}\right) = \frac{t}{t_0},$$

where we have assumed that at time t_0 the particle is at position (x_0, y_0, z_0). Exponentiating the first two equations and solving the last one for t, we get

$$\frac{x}{x_0} = \frac{1+t}{1+t_0}, \qquad \frac{y}{y_0} = \frac{(1+\frac{1}{2}t)^2}{(1+\frac{1}{2}t_0)^2} \qquad \text{and} \qquad t = t_0 \ln(z/z_0).$$

We can use the last equation to eliminate t so the particle path/trajectory through (x_0, y_0, z_0) is the curve in three dimensional space given by

$$x = x_0 \cdot \frac{\big(1 + t_0 \ln(z/z_0)\big)}{(1+t_0)}, \qquad \text{and} \qquad y = y_0 \cdot \frac{\big(1+\frac{1}{2}t_0 \ln(z/z_0)\big)^2}{(1+\frac{1}{2}t_0)^2}.$$

To find the *streamlines*, we fix time t. We must then solve the system of equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = u, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = v \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = w,$$

with t fixed, and then eliminate s between them. Hence for streamlines we have

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{x}{1+t}, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = \frac{y}{1+\frac{1}{2}t} \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = z.$$

Assuming that we are interested in the streamline that passes through the point (x_0, y_0, z_0), we again use the method of separation of variables and integrate with respect to s from s_0 to s, for each of the three equations. This gives

$$\ln\left(\frac{x}{x_0}\right) = \frac{s-s_0}{1+t}, \qquad \ln\left(\frac{y}{y_0}\right) = \frac{s-s_0}{1+\frac{1}{2}t} \qquad \text{and} \qquad \ln\left(\frac{z}{z_0}\right) = s - s_0.$$

Using the last equation, we can substitute for $s - s_0$ into the first equations. If we then multiply the first equation by $1 + t$ and the second by $1 + \frac{1}{2}t$, and use the usual log law $\ln a^b = b \ln a$, then exponentiation reveals that

$$\left(\frac{x}{x_0}\right)^{1+t} = \left(\frac{y}{y_0}\right)^{1+\frac{1}{2}t} = \frac{z}{z_0},$$

which are the equations for the streamline through (x_0, y_0, z_0).

4 Conservation of mass

4.1 Continuity equation

Recall, we suppose our fluid is contained with a region/domain $\mathcal{D} \subseteq \mathbb{R}^d$ (here we will assume $d = 3$, but everything we say is true for the collapsed two dimensional case $d = 2$). Hence $\boldsymbol{x} = (x, y, z) \in \mathcal{D}$ is a position/point in $\mathcal{D}$. At each time t we will suppose that the fluid has a well defined *mass density* $\rho(\boldsymbol{x}, t)$ at the point $\boldsymbol{x}$. Further, each fluid particle traces out a well defined path in the fluid, and its motion along that path is governed by the *velocity field* $\boldsymbol{u}(\boldsymbol{x}, t)$ at position $\boldsymbol{x}$ at time t. Consider an arbitrary subregion $\Omega \subseteq \mathcal{D}$. The total mass of fluid contained inside the region Ω at time t is

$$\int_\Omega \rho(\boldsymbol{x}, t)\,\mathrm{d}V.$$

where $\mathrm{d}V$ is the volume element in $\mathbb{R}^d$. Let us now consider the rate of change of mass inside Ω. By the principle of conservation of mass, the rate of increase of the mass in Ω is given by the mass of fluid entering/leaving the boundary $\partial\Omega$ of Ω.

To compute the total mass of fluid entering/leaving the boundary $\partial\Omega$, we consider a small area patch $\mathrm{d}S$ on the boundary of $\partial\Omega$, which has unit outward normal $\boldsymbol{n}$. The total mass of fluid flowing out of Ω through the area patch $\mathrm{d}S$ per unit time is

$$\text{mass density} \times \text{fluid volume leaving per unit time} = \rho(\boldsymbol{x}, t)\,\boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S,$$

where $\boldsymbol{x}$ is at the center of the area patch $\mathrm{d}S$ on $\partial\Omega$. Note that to estimate the fluid volume leaving per unit time we have decomposed the fluid velocity at $\boldsymbol{x} \in \partial\Omega$, time t, into velocity components normal $(\boldsymbol{u} \cdot \boldsymbol{n})$ and tangent to the surface $\partial\Omega$ at that point. The velocity component tangent to the surface pushes fluid across the surface—no fluid enters or leaves Ω via this component. Hence we only retain the normal component—see Fig. 2.

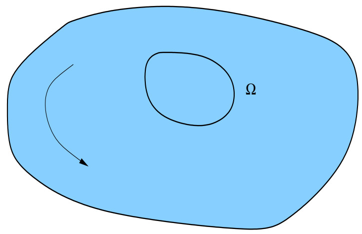

Fig. 1 The fluid of mass density $\rho(\boldsymbol{x}, t)$ swirls around inside the container $\mathcal{D}$, while Ω is an imaginary subregion.

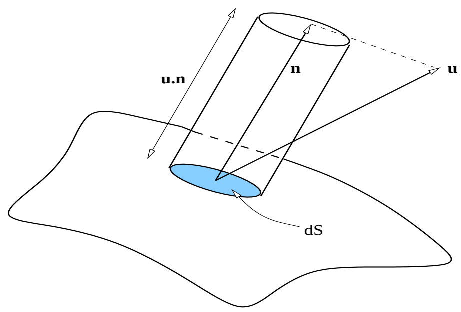

Fig. 2 The total mass of fluid moving through the patch $\mathrm{d}S$ on the surface $\partial\Omega$ per unit time, is given by the mass density $\rho(\boldsymbol{x}, t)$ times the volume of the cylinder shown which is $\boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}S$.

Returning to the principle of conservation of mass, this is now equivalent to the *integral form of the law of conservation of mass*:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \rho(\boldsymbol{x}, t)\, \mathrm{d}V = -\int_{\partial\Omega} \rho \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}S.$$

The divergence theorem and that the rate of change of the total mass inside Ω equals the total rate of change of mass density inside Ω imply, respectively,

$$\int_{\Omega} \nabla \cdot (\rho \boldsymbol{u})\, \mathrm{d}V = \int_{\partial\Omega} (\rho \boldsymbol{u}) \cdot \boldsymbol{n}\, \mathrm{d}S \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \rho\, \mathrm{d}V = \int_{\Omega} \frac{\partial \rho}{\partial t}\, \mathrm{d}V.$$

Using these two relations, the law of conservation of mass is equivalent to

$$\int_{\Omega} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u})\, \mathrm{d}V = 0.$$

Now we use that Ω is arbitrary to deduce the *differential form of the law of conservation of mass* or *continuity equation* that applies pointwise:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0.$$

This is the first of our three conservation laws.

4.2 **Incompressible flow**

Having established the continuity equation we can now define a subclass of flows which are *incompressible*. The classic examples are water, and the brake fluid in your car whose incompressibility properties are vital to the effective transmission of pedal pressure to brakepad pressure.

Definition 1 (Incompressibility) A fluid with the property $\nabla \cdot \boldsymbol{u} = 0$ is *incompressible*.

The continuity equation and the identity, $\nabla \cdot (\rho\,\boldsymbol{u}) = \nabla\rho \cdot \boldsymbol{u} + \rho\,\nabla \cdot \boldsymbol{u}$, imply

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla\rho + \rho\,\nabla \cdot \boldsymbol{u} = 0.$$

Hence since $\rho > 0$, a flow is incompressible if and only if

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla\rho = 0.$$

If the fluid is *homogeneous* so that ρ is constant in space, then the flow is incompressible if and only if ρ is constant in time.

4.3 **Stream functions**

A *stream function* exists for a given flow $\boldsymbol{u} = (u, v, w)$ if the velocity field $\boldsymbol{u}$ is *solenoidal*, i.e. $\nabla \cdot \boldsymbol{u} = 0$, *and* we have an additional symmetry that allows us to eliminate one coordinate. For example, a two dimensional incompressible fluid flow $\boldsymbol{u} = \boldsymbol{u}(x, y, t)$ is solenoidal since $\nabla \cdot \boldsymbol{u} = 0$, and has the symmetry that it is uniform with respect to z. For such a flow we see that

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \Leftrightarrow \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

This equation is satisfied if and only if there exists a function $\psi(x, y, t)$ such that

$$\frac{\partial \psi}{\partial y} = u(x, y, t) \qquad \text{and} \qquad -\frac{\partial \psi}{\partial x} = v(x, y, t).$$

The function ψ is called *Lagrange's stream function*. A stream function is always only defined up to any arbitrary additive constant. Further note that for t fixed, *streamlines* are given by constant contour lines of ψ (note also that $\nabla\psi \cdot \boldsymbol{u} = 0$ everywhere).

Note that if we use plane polar coordinates so $\boldsymbol{u} = \boldsymbol{u}(r, \theta, t)$ and the velocity components are $\boldsymbol{u} = (u_r, u_\theta)$ then

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \Leftrightarrow \qquad \frac{1}{r}\frac{\partial}{\partial r}(r\,u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0.$$

This is satisfied if and only if there exists a function $\psi(r, \theta, t)$ such that

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = u_r(r, \theta, t) \qquad \text{and} \qquad -\frac{\partial \psi}{\partial r} = u_\theta(r, \theta, t).$$

Example Suppose that in Cartesian coordinates we have the two dimensional flow $\boldsymbol{u} = (u, v)$ given by

$$(u, v) = (k\,x, -k\,y),$$

for some constant k. Note that $\nabla \cdot \boldsymbol{u} = 0$ so there exists a stream function satisfying

$$\frac{\partial \psi}{\partial y} = k\,x \qquad \text{and} \qquad -\frac{\partial \psi}{\partial x} = -k\,y.$$

Consider the first partial differential equation. Integrating with respect to y we get

$$\psi = k\,xy + C(x)$$

where $C(x)$ is an arbitrary function of x. However we know that ψ must simultaneously satisfy the second partial differential equation above. Hence we substitute this last relation into the second partial differential equation above to get

$$-\frac{\partial \psi}{\partial x} = -k\,y \qquad \Leftrightarrow \qquad -k\,y + C'(x) = -k\,y.$$

We deduce $C'(x) = 0$ and therefore C is an arbitrary constant. Since a stream function is only defined up to an arbitrary constant we take $C = 0$ for simplicity and the stream function is given by

$$\psi = k\,xy.$$

Now suppose we used plane polar coordinates instead. The corresponding flow $\boldsymbol{u} = (u_r, u_\theta)$ is given by

$$(u_r, u_\theta) = (k\,r\cos 2\theta, -k\,r\sin 2\theta).$$

First note that $\nabla \cdot \boldsymbol{u} = 0$ using the polar coordinate form for $\nabla \cdot \boldsymbol{u}$ indicated above. Hence there exists a stream function $\psi = \psi(r,\theta)$ satisfying

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = k\,r\cos 2\theta \qquad \text{and} \qquad -\frac{\partial \psi}{\partial r} = -k\,r\sin 2\theta.$$

As above, consider the first partial differential equation shown, and integrate with respect to θ to get

$$\psi = \tfrac{1}{2}\,k\,r^2 \sin 2\theta + C(r).$$

Substituting this into the second equation above reveals that $C'(r) = 0$ so that C is a constant. We can for convenience set $C = 0$ so that

$$\psi = \tfrac{1}{2}\,k\,r^2 \sin 2\theta.$$

Comparing this form with its Cartesian equivalent above, reveals they are the same.

5 Balance of momentum

5.1 Differentiation following the fluid

Recall our image of a small fluid particle moving in a fluid flow field prescribed by the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$. The velocity of the particle at time t at position $\boldsymbol{x}(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t).$$

As the particle moves in the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, say from position $\boldsymbol{x}(t)$ to a nearby position an instant in time later, two dynamical contributions change: (i) a small instant in time has elapsed and the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, which depends on time, will have changed a little; (ii) the position of the particle has changed in that short time as it moved slightly, and the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, which depends on position, will be slightly different at the new position.

Let us compute the *acceleration* of the particle to explicitly observe these two contributions. By using the chain rule we see that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{x}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}\big(\boldsymbol{x}(t),t\big)\\ &= \frac{\partial \boldsymbol{u}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial t}\\ &= \left(\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial}{\partial z}\right)\boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial t}\\ &= \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial t}.\end{aligned}$$

Indeed for any function $F(x,y,z,t)$, scalar or vector valued, the chain rule implies

$$\frac{\mathrm{d}}{\mathrm{d}t}F\big(x(t),y(t),z(t),t\big) = \frac{\partial F}{\partial t} + \boldsymbol{u}\cdot\nabla F.$$

Definition 2 (Material derivative) If the velocity field components are

$$\boldsymbol{u} = (u,v,w) \qquad \text{and} \qquad \boldsymbol{u}\cdot\nabla \equiv u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

then we define the *material derivative* following the fluid to be

$$\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla.$$

5.2 **Rate of strain tensor**

Consider a fluid flow in a region $\mathcal{D}\subseteq\mathbb{R}^3$. Suppose $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{h}$ are two nearby points in the interior of $\mathcal{D}$. How is the flow, or more precisely the velocity field, at $\boldsymbol{x}$ related to that at $\boldsymbol{x}+\boldsymbol{h}$? From a mathematical perspective, by Taylor expansion we have

$$\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{h}) = \boldsymbol{u}(\boldsymbol{x}) + \big(\nabla\boldsymbol{u}(\boldsymbol{x})\big)\cdot\boldsymbol{h} + \mathcal{O}(h^2),$$

where $(\nabla\boldsymbol{u})\cdot\boldsymbol{h}$ is simply matrix multiplication of the 3×3 matrix $\nabla\boldsymbol{u}$ by the column vector $\boldsymbol{h}$. Recall that $\nabla\boldsymbol{u}$ is given by

$$\nabla\boldsymbol{u} = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z\\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z\\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{pmatrix}.$$

In the context of fluid flow it is known as the *rate of strain tensor*. This is because, locally, it measures that rate at which neighbouring fluid particles are being pulled apart (it helps to recall that the velocity field $\boldsymbol{u}$ records the rate of change of particle position with respect to time).

Again from a mathematical perspective, we can decompose $\nabla\boldsymbol{u}$ as follows. We can always write

$$\nabla\boldsymbol{u} = \tfrac{1}{2}\big((\nabla\boldsymbol{u}) + (\nabla\boldsymbol{u})^{\mathrm{T}}\big) + \tfrac{1}{2}\big((\nabla\boldsymbol{u}) - (\nabla\boldsymbol{u})^{\mathrm{T}}\big).$$

We set

$$D := \tfrac{1}{2}\big((\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^{\mathrm{T}}\big),$$
$$R := \tfrac{1}{2}\big((\nabla \boldsymbol{u}) - (\nabla \boldsymbol{u})^{\mathrm{T}}\big).$$

Note that $D = D(\boldsymbol{x})$ is a 3×3 symmetric matrix, while $R = R(\boldsymbol{x})$ is the 3×3 skew-symmetric matrix given by

$$R = \begin{pmatrix} 0 & \partial u/\partial y - \partial v/\partial x & \partial u/\partial z - \partial w/\partial x \\ \partial v/\partial x - \partial u/\partial y & 0 & \partial v/\partial z - \partial w/\partial y \\ \partial w/\partial x - \partial u/\partial z & \partial w/\partial y - \partial v/\partial z & 0 \end{pmatrix}.$$

Note that if we set

$$\omega_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad \omega_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \qquad \text{and} \qquad \omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

then R is more simply expressed as

$$R = \tfrac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Further by direct computation we see that

$$R\,\boldsymbol{h} = \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{h},$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{x})$ is the vector with three components ω_1, ω_2 and ω_3. At this point, we have thus established the following.

Theorem 1 *If $\boldsymbol{x}$ and $\boldsymbol{x} + \boldsymbol{h}$ are two nearby points in the interior of $\mathcal{D}$, then*

$$\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{h}) = \boldsymbol{u}(\boldsymbol{x}) + D(\boldsymbol{x}) \cdot \boldsymbol{h} + \tfrac{1}{2}\boldsymbol{\omega}(\boldsymbol{x}) \times \boldsymbol{h} + \mathcal{O}(h^2).$$

The symmetric matrix D is the *deformation tensor*. Since it is symmetric, there is an orthonormal basis $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$ in which D is diagonal, i.e. if $X = [\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3]$ then

$$X^{-1} D X = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}.$$

The vector $\boldsymbol{\omega}$ is the *vorticity field* of the flow. An equivalent definition for it is

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}.$$

It encodes the magnitude of, and direction of the axis about which, the fluid rotates, locally.

Now consider the motion of a fluid particle labelled by $\boldsymbol{x} + \boldsymbol{h}$ where $\boldsymbol{x}$ is fixed and $\boldsymbol{h}$ is small (for example suppose that only a short time has elapsed). Then the position of the particle is given by

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x} + \boldsymbol{h}) = \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{h}) \\ \Leftrightarrow \quad & \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{h}) \\ \Leftrightarrow \quad & \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} \approx \boldsymbol{u}(\boldsymbol{x}) + D(\boldsymbol{x}) \cdot \boldsymbol{h} + \tfrac{1}{2}\boldsymbol{\omega}(\boldsymbol{x}) \times \boldsymbol{h}. \end{aligned}$$

Let us consider in turn each of the effects on the right shown:

1. The term $\boldsymbol{u}(\boldsymbol{x})$ is simply uniform translational velocity (the particle being pushed by the ambient flow surrounding it).
2. Now consider the second term $D(\boldsymbol{x}) \cdot \boldsymbol{h}$. If we ignore the other terms then, approximately, we have
$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = D(\boldsymbol{x}) \cdot \boldsymbol{h}.$$
Making a local change of coordinates so that $\boldsymbol{h} = X\hat{\boldsymbol{h}}$ we get
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\hat{h}_1\\ \hat{h}_2\\ \hat{h}_3\end{pmatrix} = \begin{pmatrix}d_1 & 0 & 0\\ 0 & d_2 & 0\\ 0 & 0 & d_3\end{pmatrix}\begin{pmatrix}\hat{h}_1\\ \hat{h}_2\\ \hat{h}_3\end{pmatrix}.$$
We see that we have pure expansion or contraction (depending on whether d_i is positive or negative, respectively) in each of the characteristic directions $\hat{h}_i$, $i = 1, 2, 3$. Indeed the small linearized volume element $\hat{h}_1\hat{h}_2\hat{h}_3$ satisfies
$$\frac{\mathrm{d}}{\mathrm{d}t}(\hat{h}_1\hat{h}_2\hat{h}_3) = (d_1 + d_2 + d_3)(\hat{h}_1\hat{h}_2\hat{h}_3).$$
Note that $d_1 + d_2 + d_3 = \mathrm{Tr}(D) = \nabla \cdot \boldsymbol{u}$.
3. Let us now examine the effect of the third term $\frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{h}$. Ignoring the other two terms we have
$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \tfrac{1}{2}\boldsymbol{\omega}(\boldsymbol{x}) \times \boldsymbol{h}.$$
Direct computation shows that
$$\boldsymbol{h}(t) = \varPhi(t, \boldsymbol{\omega}(\boldsymbol{x}))\boldsymbol{h}(0),$$
where $\varPhi(t, \boldsymbol{\omega}(\boldsymbol{x}))$ is the matrix that represents the rotation through an angle t about the axis $\boldsymbol{\omega}(\boldsymbol{x})$. Note also that $\nabla \cdot \big(\boldsymbol{\omega}(\boldsymbol{x}) \times \boldsymbol{h}\big) = 0$.

5.3 Internal fluid forces

Let us consider the forces that act on a small parcel of fluid in a fluid flow. There are two types:

1. *external or body forces*, these may be due to gravity or external electromagnetic fields. They exert a force per unit volume on the continuum.
2. *surface or stress forces*, these are forces, molecular in origin, that are applied by the neighbouring fluid across the surface of the fluid parcel.

The surface or stress forces are the result of molecular diffusion, we can explain them as follows. Imagine two neighbouring parcels of fluid P and P' as shown in Fig. 3, with a mutual contact surface is S as shown. Suppose both parcels of fluid are moving parallel to S and to each other, but the speed of P, say $\boldsymbol{u}$, is much faster than that of P', say $\boldsymbol{u}'$. In the kinetic theory of matter molecules jiggle about and take random walks; they diffuse into their surrounding locale and impart their kinetc energy to molecules they pass by. Hence the faster molecules in P will diffuse across S and impart momentum to the molecules in P'. Similarly, slower molecules from P' will diffuse across s to slow the fluid in P down. In regions of the flow where the velocity field changes rapidly over small length scales, this effect is important—see Chorin and Marsden [3, p. 31].

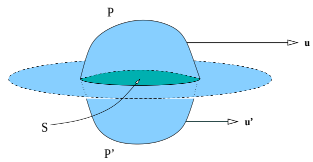

Fig. 3 Two neighbouring parcels of fluid P and P'. Suppose S is the surface of mutual contact between them. Their respective velocities are $\boldsymbol{u}$ and $\boldsymbol{u}'$ and in the same direction and parallel to S, but with $|\boldsymbol{u}| \gg |\boldsymbol{u}'|$. The faster molecules in P will diffuse across the surface S and impart momentum to P'.

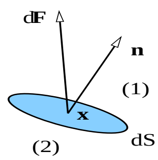

Fig. 4 The force $\mathrm{d}\boldsymbol{F}$ on side (2) by side (1) of $\mathrm{d}S$ is given by $\boldsymbol{\Sigma}(\boldsymbol{n})\,\mathrm{d}S$.

We now proceed more formally. The force per unit area exerted across a surface (imaginary in the fluid) is called the *stress*. Let $\mathrm{d}S$ be a small imaginary surface in the fluid centered on the point $\boldsymbol{x}$—see Fig. 4. The force $\mathrm{d}\boldsymbol{F}$ on side (2) by side (1) of $\mathrm{d}S$ in the fluid/material is given by

$$\mathrm{d}\boldsymbol{F} = \boldsymbol{\Sigma}(\boldsymbol{n})\,\mathrm{d}S.$$

Here $\boldsymbol{\Sigma}$ is the stress at the point $\boldsymbol{x}$. It is a function of the normal direction $\boldsymbol{n}$ to the surface $\mathrm{d}S$, in fact it is given by:

$$\boldsymbol{\Sigma}(\boldsymbol{n}) = \sigma(\boldsymbol{x})\,\boldsymbol{n}.$$

Note $\sigma = [\sigma_{ij}]$ is a 3×3 matrix known as the *stress tensor*. The diagonal components of σ_{ij}, with $i = j$, generate *normal stresses*, while the off-diagonal components, with $i \neq j$, generate *tangential* or *shear stresses*. Indeed let us decompose the stress tensor $\sigma = \sigma(\boldsymbol{x})$ as follows (here I is the 3×3 identity matrix):

$$\sigma = -p\,I + \hat{\sigma}.$$

Here the scalar quantity $p = p(\boldsymbol{x}) > 0$ is defined to be

$$p := -\tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

and represents the fluid *pressure*. The remaining part of the stress tensor $\hat{\sigma} = \hat{\sigma}(\boldsymbol{x})$ is known as the *deviatoric stress tensor*. In this decomposition, the term $-p\,I$ generates the normal stresses, since if this were the only term present,

$$\sigma = -p\,I \qquad \Rightarrow \qquad \boldsymbol{\Sigma}(\boldsymbol{n}) = -p\,\boldsymbol{n}.$$

The deviatoric stress tensor $\hat{\sigma}$ on the other hand, generates the shear stresses.

We shall make three assumptions about the deviatoric stress tensor $\hat{\sigma}$, it is:

1. *A linear homogeneous function of the velocity gradients* $\nabla \boldsymbol{u}$; i.e. at each point $\hat{\sigma}$ is linearly related to the rate of strain tensor $\nabla \boldsymbol{u}$. This is the key property of what is known as a *Newtonian fluid*: the stress is proportional to the rate of strain.
2. *Invariant under rigid body rotations*; i.e. if U is an orthogonal matrix, then

$$\hat{\sigma}\big(U \cdot \nabla \boldsymbol{u} \cdot U^{-1}\big) \equiv U \cdot \hat{\sigma}(\nabla \boldsymbol{u}) \cdot U^{-1}.$$

   When the fluid performs rigid body rotation, there should be no diffusion of momentum (the whole mass of fluid is behaving like a solid body).
3. *Symmetric*; i.e. it is a symmetric matrix: $\hat{\sigma}_{ij} = \hat{\sigma}_{ji}$. This can be deduced as a result of balance of angular momentum.

These three assumptions imply that $\hat{\sigma}$ only depends on the symmetric part of $\nabla \boldsymbol{u}$, i.e. it is a homogeneous linear function of the deformation tensor D. Further, we can deduce that $\hat{\sigma}$ and D commute, i.e. $\hat{\sigma} D = D \hat{\sigma}$. Therefore $\hat{\sigma}$ and D can be simultaneously diagonalized by $X = [\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3]$. To see this, suppose d_i is an eigenvalue of D with eigenvector $\hat{\boldsymbol{e}}_i$, then

$$\hat{\sigma} D \, \hat{\boldsymbol{e}}_i = D \hat{\sigma} \, \hat{\boldsymbol{e}}_i = d_i \, \hat{\sigma} \, \hat{\boldsymbol{e}}_i.$$

Hence $\hat{\sigma} \hat{\boldsymbol{e}}_i$ is an eigenvector of D with eigenvalue d_i. This implies $\hat{\sigma} \hat{\boldsymbol{e}}_i \propto \hat{\boldsymbol{e}}_i$, so $\hat{\boldsymbol{e}}_i$ is an eigenvector of $\hat{\sigma}$ also. We thus conclude that we can simultaneously diagonalize $\hat{\sigma}$ and D, and the eigenvalues of $\hat{\sigma}$ are homogeneous linear functions of the eigenvalues of D. Furthermore, property 2 above implies that the eigenvalues of $\hat{\sigma}$ must be symmetric functions of the eigenvalues of D (we can choose U to permute two eigenvalues of D if it permutes two coordinate axes, and this must permute the eigenvalues of $\hat{\sigma}$). Since each eigenvalue $\hat{\sigma}_i$ of $\hat{\sigma}$ is a linear homogeneous symmetric function of the d_i, it must have the form

$$\hat{\sigma}_i = \lambda(d_1 + d_2 + d_3) + 2\mu d_i$$

for $i = 1, 2, 3$. Recall that $d_1 + d_2 + d_3 = \nabla \cdot \boldsymbol{u}$. If we thus use X^{-1} to transform back to our original basis then we have

$$\hat{\sigma} = \lambda(\nabla \cdot \boldsymbol{u}) I + 2\mu D.$$

If we set $\zeta = \lambda + \frac{2}{3}\mu$ this last relation becomes

$$\hat{\sigma} = 2\mu\big(D - \tfrac{1}{3}(\nabla \cdot \boldsymbol{u}) I\big) + \zeta(\nabla \cdot \boldsymbol{u}) I,$$

where μ and ζ are the first and second *coefficients of viscosity*, respectively.

5.4 **Navier–Stokes equations**

Consider again an arbitrary imaginary subregion Ω of $\mathcal{D}$ as in Fig. 1. At any instant in time t, the total force exerted on the fluid inside Ω through the stresses exerted across its boundary $\partial\Omega$ is given by

$$\int_{\partial\Omega} (-pI + \hat{\sigma}) \, \boldsymbol{n} \, \mathrm{d}S \equiv \int_{\Omega} (-\nabla p + \nabla \cdot \hat{\sigma}) \, \mathrm{d}V,$$

where (for convenience here we set $(x_1, x_2, x_3) \equiv (x, y, z)$ and $(u_1, u_2, u_3) \equiv (u, v, w)$)

$$
\begin{aligned}
[\nabla \cdot \hat{\sigma}]_i &= \sum_{j=1}^{3} \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} \\
&= \lambda [\nabla(\nabla \cdot \boldsymbol{u})]_i + 2\mu \sum_{j=1}^{3} \frac{\partial D_{ij}}{\partial x_j} \\
&= \lambda [\nabla(\nabla \cdot \boldsymbol{u})]_i + \mu \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \\
&= \lambda [\nabla(\nabla \cdot \boldsymbol{u})]_i + \mu \sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial x_j^2} - \frac{\partial^2 u_j}{\partial x_i \partial x_j} \\
&= (\lambda + \mu) [\nabla(\nabla \cdot \boldsymbol{u})]_i + \mu \nabla^2 u_i .
\end{aligned}
$$

If $\boldsymbol{f}(\boldsymbol{x}, t)$ is a body force (external force) per unit mass then the total body force on the fluid inside Ω is

$$\int_{\Omega} \rho\, \boldsymbol{f} \, \mathrm{d}V.$$

Thus on any parcel of fluid, the force per unit volume acting on it is

$$-\nabla p + \nabla \cdot \hat{\sigma} + \rho\, \boldsymbol{f}.$$

Hence using Newton's 2nd law (force = mass $\times$ acceleration) we can deduce the following relation—Cauchy's equation of motion—the differential form of the *balance of momentum*:

$$\rho \, \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \nabla \cdot \hat{\sigma} + \rho\, \boldsymbol{f}.$$

Combining this with the form for $\nabla \cdot \hat{\sigma}$ we deduced above, we arrive at

$$\rho \, \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + (\lambda + \mu) \nabla(\nabla \cdot \boldsymbol{u}) + \mu \Delta \boldsymbol{u} + \rho\, \boldsymbol{f},$$

where $\Delta = \nabla^2$ is the Laplacian operator. These are the *Navier–Stokes equations*. If we assume we are in three dimensional space so $d = 3$, then together with the continuity equation we have four equations, but five unknowns—namely $\boldsymbol{u}$, p and ρ. Thus for a *compressible* fluid flow, we cannot specify the fluid motion completely without specifying one more condition/relation.

For an *incompressible homogeneous* flow for which the density $\rho = \rho_0$ is constant, we get a complete set of equations known as the *Navier–Stokes equations for an incompressible flow*:

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= \nu \, \Delta \boldsymbol{u} - \nabla p + \boldsymbol{f}, \\
\nabla \cdot \boldsymbol{u} &= 0,
\end{aligned}
$$

where $\nu = \mu / \rho_0$ is the coefficient of *kinematic viscosity*. Note that the pressure field here is the rescaled pressure by a factor $1/\rho_0$: since ρ_0 is constant $(\nabla p)/\rho_0 \equiv \nabla(p/\rho_0)$, and we re-label the term p/ρ_0 to be p. Note that we have a closed system of equations: we have four equations in four unknowns, $\boldsymbol{u}$ and p.

For any motion of an *ideal fluid* we only include normal stresses and completely ignore any shear stresses. Hence instead of the the Navier–Stokes equation above we get the Euler equations of motion for an ideal fluid (derived by Euler in 1755) given by (take $\lambda = \mu = 0$):

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \boldsymbol{f},$$

The fact that there are no tangential forces in an ideal fluid has some important consequences, quoting from Chorin and Marsden [3, p. 5]:

> ...there is no way for rotation to start in a fluid, nor, if there is any at the beginning, to stop... ... even here we can detect trouble for ideal fluids because of the abundance of rotation in real fluids (near the oars of a rowboat, in tornadoes, etc.).

We discuss the Euler equations in more detail in Section 13.2.

5.5 **Boundary conditions**

Now that we have the partially differential equations that determine how fluid flows evolve, we complement them with the boundary and initial conditions. The initial condition is the velocity profile $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, 0)$ at time $t = 0$. It is the state in which the flow starts. To have a *well-posed* evolutionary partial differential system for the evolution of the fluid flow, we also need to specify how the flow behaves near boundaries. Here a boundary could be a rigid boundary, for example the walls of the container the fluid is confined to or the surface of an obstacle in the fluid flow. Another example of a boundary is the free surface between two immiscible fluids—such as between seawater and air on the ocean surface. Here we will focus on rigid boundaries.

For an *ideal fluid flow*, i.e. one evolving according to the Euler equations, we simply need to specify that there is no net flow normal to the boundary—the fluid does not cross the boundary but can move tangentially to it. Mathematically this is means that we specify that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ everywhere on the rigid boundary.

For *viscous flow*, i.e. evolving according to the Navier–Stokes equations, we need to specify additional boundary conditions. This is due to the inclusion of the extra term $\nu \Delta \boldsymbol{u}$ which increases the number of spatial derivatives in the governing evolution equations from one to two. Mathematically, we specify that

$$\boldsymbol{u} = \boldsymbol{0}$$

everywhere on the rigid boundary, i.e. in addition to the condition that there must be no net normal flow at the boundary, we also specify there is no tangential flow there. The fluid velocity is simply zero at a rigid boundary; it is sometimes called *no-slip* boundary conditions. Experimentally this is observed as well, to a very high degree of precision; see Chorin and Marsden [3, p. 34]. (Dye can be introduced into a flow near a boundary and how the flow behaves near it observed and measured very accurately.) Further, recall that in a viscous fluid flow we are incorporating the effect of molecular diffusion between neighbouring fluid parcels—see Fig. 3. The rigid non-moving boundary should impart a zero tangential flow condition to the fluid particles right up against it. The no-slip boundary condition is crucially repsresents the mechanism for vorticity production in nature that can be observed everywhere. Just look at the flow of a river close to the river bank.

Remark 1 At a material boundary (or free surface) between two immiscible fluids, we would specify that there is no jump in the velocity across the surface boundary. This is true if there is no surface tension or at least if it is negligible—for example at the seawater-air boundary of the ocean. However at the surface of melting wax at the top of a candle, there is surface tension, and there is a jump in the stress $\sigma\,\boldsymbol{n}$ at the boundary surface. Surface tension is also responsible for the phenomenon of being able to float a needle on the surface of a bowl of water as well as many other interesting effects such as the shape of water drops.

5.6 **Evolution of vorticity**

Recall from our discussion in Section 5.2, that the *vorticity field* of a flow with velocity field $\boldsymbol{u}$ is defined as

$$\boldsymbol{\omega} := \nabla \times \boldsymbol{u}.$$

It encodes the magnitude of, and direction of the axis about which, the fluid rotates, locally. Note that $\nabla \times \boldsymbol{u}$ can be computed as follows

$$\nabla \times \boldsymbol{u} = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{pmatrix} = \begin{pmatrix} \partial w/\partial y - \partial v/\partial z \\ \partial u/\partial z - \partial w/\partial x \\ \partial v/\partial x - \partial u/\partial y \end{pmatrix}.$$

Using the Navier–Stokes equations for a homogeneous incompressible fluid, we can in fact derive a closed system of equations governing the evolution of vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ as follows. Using the identity $\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u})$ we see that we can equivalently represent the Navier–Stokes equations in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \tfrac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big) - \boldsymbol{u} \times \boldsymbol{\omega} = \nu\,\Delta \boldsymbol{u} - \nabla p + \boldsymbol{f}.$$

If we take the curl of this equation and use the identity

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \boldsymbol{u}\,(\nabla \cdot \boldsymbol{\omega}) - \boldsymbol{\omega}\,(\nabla \cdot \boldsymbol{u}) + (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega},$$

noting that $\nabla \cdot \boldsymbol{u} = 0$ and $\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \boldsymbol{u}) \equiv 0$, we find that we get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu\,\Delta \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nabla \times \boldsymbol{f}.$$

Note that we can recover the velocity field $\boldsymbol{u}$ from the vorticity $\boldsymbol{\omega}$ by using the identity $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \Delta \boldsymbol{u}$. This implies

$$\Delta \boldsymbol{u} = -\nabla \times \boldsymbol{\omega},$$

and closes the system of partial differential equations for $\boldsymbol{\omega}$ and $\boldsymbol{u}$. However, we can also simply observe that

$$\boldsymbol{u} = \big(-\Delta\big)^{-1}(\nabla \times \boldsymbol{\omega}).$$

If the body force is conservative so that $\boldsymbol{f} = \nabla \Phi$ for some potential Φ, then $\nabla \times \boldsymbol{f} \equiv \boldsymbol{0}$.

Remark 2 We can replace the 'vortex stretching' term $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$ in the evolution equation for the vorticity by $D\boldsymbol{\omega}$, where D is the 3×3 deformation matrix, since

$$\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = (\nabla \boldsymbol{u})\boldsymbol{\omega} = D\boldsymbol{\omega} + R\boldsymbol{\omega} = D\boldsymbol{\omega},$$

as direct computation reveals that $R\boldsymbol{\omega} \equiv \boldsymbol{0}$.

6 Transport theorem

Suppose that the region within which the fluid is moving is $\mathcal{D}$. Suppose Ω is a subregion of $\mathcal{D}$ identified at time $t=0$. As the fluid flow evolves the fluid particles that originally made up Ω will subsequently fill out a volume Ω_t at time t. We think of Ω_t as the volume *moving with the fluid*.

Theorem 2 (Transport theorem) *For any function F and density function ρ satisfying the continuity equation, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \rho\, F\,\mathrm{d}V = \int_{\Omega_t} \rho\,\frac{\mathrm{D}F}{\mathrm{D}t}\,\mathrm{d}V.$$

The transport theorem is useful because it allows us to deduce Cauchy's equation of motion from the *primitive* integral form of the balance of momentum. From Newton's second law this is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \rho\,\boldsymbol{u}\,\mathrm{d}V = \int_{\Omega_t} -\nabla p + \nabla\cdot\hat{\sigma} + \rho\,\boldsymbol{f}\,\mathrm{d}V.$$

Hence, using the transport theorem with $F \equiv \boldsymbol{u}$ and that Ω and thus Ω_t are arbitrary, we see that Cauchy's equation of motion must hold pointwise at each $\boldsymbol{x}\in\mathcal{D}$.

Proof There are four steps; see Chorin and Marsden [3, pp. 6–11].

Step 1: Fluid flow map. For a fixed position $\boldsymbol{x}\in\mathcal{D}$ we denote by $\boldsymbol{\xi}(\boldsymbol{x},t)=(\xi,\eta,\zeta)$ the position of the particle at time t, which at time $t=0$ was at $\boldsymbol{x}$. We use φ_t to denote the map $\boldsymbol{x}\mapsto\boldsymbol{\xi}(\boldsymbol{x},t)$, i.e. φ_t is the map that advances each particle at position $\boldsymbol{x}$ at time $t=0$ to its position at time t later; it is the *fluid flow-map*. Hence, for example $\varphi_t(\Omega)=\Omega_t$. We assume φ_t is sufficiently smooth and invertible for all our subsequent manipulations.

Step 2: Change of variables. For any two functions ρ and F we can perform the change of variables from $(\boldsymbol{\xi},t)$ to $(\boldsymbol{x},t)$—with $J(\boldsymbol{x},t)$ the Jacobian for this transformation given by definition as $J(\boldsymbol{x},t):=\det\bigl(\nabla\boldsymbol{\xi}(\boldsymbol{x},t)\bigr)$. Here the gradient operator is with respect to the $\boldsymbol{x}$ coordinates, i.e. $\nabla=\nabla_{\boldsymbol{x}}$. Note for Ω_t we integrate over volume elements $\mathrm{d}V=\mathrm{d}V(\boldsymbol{\xi})$, i.e. with respect to the $\boldsymbol{\xi}$ coordinates, whereas for Ω we integrate over volume elements $\mathrm{d}V=\mathrm{d}V(\boldsymbol{x})$, i.e. with respect to the fixed coordinates $\boldsymbol{x}$. Hence by direct computation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \rho\,F\,\mathrm{d}V &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} (\rho\,F)(\boldsymbol{\xi},t)\,\mathrm{d}V(\boldsymbol{\xi})\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} (\rho\,F)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\,J(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x})\\
&= \int_{\Omega} \frac{\mathrm{d}}{\mathrm{d}t}\Bigl((\rho\,F)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\,J(\boldsymbol{x},t)\Bigr)\,\mathrm{d}V\\
&= \int_{\Omega} \frac{\mathrm{d}}{\mathrm{d}t}(\rho\,F)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\,J(\boldsymbol{x},t) + (\rho\,F)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\frac{\mathrm{d}}{\mathrm{d}t}J(\boldsymbol{x},t)\,rdV\\
&= \int_{\Omega} \biggl(\frac{\mathrm{D}}{\mathrm{D}t}(\rho\,F)\biggr)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\,J(\boldsymbol{x},t) + (\rho\,F)(\boldsymbol{\xi}(\boldsymbol{x},t),t)\frac{\mathrm{d}}{\mathrm{d}t}J(\boldsymbol{x},t)\,\mathrm{d}V.
\end{aligned}$$

Step 3: Evolution of the Jacobian. We establish the following result for the Jacobian:

$$\frac{\mathrm{d}}{\mathrm{d}t}J(\boldsymbol{x},t) = \bigl(\nabla\cdot\boldsymbol{u}(\boldsymbol{\xi}(\boldsymbol{x},t),t)\bigr)\,J(\boldsymbol{x},t).$$

We know that a particle at position $\boldsymbol{\xi}(\boldsymbol{x},t) = \big(\xi(\boldsymbol{x},t), \eta(\boldsymbol{x},t), \zeta(\boldsymbol{x},t)\big)$, which started at $\boldsymbol{x}$ at time $t=0$, evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi}(\boldsymbol{x},t) = \boldsymbol{u}\big(\boldsymbol{\xi}(\boldsymbol{x},t),t\big).$$

Taking the gradient with respect to $\boldsymbol{x}$ of this relation, and swapping over the gradient and $\mathrm{d}/\mathrm{d}t$ operations on the left, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla\boldsymbol{\xi}(\boldsymbol{x},t) = \nabla\boldsymbol{u}\big(\boldsymbol{\xi}(\boldsymbol{x},t),t\big).$$

Using the chain rule we have

$$\nabla_{\boldsymbol{x}}\,\boldsymbol{u}\big(\boldsymbol{\xi}(\boldsymbol{x},t),t\big) = \Big(\nabla_{\boldsymbol{\xi}}\,\boldsymbol{u}\big(\boldsymbol{\xi}(\boldsymbol{x},t),t\big)\Big)\cdot\big(\nabla_{\boldsymbol{x}}\,\boldsymbol{\xi}(\boldsymbol{x},t)\big).$$

Combining the last two relations we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla\boldsymbol{\xi} = (\nabla_{\boldsymbol{\xi}}\,\boldsymbol{u})\,\nabla\boldsymbol{\xi}.$$

Abel's Theorem then tells us that $J = \det\nabla\boldsymbol{\xi}$ evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\det\nabla\boldsymbol{\xi} = \big(\mathrm{Tr}(\nabla_{\boldsymbol{\xi}}\,\boldsymbol{u})\big)\,\det\nabla\boldsymbol{\xi}.$$

Since $\mathrm{Tr}(\nabla_{\boldsymbol{\xi}}\,\boldsymbol{u}) \equiv \nabla\cdot\boldsymbol{u}$ we have established the required result.

Step 4: Conservation of mass. We see that we thus have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\rho\,F\,\mathrm{d}V &= \int_{\Omega}\bigg(\frac{\mathrm{D}}{\mathrm{D}t}(\rho\,F) + (\rho\,F)\big(\nabla\cdot\boldsymbol{u}\big)\bigg)\big(\boldsymbol{\xi}(\boldsymbol{x},t),t\big)\,J(\boldsymbol{x},t)\,\mathrm{d}V \\
&= \int_{\Omega_t}\bigg(\frac{\mathrm{D}}{\mathrm{D}t}(\rho\,F) + \big(\rho\,\nabla\cdot\boldsymbol{u}\big)\,F\bigg)\,\mathrm{d}V \\
&= \int_{\Omega_t}\rho\,\frac{\mathrm{D}F}{\mathrm{D}t}\,\mathrm{d}V,
\end{aligned}$$

where in the last step we have used the conservation of mass equation. □

Corollary 1 (Equivalent incompressibility statements) *The following statements are equivalent, for any subregion Ω of the fluid, the:*

1. *Fluid is incompressible;*
2. *Jacobian $J \equiv 1$;*
3. *Volume of Ω_t is constant in time.*

Proof Using the result in Step 3 of the proof of the transport theorem, we see that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(\Omega_t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\mathrm{d}V(\boldsymbol{\xi}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}J(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x}) \\
&= \int_{\Omega}\big(\nabla\cdot\boldsymbol{u}(\boldsymbol{\xi}(\boldsymbol{x},t),t)\big)\,J(\boldsymbol{x},t)\,\mathrm{d}V(\boldsymbol{x}) \\
&= \int_{\Omega_t}\big(\nabla\cdot\boldsymbol{u}(\boldsymbol{\xi},t)\big)\,\mathrm{d}V(\boldsymbol{\xi}).
\end{aligned}$$

Further, noting that by definition $J(\boldsymbol{x},0)=1$, establishes the result. □

7 Simple example flows

We roughly follow an illustrative sequence of examples given in Majda and Bertozzi [11, pp. 8–15]. The first few are example flows of a class of exact solutions to both the Euler and Navier–Stokes equations.

Lemma 1 (Majda and Bertozzi, p. 8) *Let $D = D(t) \in \mathbb{R}^3$ be a real symmetric matrix such that $\mathrm{Tr}(D) = 0$ (respresenting the deformation matrix). Suppose that the vorticity $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ solves the ordinary differential system*

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = D(t)\,\boldsymbol{\omega}$$

for some initial data $\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 \in \mathbb{R}^3$. If the three components of vorticity are thus $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, set

$$R \coloneqq \tfrac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Then we have that

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t) &= \tfrac{1}{2}\boldsymbol{\omega}(t) \times \boldsymbol{x} + D(t)\,\boldsymbol{x}, \\ p(\boldsymbol{x}, t) &= -\tfrac{1}{2}\left(\frac{\mathrm{d}D}{\mathrm{d}t} + D^2(t) + R^2(t)\right)\boldsymbol{x} \cdot \boldsymbol{x}, \end{aligned}$$

are exact solutions to the incompressible Euler and Navier–Stokes equations.

Remark 3 Since the pressure is a quadratic function of the spatial coordinates $\boldsymbol{x}$, these solutions only have meaningful interpretations locally. Further note that the velocity solution field $\boldsymbol{u}$ only depends linearly on the spatial coordinates $\boldsymbol{x}$; this explains why once we established these are exact solutions of the Euler equations, they are also solutions of the Navier–Stokes equations.

Proof Recall that $\nabla\boldsymbol{u}$ is the rate of strain tensor. It can be decomposed into a direct sum of its symmetric and skew-symmetric parts which are the 3×3 matrices

$$\begin{aligned} D &\coloneqq \tfrac{1}{2}\big((\nabla\boldsymbol{u}) + (\nabla\boldsymbol{u})^{\mathrm{T}}\big), \\ R &\coloneqq \tfrac{1}{2}\big((\nabla\boldsymbol{u}) - (\nabla\boldsymbol{u})^{\mathrm{T}}\big). \end{aligned}$$

We can determine how $\nabla\boldsymbol{u}$ evolves by taking the gradient of the homogeneous (no body force) Navier–Stokes equations so that

$$\frac{\partial}{\partial t}(\nabla\boldsymbol{u}) + \boldsymbol{u} \cdot \nabla(\nabla\boldsymbol{u}) + (\nabla\boldsymbol{u})^2 = \nu\,\Delta(\nabla\boldsymbol{u}) - \nabla\nabla p.$$

Note here $(\nabla\boldsymbol{u})^2 = (\nabla\boldsymbol{u})(\nabla\boldsymbol{u})$ is simply matrix multiplication. By direct computation

$$(\nabla\boldsymbol{u})^2 = (D + R)^2 = (D^2 + R^2) + (DR + RD),$$

where the first term on the right is symmetric and the second is skew-symmetric. Hence we can decompose the evolution of $\nabla\boldsymbol{u}$ into the coupled evolution of its symmetric and skew-symmetric parts

$$\begin{aligned} \frac{\partial D}{\partial t} + \boldsymbol{u} \cdot \nabla D + D^2 + R^2 &= \nu\,\Delta D - \nabla\nabla p, \\ \frac{\partial R}{\partial t} + \boldsymbol{u} \cdot \nabla R + DR + RD &= \nu\,\Delta R. \end{aligned}$$

Directly computing the evolution for the three components of $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ from the second system of equations we would arrive at the following equation for vorticity,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu\, \Delta \boldsymbol{\omega} + D\boldsymbol{\omega},$$

which we derived more directly in Section 5.6.

Thusfar we have not utilized the ansatz (form) for the velocity or pressure we assume in the statement of the theorem. Assuming $\boldsymbol{u}(\boldsymbol{x}, t) = \frac{1}{2}\boldsymbol{\omega}(t) \times \boldsymbol{x} + D(t)\,\boldsymbol{x}$, for a given deformation matrix $D = D(t)$, then $\nabla \times \boldsymbol{u} = \boldsymbol{\omega}(t)$, independent of $\boldsymbol{x}$, and substituting this into the evolution equation for $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ above we obtain the following system of ordinary differential equations governing the evolution of $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$:

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = D(t)\boldsymbol{\omega}.$$

Now the symmetric part governing the evolution of $D = D(t)$, which is independent of $\boldsymbol{x}$, reduces to the system of differential equations

$$\frac{\mathrm{d}D}{\mathrm{d}t} + D^2 + R^2 = -\nabla\nabla p.$$

Note that $R = R(t)$ only as well, since $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$, and thus $\nabla\nabla p$ must be a function of t only. Hence $p = p(\boldsymbol{x}, t)$ can only quadratically depend on $\boldsymbol{x}$. Indeed after integrating we must have $p(\boldsymbol{x}, t) = -\frac{1}{2}(\mathrm{d}D/\mathrm{d}t + D^2 + R^2)\,\boldsymbol{x} \cdot \boldsymbol{x}$. □

Example (jet flow) Suppose the initial vorticity $\boldsymbol{\omega}_0 = \mathbf{0}$ and $D = \mathrm{diag}\{d_1, d_2, d_3\}$ is a constant diagonal matrix where $d_1 + d_2 + d_3 = 0$ so that $\mathrm{Tr}(D) = 0$. Then from Lemma 1, we see that the flow is irrotational, i.e. $\boldsymbol{\omega}(t) = 0$ for all $t \geqslant 0$. Hence the velocity field $\boldsymbol{u}$ is given by

$$\boldsymbol{u}(\boldsymbol{x}, t) = D(t)\boldsymbol{x} = \begin{pmatrix} d_1\, x \\ d_2\, y \\ d_3\, z \end{pmatrix}.$$

The particle path for a particle at (x_0, y_0, z_0) at time $t = 0$ is given by: $x(t) = \mathrm{e}^{d_1 t} x_0$, $y(t) = \mathrm{e}^{d_2 t} y_0$ and $z(t) = \mathrm{e}^{d_3 t} z_0$. If $d_1 < 0$ and $d_2 < 0$, then $d_3 > 0$ and we see the flow resembles two jets streaming in opposite directions away from the $z = 0$ plane.

Example (strain flow) Suppose the initial vorticity $\boldsymbol{\omega}_0 = \mathbf{0}$ and $D = \mathrm{diag}\{d_1, d_2, 0\}$ is a constant diagonal matrix such that $d_1 + d_2 = 0$. Then as in the last example, the flow is irrotational with $\boldsymbol{\omega}(t) = 0$ for all $t \geqslant 0$ and

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} d_1\, x \\ d_2\, y \\ 0 \end{pmatrix}.$$

The particle path for a particle at (x_0, y_0, z_0) at time $t = 0$ is given by: $x(t) = \mathrm{e}^{d_1 t} x_0$, $y(t) = \mathrm{e}^{d_2 t} y_0$ and $z(t) = z_0$. Since $d_2 = -d_1$, the flow forms a strain flow as shown in Fig. 5—neighbouring particles are pushed together in one direction while being pulled apart in the other orthogonal direction.

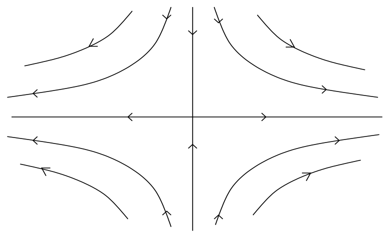

Fig. 5 Strain flow example.

Example (vortex) Suppose the initial vorticity $\boldsymbol{\omega}_0 = (0, 0, \omega_0)$ and $D = O$. Then from Lemma 1 the velocity field $\boldsymbol{u}$ is given by

$$\boldsymbol{u}(\boldsymbol{x}, t) = \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{x} = \begin{pmatrix} -\frac{1}{2}\omega_0 y \\ \frac{1}{2}\omega_0 x \\ 0 \end{pmatrix}.$$

The particle path for a particle at (x_0, y_0, z_0) at time $t = 0$ is given by: $x(t) = \cos(\frac{1}{2}\omega_0 t)x_0 - \sin(\frac{1}{2}\omega_0 t)y_0$, $y(t) = \sin(\frac{1}{2}\omega_0 t)x_0 + \cos(\frac{1}{2}\omega_0 t)y_0$ and $z(t) = z_0$. These are circular trajectories, and indeed the flow resembles a solid body rotation; see Fig. 6.

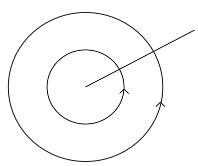

Fig. 6 When a fluid flow is a rigid body rotation, the fluid particles flow on circular streamlines. The fluid particles on paths further from the origin or axis of rotation, circulate faster at just the right speed that they remain alongside their neighbours on the paths just inside them.

Example (jet flow with swirl) Now suppose the initial vorticity $\boldsymbol{\omega}_0 = (0, 0, \omega_0)$ and $D = \mathrm{diag}\{d_1, d_2, d_3\}$ is a constant diagonal matrix where $d_1 + d_2 + d_3 = 0$. Then from Lemma 1, we see that the only non-zero component of vorticity is the third component, say $\omega = \omega(t)$, where

$$\omega(t) = \omega_0 \mathrm{e}^{d_3 t}.$$

The velocity field $\boldsymbol{u}$ is given by

$$\boldsymbol{u}(\boldsymbol{x}, t) == \begin{pmatrix} d_1\, x - \frac{1}{2}\omega(t)y \\ d_2\, y + \frac{1}{2}\omega(t)x \\ d_3\, z \end{pmatrix}.$$

The particle path for a particle at (x_0, y_0, z_0) at $t = 0$ can be described as follows. We see that $z(t) = z_0 \mathrm{e}^{d_3 t}$ while $x = x(t)$ and $y = y(t)$ satisfy the coupled system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_1 & -\frac{1}{2}\omega(t) \\ \frac{1}{2}\omega(t) & d_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

If we assume $d_1 < 0$ and $d_2 < 0$ then the particles spiral around the z-axis with decreasing radius and increasing angular velocity $\frac{1}{2}\omega(t)$. The flow thus resembles a rotating jet flow; see Fig. 7.

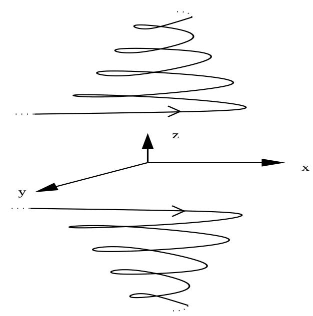

Fig. 7 Jet flow with swirl example. Fluid particles rotate around and move closer to the z-axis whilst moving further from the $z = 0$ plane.

Example (shear-layer flows) We derive a simple class of solutions that retain the three underlying mechanisms of Navier–Stokes flows: convection, vortex stretching and diffusion. Recall that the vorticity $\boldsymbol{\omega}$ evolves according to the partial differential system

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu \, \Delta \boldsymbol{\omega} + D\boldsymbol{\omega},$$

with $\Delta \boldsymbol{u} = -\nabla \times \boldsymbol{\omega}$. The material derivative term $\partial \boldsymbol{\omega}/\partial t + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega}$ convects vorticity along particle paths, while the term $\nu \, \Delta \boldsymbol{\omega}$ is responsible for the diffusion of vorticity and $D\boldsymbol{u}$ represents vortex stretching—the vorticity $\boldsymbol{\omega}$ increases/decreases when aligns along eigenvectors of D corresponding to positive/negative eigenvalues of D.

We seek an exact solution to the incompressible Navier–Stokes equations of the following form (the first two velocity components represent a strain flow)

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} -\gamma x \\ \gamma y \\ w(x, t) \end{pmatrix}$$

where γ is a constant, with $p(\boldsymbol{x}, t) = \frac{1}{2}\gamma\big(x^2 + y^2\big)$. This represents a solution to the Navier–Stokes equations if we can determine the solution $w = w(x, t)$ to the linear diffusion equation

$$\frac{\partial w}{\partial t} - \gamma x \frac{\partial w}{\partial x} = \nu \frac{\partial^2 w}{\partial x^2},$$

with $w(x, 0) = w_0(x)$. Computing the vorticity directly we get

$$\boldsymbol{\omega}(\boldsymbol{x}, t) = \begin{pmatrix} 0 \\ -\big(\partial w/\partial x\big)(x, t) \\ 0 \end{pmatrix}.$$

If we differentiate the equation above for the velocity field component w with respect to x, then if $\omega := -\partial w/\partial x$, we get

$$\frac{\partial \omega}{\partial t} - \gamma x \frac{\partial \omega}{\partial x} = \gamma \omega + \nu \frac{\partial^2 \omega}{\partial x^2},$$

with $\omega(x,0) = \omega_0(x) = -(\partial w_0/\partial x)(x)$. For this simpler flow we can see simpler signatures of the three effects we want to isolate: there is the convecting velocity $-\gamma x$; vortex stretching from the term $\gamma\omega$ and diffusion in the term $\nu\partial^2\omega/\partial x^2$. Note that is in the general case, the velocity field w can be recovered from the vorticity field ω by

$$w(x,t) = -\int_{-\infty}^{x} \omega(\xi,t)\,\mathrm{d}\xi.$$

Let us consider a special case: the viscous shear-layer solution where $\gamma = 0$. In this case we see that the partial differential equation above for ω reduces to the simple heat equation with solution

$$\omega(x,t) = \int_{\mathbb{R}} G(x-\xi,\nu t)\,\omega_0(\xi)\,\mathrm{d}\xi,$$

where G is the Gaussian heat kernel

$$G(\xi,t) \coloneqq \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\xi^2/4t}.$$

Indeed the velocity field w is given by

$$w(x,t) = \int_{\mathbb{R}} G(x-\xi,\nu t)\,w_0(\xi)\,\mathrm{d}\xi,$$

so that both the vorticity ω and velocity w fields diffuse as time evolves; see Fig. 8.

It is possible to write down the exact solution for the general case in terms of the Gaussian heat kernel, indeed, a very nice exposition can be found in Majda and Bertozzi [11, p. 18].

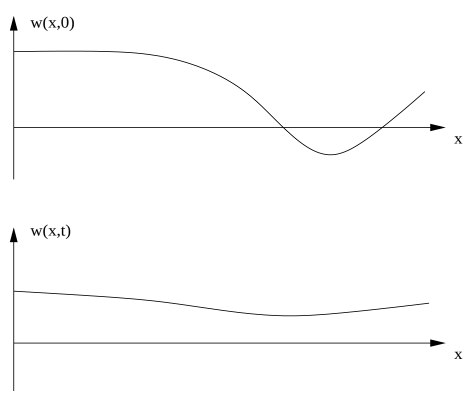

Fig. 8 Viscous shear flow example. The effect of diffusion on the velocity field $w = w(x,t)$ is to smooth out variations in the field as time progresses.

Example (channel shear flow) Consider the two-dimensional flow given by $u = 1 - y^2$ and $v = 0$ for $-1 \leqslant y \leqslant 1$ and all $x \in \mathbb{R}$ (which is an exact solution of the incompressible Navier–Stokes equations). For this flow the vorticity is given by

$$\nabla \times \boldsymbol{u} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{k} = 2y\,\boldsymbol{k}.$$

See the shape of the flow in Fig. 9. The flow is stationary near the channel walls (no-slip boundary conditions are satisfied there) and the flow rate a maximum in the middle of the channel. The gradient of the horizontal velocity u with respect to y is non-zero and thus the vorticity is non-zero (the vertical velocity component is zero).

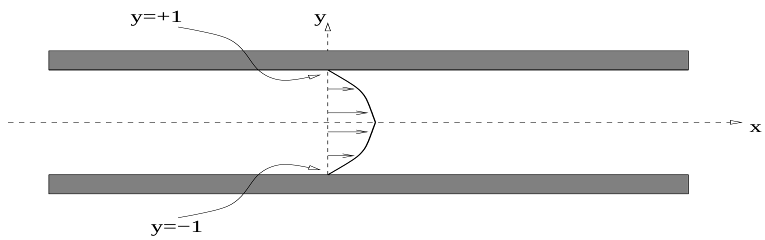

Fig. 9 Shear flow in a two-dimensional horizontal channel.

Example (sink or bath drain) As the water (of uniform density ρ) flows out through a hole at the bottom of a bath the residual rotation is confined to a core of radius a, so that the water particles may be taken to move on horizontal circles with

$$u_\theta = \begin{cases} \Omega r, & r \leqslant a, \\ \frac{\Omega a^2}{r}, & r > a. \end{cases}$$

As we have all observed when water runs out of a bath or sink, the free surface of the water directly over the drain hole has a depression in it—see Fig. 10. The question is, what is the form/shape of this free surface depression?

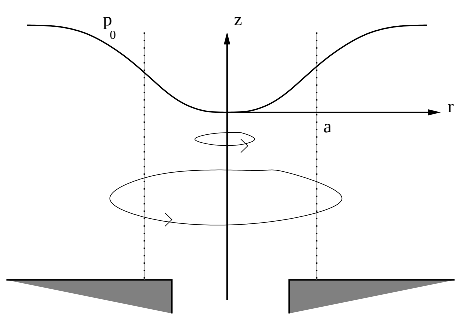

Fig. 10 Water draining from a bath.

We know that the pressure at the free surface is uniform, it is atmospheric pressure, say P_0. We need the Euler equations for a homogeneous incompressible fluid in *cylindrical coordinates* (r,θ,z) with the velocity field $\boldsymbol{u}=(u_r,u_\theta,u_z)$. These are

$$\begin{aligned}\frac{\partial u_r}{\partial t}+(\boldsymbol{u}\cdot\nabla)u_r-\frac{u_\theta^2}{r}&=-\frac{1}{\rho}\frac{\partial p}{\partial r}+f_r,\\ \frac{\partial u_\theta}{\partial t}+(\boldsymbol{u}\cdot\nabla)u_\theta+\frac{u_r u_\theta}{r}&=-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}+f_\theta,\\ \frac{\partial u_z}{\partial t}+(\boldsymbol{u}\cdot\nabla)u_z&=-\frac{1}{\rho}\frac{\partial p}{\partial z}+f_z,\end{aligned}$$

where $p=p(r,\theta,z,t)$ is the pressure, ρ is the uniform constant density and $\boldsymbol{f}=(f_r,f_\theta,f_z)$ is the body force per unit mass. Here we also have

$$\boldsymbol{u}\cdot\nabla=u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}.$$

Further the incompressibility condition $\nabla\cdot\boldsymbol{u}=0$ is given in cylindrical coordinates by

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}=0.$$

Now we look at the setting we are presented with for this problem. Note the flow is steady and $u_r=u_z=0$, $f_r=f_\theta=0$. The force due to gravity implies $f_z=-g$. The whole problem is also symmetric with respect to θ, so that all partial derivatives with respect to θ should be zero. Combining all these facts reduces Euler's equations above to

$$-\frac{u_\theta^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r},\qquad 0=-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}\qquad\text{and}\qquad 0=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g.$$

The incompressibility condition is satisfied trivially. The second equation above tells us the pressure p is independent of θ, as we might have already suspected. Hence we assume $p=p(r,z)$ and focus on the first and third equation above.

Assume $r\leqslant a$. Using that $u_\theta=\Omega r$ in the first equation we see that

$$\frac{\partial p}{\partial r}=\rho\Omega^2 r\qquad\Leftrightarrow\qquad p(r,z)=\tfrac{1}{2}\rho\Omega^2 r^2+C(z),$$

where $C(z)$ is an arbitrary function of z. If we then substitute this into the third equation above we see that

$$\frac{1}{\rho}\frac{\partial p}{\partial z}=-g\qquad\Leftrightarrow\qquad C'(z)=-\rho g,$$

and hence $C(z)=-\rho gz+C_0$ where C_0 is an arbitrary constant. Thus we now deduce that the pressure function is given by

$$p(r,z)=\tfrac{1}{2}\rho\Omega^2 r^2-\rho gz+C_0.$$

At the free surface of the water, the pressure is constant atmospheric pressure P_0 and so if we substitute this into this expression for the pressure we see that

$$P_0=\tfrac{1}{2}\rho\Omega^2 r^2-\rho gz+C_0\qquad\Leftrightarrow\qquad z=(\Omega^2/2g)\,r^2-(C_0-P_0)/\rho g.$$

Hence the depression in the free surface for $r\leqslant a$ is a *parabolic surface* of revolution. Note that pressure is only ever globally defined up to an additive constant so we are at liberty to take $C_0=0$ or $C_0=P_0$ if we like.

For $r > a$ a completely analogous argument using $u_\theta = \Omega a^2/r$ shows that

$$p(r,z) = -\frac{\rho\Omega^2 a^4}{2\,r^2} - \rho g z + K_0,$$

where K_0 is an arbitrary constant. Since the pressure must be continuous at $r = a$, we substitute $r = a$ into the expression for the pressure here for $r > a$ and the expression for the pressure for $r \leqslant a$, and equate the two. This gives

$$-\tfrac{1}{2}\rho\Omega^2 a^2 - \rho g z + K_0 = \tfrac{1}{2}\rho\Omega^2\, a^2 - \rho g z \qquad \Leftrightarrow \qquad K_0 = \rho\Omega^2\, a^2.$$

Hence the pressure for $r > a$ is given by

$$p(r,z) = -\frac{\rho\Omega^2 a^4}{2\,r^2} - \rho g z + \rho\Omega^2\, a^2.$$

Using that the pressure at the free surface is $p(r,z) = P_0$, we see that for $r > a$ the free surface is given by

$$z = -\frac{\Omega^2 a^4}{g\,r^2} + \frac{\Omega^2\, a^2}{g}.$$

8 Kelvin's circulation theorem, vortex lines and tubes

We turn our attention to important concepts centred on vorticity in a flow.

Definition 3 (Circulation) Let $\mathcal{C}$ be a simple closed contour in the fluid at time $t = 0$. Suppose that $\mathcal{C}$ is carried along by the flow to the closed contour $\mathcal{C}_t$ at time t, i.e. $\mathcal{C}_t = \varphi_t(\mathcal{C})$. The *circulation* around $\mathcal{C}_t$ is defined to be the line integral

$$K = \oint_{\mathcal{C}_t} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}.$$

Using Stokes' Theorem an equivalent definition for the circulation is

$$K = \oint_{\mathcal{C}_t} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \int_S (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{n}\,\mathrm{d}S = \int_S \boldsymbol{\omega} \cdot \boldsymbol{n}\,\mathrm{d}S$$

where S is any surface with perimeter $\mathcal{C}_t$; see Fig. 12. In other words the circulation is equivalent to the flux of vorticity through the surface with perimeter $\mathcal{C}_t$.

Theorem 3 (Kelvin's circulation theorem (1869)) *For ideal, incompressible flow without external forces, the circulation K for any closed contour $\mathcal{C}_t$ is constant in time.*

Proof Using a variant of the Transport Theorem for closed loops of fluid particles, and the Euler equations, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathcal{C}_t} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \oint_{\mathcal{C}_t} \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \mathrm{d}\boldsymbol{x} = -\oint_{\mathcal{C}_t} \nabla p \cdot \mathrm{d}\boldsymbol{x} = 0$$

since $\mathcal{C}_t$ is closed. □

Corollary 2 *The flux of vorticity across a surface moving with the fluid is constant in time.*

Definition 4 (Vortex lines) These are the lines that are everywhere parallel to the local vorticity $\boldsymbol{\omega}$, i.e. with t fixed they solve

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{x}(s) = \boldsymbol{\omega}(\boldsymbol{x}(s), t).$$

These are the trajectories for the field $\boldsymbol{\omega}$ for t fixed.

Definition 5 (Vortex tube) This is the surface formed by the vortex lines through the points of a simple closed curve $\mathcal{C}$; see Fig. 12. We can define the *strength* of the vortex tube to be

$$\int_S \boldsymbol{\omega}\cdot\boldsymbol{n}\,\mathrm{d}S \equiv \oint_{\mathcal{C}_t} \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x}.$$

Remark 4 This is a good definition because it is independent of the precise cross-sectional area S, and the precise circuit $\mathcal{C}$ around the vortex tube taken (because $\nabla\cdot\boldsymbol{\omega}\equiv 0$); see Fig. 12. Vorticity is larger where the cross-sectional area is smaller and vice-versa. Further, for an ideal fluid, vortex tubes move with the fluid and the strength of the vortex tube is constant in time as it does so (Helmholtz's theorem; 1858); see Chorin and Marsden [3, p. 26].

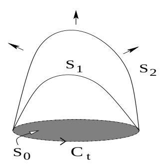

Fig. 11 Stokes' theorem tells us that the circulation around the closed contour $\mathcal{C}$ equals the flux of vorticity through any surface whose perimeter is $\mathcal{C}$. For example here the flux of vorticity through S_0, S_1 and S_2 is the same.

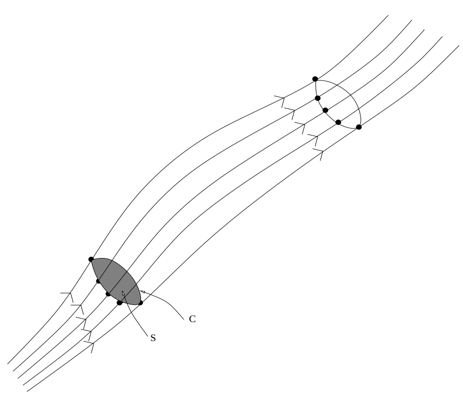

Fig. 12 The strength of the vortex tube is given by the circulation around any curve $\mathcal{C}$ that encircles the tube once.

9 Bernoulli's Theorem

Theorem 4 (Bernoulli's Theorem) *Suppose we have an ideal homogeneous incompressible stationary flow with a conservative body force $\boldsymbol{f} = -\nabla\Phi$, where Φ is the potential function. Then the quantity*

$$H := \tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \Phi$$

is constant along streamlines.

Proof We need the following identity that can be found in Appendix A:

$$\tfrac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big) = \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\times(\nabla\times\boldsymbol{u}).$$

Since the flow is stationary, Euler's equation of motion for an ideal fluid imply

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla\Big(\frac{p}{\rho}\Big) - \nabla\Phi.$$

Using the identity above we see that

$$\begin{aligned}
& & \tfrac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) &= -\nabla\Big(\frac{p}{\rho}\Big) - \nabla\Phi \\
\Leftrightarrow & & \nabla\Big(\tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \Phi\Big) &= \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) \\
\Leftrightarrow & & \nabla H &= \boldsymbol{u}\times(\nabla\times\boldsymbol{u}),
\end{aligned}$$

using the definition for H given in the theorem. Now let $\boldsymbol{x}(s)$ be a streamline that satisfies $\boldsymbol{x}'(s) = \boldsymbol{u}\big(\boldsymbol{x}(s)\big)$. By the fundamental theorem of calculus, for any s_1 and s_2,

$$\begin{aligned}
H\big(\boldsymbol{x}(s_2)\big) - H\big(\boldsymbol{x}(s_1)\big) &= \int_{s_1}^{s_2} \mathrm{d}H\big(\boldsymbol{x}(s)\big) \\
&= \int_{s_1}^{s_2} \nabla H\cdot\boldsymbol{x}'(s)\,\mathrm{d}s \\
&= \int_{s_1}^{s_2} \big(\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\big)\cdot\boldsymbol{u}\big(\boldsymbol{x}(s)\big)\,\mathrm{d}s \\
&= 0,
\end{aligned}$$

where we used that $(\boldsymbol{u}\times\boldsymbol{a})\cdot\boldsymbol{u} \equiv \boldsymbol{0}$ for any vector $\boldsymbol{a}$ (since $\boldsymbol{u}\times\boldsymbol{a}$ is orthogonal to $\boldsymbol{u}$). Since s_1 and s_2 are arbitrary we deduce that H does change along streamlines. □

Example (Torricelli 1643). Consider the problem of an oil drum full of water that has a small hole punctured into it near the bottom. The problem is to determine the velocity of the fluid jetting out of the hole at the bottom and how that varies with the amount of water left in the tank—the setup is shown in Fig 13. We shall assume the hole has a small cross-sectional area α. Suppose that the cross-sectional area of the drum, and therefore of the free surface (water surface) at $z = 0$, is A. We naturally assume $A \gg \alpha$. Since the rate at which the amount of water is dropping inside the drum must equal the rate at which water is leaving the drum through the punctured hole, we have

$$\Big(-\frac{\mathrm{d}h}{\mathrm{d}t}\Big)\cdot A = U\cdot\alpha \qquad \Leftrightarrow \qquad \Big(-\frac{\mathrm{d}h}{\mathrm{d}t}\Big) = \Big(\frac{\alpha}{A}\Big)\cdot U.$$

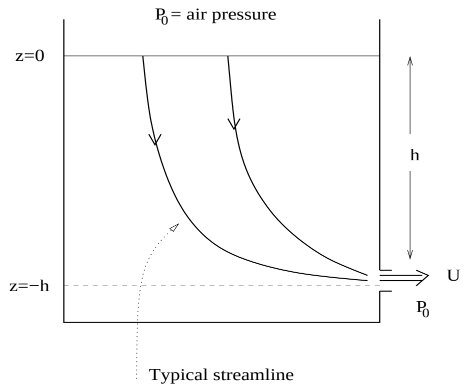

Fig. 13 *Torricelli problem*: the pressure at the top surface and outside the puncture hole is atmospheric pressure P_0. Suppose the height of water above the puncture is h. The goal is to determine how the velocity of water U out of the puncture hole varies with h.

We observe that $A \gg \alpha$, i.e. $\alpha/A \ll 1$, and hence we can deduce

$$\frac{1}{U^2}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 = \left(\frac{\alpha}{A}\right)^2 \ll 1.$$

Since the flow is quasi-stationary, incompressible as it's water, and there is conservative body force due to gravity, we apply Bernoulli's Theorem for one of the typical streamlines shown in Fig. 13. This implies that the quantity H is the same at the free surface and at the puncture hole outlet, hence

$$\tfrac{1}{2}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 + \frac{P_0}{\rho} = \tfrac{1}{2}U^2 + \frac{P_0}{\rho} - gh.$$

Thus cancelling the P_0/ρ terms then we can deduce that

$$\begin{aligned} gh &= \tfrac{1}{2}U^2 - \tfrac{1}{2}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 \\ &= \tfrac{1}{2}U^2\left(1 - \frac{1}{U^2}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2\right) \\ &= \tfrac{1}{2}U^2\left(1 - \left(\frac{\alpha}{A}\right)^2\right) \\ &\sim \tfrac{1}{2}U^2 \end{aligned}$$

for $\alpha/A \ll 1$. Thus in the asymptotic limit $gh = \frac{1}{2}U^2$ so that

$$U = \sqrt{2gh}.$$

Remark 5 Note the pressure inside the container at the puncture hole level is $P_0 + \rho gh$. The difference between this and the atmospheric pressure P_0 outside, accelerates the water through the puncture hole.

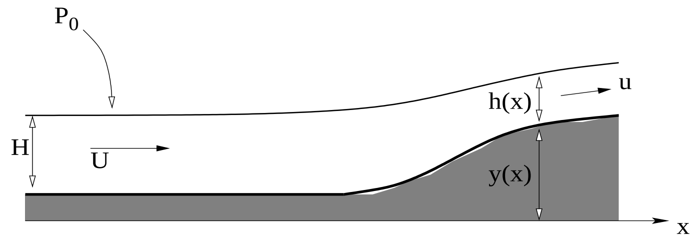

Fig. 14 *Channel flow problem*: a steady flow of water, uniform in cross-section, flows over a gently undulating bed of height $y = y(x)$ as shown. The depth of the flow is given by $h = h(x)$. Upstream the flow is characterized by flow velocity U and depth H.

Example (Channel flow: Froude number). Consider the problem of a steady flow of water in a channel over a gently underlating bed—see Fig 14. We assume that the flow is shallow and uniform in cross-section. Upstream the flow is characterized by flow velocity U and depth H. The flow then impinges on a gently undulating bed of height $y = y(x)$ as shown in Fig 14, where x measures distance downstream. The depth of the flow is given by $h = h(x)$ whilst the fluid velocity at that point is $u = u(x)$, which is uniform over the depth throughout. Re-iterating slightly, our assumptions are thus,

$$\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| \ll 1 \qquad \text{(bed gently undulating)}$$

and

$$\left|\frac{\mathrm{d}h}{\mathrm{d}x}\right| \ll 1 \qquad \text{(small variation in depth).}$$

The continuity equation (incompressibility here) implies that for all x,

$$uh = UH.$$

Then Euler's equations for a steady flow imply Bernoulli's theorem which we apply to the surface streamline, for which the pressure is constant and equal to atmospheric pressure P_0, hence we have for all x:

$$\tfrac{1}{2}U^2 + gH = \tfrac{1}{2}u^2 + g(y + h).$$

Substituting for $u = u(x)$ from the incompressibility condition above, and rearranging, Bernoulli's theorem implies that for all x we have the constraint

$$y = \frac{U^2}{2g} + H - h - \frac{(UH)^2}{2gh^2}.$$

We can think of this as a parametric equation relating the fluid depth $h = h(x)$ to the undulation height $h = h(x)$ where the parameter x runs from $x = -\infty$ far upstream to $x = +\infty$ far downstream. We plot this relation, y as a function of h, in Fig 15. Note that y has a unique global maximum y_0 coinciding with the local maximum and given by

$$\frac{\mathrm{d}y}{\mathrm{d}h} = 0 \qquad \Leftrightarrow \qquad h = h_0 = \frac{(UH)^{2/3}}{g^{1/3}}.$$

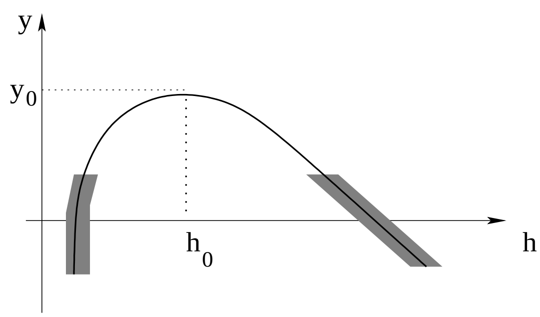

Fig. 15 *Channel flow problem*: The flow depth $h = h(x)$ and undulation height $y = y(x)$ are related as shown, from Bernoulli's theorem. Note that y has a maximum value y_0 at height $h_0 = H\mathrm{F}^{2/3}$ where $\mathrm{F} = U/\sqrt{gH}$ is the Froude number.

Note that if we set

$$\mathrm{F} := U/\sqrt{gH}$$

then $h_0 = H\mathrm{F}^{2/3}$, where F is known as the *Froude number*. It is a dimensionless function of the upstream conditions and represents the ratio of the oncoming fluid speed to the wave (signal) speed in fluid depth H.

Note that when $y = y(x)$ attains its maximum value at h_0, then $y = y_0$ where

$$y_0 := H\left(1 + \tfrac{1}{2}\mathrm{F}^2 - \tfrac{3}{2}\mathrm{F}^{2/3}\right).$$

This puts a bound on the height of the bed undulation that is compatible with the upstream conditions. In Fig 16 we plot the maximum permissible height y_0 the undulation is allowed to attain as a function of the Froude number F. Note that two different values of the Froude number F give the same maximum permissible undulation height y_0, one of which is slower and one of which is faster (compared with $\sqrt{gH}$).

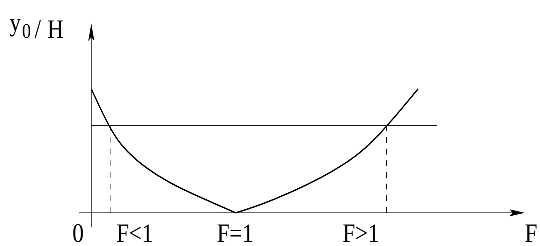

Fig. 16 *Channel flow problem*: Two different values of the Froude number F give the same maximum permissible undulation height y_0. Note we actually plot the normalized maximum possible height y_0/H on the ordinate axis.

Let us now consider and actual given undulation $y = y(x)$. Suppose that it attains an *actual* maximum value $y_{\max}$. There are three cases to consider, in turn we shall consider $y_{\max} < y_0$, the more interesting case, and then $y_{\max} > y_0$. The third case $y_{\max} = y_0$ is an exercise (see the Exercises section at the end of these notes).

In the first case, $y_{\max} < y_0$, as x varies from $x = -\infty$ to $x = +\infty$, the undulation height $y = y(x)$ varies but is such that $y(x) \leqslant y_{\max}$. Refer to Fig. 15, which plots the constraint relationship between y and h resulting from Bernoulli's theorem. Since $y(x) \leqslant y_{\max}$ as x varies from $-\infty$ to $+\infty$, the values of (h, y) are restricted to part of the branches of the graph either side of the global maximum (h_0, y_0). In the figure these parts of the branches are the locale of the shaded sections shown. Note that the derivative $\mathrm{d}y/\mathrm{d}h = 1/(\mathrm{d}h/\mathrm{d}y)$ has the same fixed (and opposite) sign in each of the branches. In the branch for which h is small, $\mathrm{d}y/\mathrm{d}h > 0$, while the branch for which h is larger, $\mathrm{d}y/\mathrm{d}h < 0$. Indeed note the by differentiating the constraint condition, we have

$$\frac{\mathrm{d}y}{\mathrm{d}h} = -\left(1 - \frac{(UH)^2}{gh^3}\right).$$

Using the incompressibility condition to substitute for UH we see that this is equivalent to

$$\frac{\mathrm{d}y}{\mathrm{d}h} = -\left(1 - \frac{u^2}{gh}\right).$$

We can think of $u/\sqrt{gh}$ as a local Froude number if we like. In any case, note that since we are in one branch or the other, and in either case the sign of $\mathrm{d}y/\mathrm{d}h$ is fixed, this means that using the expression for $\mathrm{d}y/\mathrm{d}h$ we just derived, for any flow realization the sign of $1 - u^2/gh$ is also fixed. When $x = -\infty$ this quantity has the value $1 - U^2/gH$. Hence the sign of $1 - U^2/gH$ determines the sign of $1 - u^2/gh$. Hence if $\mathrm{F} < 1$ then $U^2/gH = \mathrm{F}^2 < 1$ and therefore for all x we must have $u^2/gh < 1$. And we also deduce in this case that we must be on the branch for which h is relatively large as $\mathrm{d}y/\mathrm{d}h$ is negative. The flow is said to be *subcritical* throughout and indeed we see that

$$\frac{\mathrm{d}h}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{-1} = -\left(1 - \frac{u^2}{gh}\right)^{-1} < -1 \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}y}(h + y) < 0.$$

Hence in this case, as the bed height y increases, the fluid depth h decreases and vice-versa. On the otherhand if $\mathrm{F} > 1$ then $U^2/gH > 1$ and thus $u^2/gh > 1$. We must be on the branch for which h is relatively small as $\mathrm{d}y/\mathrm{d}h$ is positive. The flow is said to be *supercritical* throughout and we have

$$\frac{\mathrm{d}h}{\mathrm{d}y} = -\left(1 - \frac{u^2}{gh}\right)^{-1} > 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}y}(h + y) > 1.$$

Hence in this case, as the bed height y increases, the fluid depth h increases and vice-versa. Both cases, $\mathrm{F} < 1$ and $\mathrm{F} > 1$, are illustrated by a typical scenario in Fig. 17.

In the second case, $y_{\max} > y_0$, the undulation height is larger than the maximum permissibe height y_0 compatible with the upstream conditions. Under the conditions we assumed, there is no flow realized here. In a real situation we may imagine a flow impinging on a large barrier with height $y_{\max} > y_0$, and the result would be some sort of *reflection* of the flow occurs to change the upstream conditions in an attempt to make them compatible with the *obstacle*. (Our steady flow assumption obviously breaks down here.)

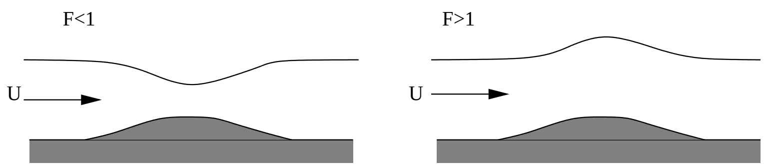

Fig. 17 *Channel flow problem*: for the case $y_{\max} < y_0$, when $\mathrm{F} < 1$, as the bed height y increases, the fluid depth h decreases and vice-versa. Hence we see a depression in the fluid surface above a bump in the bed. On the other hand, when $\mathrm{F} > 1$, as the bed height y increases, the fluid depth h increases and vice-versa. Hence we see an elevation in the fluid surface above a bump in the bed.

10 Irrotational/potential flow

Many flows have extensive regions where the vorticity is zero; some have zero vorticity everywhere. We would call these, respectively, *irrotational* regions of the flow and irrotational flows. In such regions

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \mathbf{0}.$$

Hence the field $\boldsymbol{u}$ is *solenoidal* and there exists a scalar function ϕ such that

$$\boldsymbol{u} = \nabla\phi.$$

The function ϕ is known as the *flow potential*. In turn this implies that

$$K = \oint_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = 0$$

for all simple closed curves $\mathcal{C}$ in the region (the reverse implication is also true).

If the fluid is also incompressible, then ϕ is *harmonic* since $\nabla \cdot \boldsymbol{u} = 0$ implies

$$\nabla^2\phi = 0.$$

Hence for such situations, we in essense need to solve Laplace's equation $\Delta\phi = 0$ subject to certain boundary conditions. For example for an ideal flow, $\boldsymbol{u}\cdot\boldsymbol{n} = \nabla\phi\cdot\boldsymbol{n} = \partial\phi/\partial n$ is given on the boundary, and this would consitute a Neumann problem for Laplace's equation.

Example (linear two-dimensional flow) Consider the flow field $\boldsymbol{u} = (kx, -ky)$ where k is a constant. It is irrotational. Hence there exists a flow potential $\phi = \frac{1}{2}k(x^2 - y^2)$. Since $\nabla \cdot \boldsymbol{u} = 0$ as well, we have $\Delta\phi = 0$. Further, since this flow is two-dimensional, there also exists a streamfunction $\psi = kxy$.

Example (line vortex) Consider the flow field $(u_r, u_\theta, u_z) = (0, k/r, 0)$ where $k > 0$ is a constant. This is the idealization of a thin vortex tube. Direct computation shows that $\nabla \times \boldsymbol{u} = \mathbf{0}$ everywhere except at $r = 0$, where $\nabla \times \boldsymbol{u}$ is infinite. For $r > 0$, there exists a flow potential $\phi = k\theta$. For any closed circuit $\mathcal{C}$ in this region, we have

$$K = \oint_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = 2\pi k\, N$$

where N is the number of times the closed curve $\mathcal{C}$ winds round the origin $r = 0$. The circulation K will be zero for all circuits reducible continuously to a point without breaking the vortex.

Example (D'Alembert's paradox) Consider a uniform flow into which we place an obstacle. We would naturally expect that the obstacle represents an obstruction to the fluid flow and that the flow would exert a force on the obstacle, which if strong enough, might dislodge it and subsequently carry it downstream. However for an ideal flow, as we are just about to prove, this is not the case. There is no net force exerted on an obstacle placed in the midst of a uniform flow.

We thus consider a uniform ideal flow into which is placed a sphere, radius a. The set up is shown in Fig. 18. We assume that the flow around the sphere is steady, incompressible and irrotational. Suppose further that the flow is axisymmetric. By this we mean the following. Use spherical polar coordinates to represent the flow with the south-north pole axis passing through the centre of the sphere and aligned with the uniform flow U at infinity; see Fig. 18. Then the flow is axisymmetric if it is independent of the azimuthal angle φ of the spherical coordinates (r, θ, φ). Further we also assume no swirl so that $u_\varphi = 0$. Since the flow is incompressible and irrotational, it is a

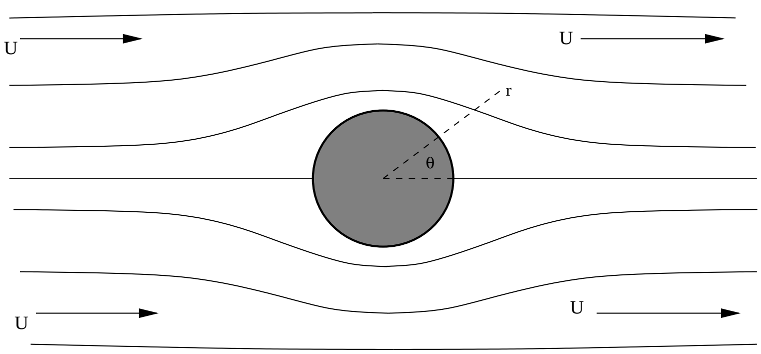

Fig. 18 Consider an ideal steady, incompressible, irrotational and axisymmetric flow past a sphere as shown. The net force exerted on the sphere (obstacle) in the flow is zero. This is D'Alembert's paradox.

potential flow. Hence we seek a potential function ϕ such that $\nabla^2 \phi = 0$. In spherical polar coordinates this is equivalent to

$$\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right)\right) = 0.$$

The general solution to Laplace's equation is well known, and in the case of axisymmetry the general solution is given by

$$\phi(r, \theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta)$$

where P_n are the Legendre polynomials; with $P_1(x) = x$. The coefficients A_n and B_n are constants, most of which, as we shall see presently, are zero. For our problem we have two sets of boundary data. First, that as $r \to \infty$ in any direction, the flow field is uniform and given by $\boldsymbol{u} = (0, 0, U)$ (expressed in Cartesian coordinates with the z-axis aligned along the south-north pole) so that as $r \to \infty$

$$\phi \to Ur \cos\theta.$$

Second, on the sphere $r = a$ itself we have a no normal flow condition

$$\frac{\partial \phi}{\partial r} = 0.$$

Using the first boundary condition for $r \to \infty$ we see that all the A_n must be zero except $A_1 = U$. Using the second boundary condition on $r = a$ we see that all the B_n must be zero except for $B_1 = \frac{1}{2}Ua^3$. Hence the potential for this flow around the sphere is

$$\phi = U(r + a^3/2r^2)\cos\theta.$$

In spherical polar coordinates, the velocity field $\boldsymbol{u} = \nabla\phi$ is given by

$$\boldsymbol{u} = (u_r, u_\theta) = \big(U(1 - a^3/r^3)\cos\theta, -U(1 + a^3/2r^3)\sin\theta\big).$$

Since the flow is ideal and steady as well, Bernoulli's theorem applies and so along a typical streamline $\frac{1}{2}|\boldsymbol{u}|^2 + P/\rho$ is constant. Indeed since the conditions at infinity are uniform so that the pressure P_∞ and velocity field U are the same everywhere there, this means that for any streamline and in fact everywhere for $r \geqslant a$ we have

$$\frac{1}{2}|\boldsymbol{u}|^2 + P/\rho = \frac{1}{2}U^2 + P_\infty/\rho.$$

Rearranging this equation and using our expression for the velocity field above we have

$$\frac{P - P_\infty}{\rho} = \tfrac{1}{2}U^2\big(1 - (1 - a^3/r^3)^2\cos^2\theta - (1 + a^3/2r^3)^2\sin^2\theta\big).$$

On the sphere $r = a$ we see that

$$\frac{P - P_\infty}{\rho} = \tfrac{1}{2}U^2\big(1 - \tfrac{9}{4}\sin^2\theta\big).$$

Note that on the sphere, the pressure is symmetric about $\theta = 0, \pi/2, \pi, 3\pi/2$. Hence the fluid exerts no net force on the sphere! (There is no drag or lift.) This result, in principle, applies to any shape of obstacle in such a flow. In reality of course this cannot be the case, the presence of viscosity remedies this paradox (and crucially generates vorticity).

11 Dynamical similarity and Reynolds number

Our goal in this section is to demonstrate an important scaling property of the Navier–Stokes equations for a homogeneous incompressible fluid without body force:

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= \nu\, \Delta \boldsymbol{u} - \frac{1}{\rho}\nabla p, \\ \nabla \cdot \boldsymbol{u} &= 0. \end{aligned}$$

Note that two physical properties inherent to the fluid modelled are immediately apparent, the mass density ρ, which is constant throughout the flow, and the kinematic viscosity ν. Suppose we consider such a flow which is characterized by a typical length scale L and velocity U. For example we might imagine a flow past an obstacle such a sphere whose diameter is characterized by L and the impinging/undisturbed far-field flow is uniform and given by U. These two scales naturally determine a typically time scale $T = L/U$. Using these scales we can introduce the dimensionless variables

$$\boldsymbol{x}' = \frac{\boldsymbol{x}}{L}, \qquad \boldsymbol{u}' = \frac{\boldsymbol{u}}{U} \qquad \text{and} \qquad t' = \frac{t}{T}.$$

Directly substituting for $\boldsymbol{u} = U\boldsymbol{u}'$ and using the chain rule to replace t by t' and $\boldsymbol{x}$ by $\boldsymbol{x}'$ in the Navier–Stokes equations, we obtain:

$$\frac{U}{T}\frac{\partial \boldsymbol{u}'}{\partial t'} + \frac{U^2}{L}\boldsymbol{u}' \cdot \nabla_{\boldsymbol{x}'}\boldsymbol{u}' = \frac{\nu U}{L^2}\,\Delta_{\boldsymbol{x}'}\boldsymbol{u}' - \frac{1}{\rho L}\nabla_{\boldsymbol{x}'}p.$$

The incompressibility condition becomes $\nabla_{\boldsymbol{x}'} \cdot \boldsymbol{u}' = 0$. Using that $T = L/U$ and dividing through by U^2/L we get

$$\frac{\partial \boldsymbol{u}'}{\partial t'} + \boldsymbol{u}' \cdot \nabla_{\boldsymbol{x}'}\boldsymbol{u}' = \frac{\nu}{UL}\,\Delta_{\boldsymbol{x}'}\boldsymbol{u}' - \frac{1}{\rho U^2}\nabla_{\boldsymbol{x}'}p.$$

If we set $p' = p/\rho U^2$ and then drop the primes, we get

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{1}{\mathrm{Re}}\,\Delta \boldsymbol{u} - \nabla p,$$

which is the representation for the Navier–Stokes equations in dimensionless variables. The dimensionless number

$$\mathrm{Re} := \frac{UL}{\nu}$$

is the *Reynolds number*. Its practical significance is as follows. Suppose we want to design a jet plane (or perhaps just a wing). It might have a characteristic scale L_1 and typically cruise at speeds U_1 with surrounding air having viscosity ν_1. Rather than build the plane to test its airflow properties it would be cheaper to build a scale model of the aircraft—with exactly the same shape/geometry but smaller, with characteristic scale L_2. Then we could test the airflow properties in a wind tunnel for example, by using a driving impinging wind of characteristic velocity U_2 and air of viscosity ν_2 so that

$$\frac{U_1 L_1}{\nu_1} = \frac{U_2 L_2}{\nu_2}.$$

The Reynolds number in the two scenarios are the same and the dimensionless Navier–Stokes equations for the two flows identical. Hence the shape of the flows in the two scenarios will be the same. We could also for example, replace the wind tunnel by a water tunnel: the viscosity of air is $\nu_1 = 0.15\,\mathrm{cm}^2/\mathrm{s}$ and of water $\nu_2 = 0.0114\,\mathrm{cm}^2/\mathrm{s}$, i.e. $\nu_1/\nu_2 \approx 13$. Hence for the same geometry and characteristic scale $L_1 = L_2$, if we choose $U_1 = 13\,U_2$, the Reynolds numbers for the two flows will be the same. Such flows, with the same geometry and the same Reynolds number are said to be *similar*.

Remark 6 Some typical Reynolds are as follows: aircraft: 10^8 to 10^9; cricket ball: 10^5; blue whale: 10^8; cruise ship: 10^9; canine artery: 10^3; nematode: 0.6; capilliaries: 10^{-3}.

12 Exercises

Exercise (streamlines: Cartesian coordinates) Sketch streamlines for the following steady flow fields: (a) $(u,v) = (U + \alpha y, 0)$; (b) $(u,v) = (kx, -ky)$; (c) $(u,v) = (-\Omega y, \Omega x)$; (d) $(u,v,w) = (\alpha x, \alpha y, -2\alpha z)$; where U, α, k, and Ω are all constants.

Exercise (streamlines: plane/cylindrical polar coordinates) Sketch streamlines for the steady flow fields: (a) $(u_r, u_\theta) = (0, \Omega r)$; (b) $(u_r, u_\theta) = (0, k/r)$; (c) $(u_r, u_\theta, u_z) = \alpha(t)\cdot(x-y, x+y, 0)$—show that the streamlines are exponential spirals (*Hint:* convert to polar coordinates first). Here Ω and k are constants, and $\alpha = \alpha(t)$ is an arbitrary function of t. We use (r,θ) as plane polar coordinates, and (r,θ,z) as cylindrical polar coordinates. Note that in these coordinates the equations for trajectories are

$$\frac{\mathrm{d}r}{\mathrm{d}t} = u_r, \qquad r\frac{\mathrm{d}\theta}{\mathrm{d}t} = u_\theta, \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = u_z.$$

Exercise (trajectories and streamlines: two dimensions) For a given velocity flow field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ prescribed at position $\boldsymbol{x}$ and time t, the particle trajectories are given by the solutions to the system of ordinary differential equations

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}(t), t).$$

Streamlines are given by the solutions to the system of ordinary differential equations (where t is fixed)

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{u}\big(\boldsymbol{x}(s), t\big).$$

(a) Explain what particle trajectories and streamlines are, and their difference.
(b) For the two-dimensional flow in Cartesian coordinates,

$$(u,v) = \big(u_0, v_0\cos(kx - \alpha t)\big),$$

where u_0, v_0, k and α are constants, find the general equation for a streamline. Show that the streamline passing through $(x,y) = (0,0)$ at $t = 0$ is

$$y = \frac{v_0}{ku_0}\sin(kx).$$

Find the equation for the path of the particle which is at $(x,y) = (0,0)$ at $t = 0$. Comment briefly on the contrast between the above streamline and particle path in the two separate limiting cases: first $\alpha \to 0$; second $k \to 0$.

Exercise (trajectories and streamlines: three dimensions) Find the trajectories and streamlines when $\boldsymbol{u} = (x\mathrm{e}^{2t-z}, y\mathrm{e}^{2t-z}, 2\mathrm{e}^{2t-z})$. What is the track of the particle passing through $(1,1,0)$ at time $t = 0$?

Exercise (channel flow) Consider the two-dimensional channel flow (with U a given constant)

$$\boldsymbol{u} = \left(0, U\left(1 - \frac{x^2}{a^2}\right), 0\right),$$

between the two walls $x = \pm a$. Show that there is a *stream function* and find it. (*Hint:* a stream function ψ exists for a velocity field $\boldsymbol{u} = (u,v,w)$ when $\nabla\cdot\boldsymbol{u} = 0$ and we have an additional symmetry. Here the additional symmetry is uniformity with respect to z. You thus need to verify that if

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x},$$

then $\nabla \cdot \boldsymbol{u} = 0$ and then solve this system of equations to find ψ.)

Show that approximately 91% of the volume flux across $y = y_0$ for some constant y_0 flows through the central part of the channel $|x| \leqslant \frac{3}{4}a$.

Exercise (flow inside and around a disc) Calculate the stream function ψ for the flow field

$$\boldsymbol{u} = \big(U\cos\theta \cdot (1 - a^2/r^2), -U\sin\theta \cdot (1 + a^2/r^2) - \Gamma/2\pi r\big)$$

in plane polar coordinates, where U, a, γ are constants.

Exercise (steady oscillating channel flow) An incompressible fluid is in steady two-dimensional flow in the channel

$$-\infty < x < \infty, \quad -\pi/2 < y < \pi/2,$$

with velocity

$$\boldsymbol{u} = (1 + x\sin y, \cos y).$$

Find the equation of the streamlines and sketch them. Show that the flow has stagnation points at $(1, -\pi/2)$ and $(-1, \pi/2)$.

Exercise (Flow in an infinite pipe: Poiseuille flow) Consider an infinite pipe with circular cross-section of radius a, whose centre line is aligned along the z-axis. Assume *no-slip* boundary conditions at $r = a$, for all z, i.e. on the inside surface of the cylinder. Using cylindrical polar coordinates, look for a solution to the fluid flow in the pipe of the following form. Assume there is no radial flow, $u_r = 0$, and no swirl, $u_\theta = 0$. Further assume there is a constant pressure gradient down the pipe, i.e. that $p = -Cz$ for some constant C. Lastly, suppose that the flow down the pipe, i.e. the velocity component u_z, has the form $u_z = u_z(r)$ (it is a function of r only).

(a) Using the Navier–Stokes equations, show that

$$C = -\rho\nu \Delta u_z = -\rho\nu \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right).$$

(b) Integrating the equation above yields

$$u_z = -\frac{C\rho}{4\nu}r^2 + A\log r + B,$$

where A and B are constants. We naturally require that the solution be bounded. Explain why this implies $A = 0$. Now use the no-slip boundary condition to determine B. Hence show that

$$u_z = \frac{C\rho}{4\nu}(a^2 - r^2).$$

(c) Show that the *mass-flow rate* across any cross section of the pipe is given by

$$\int \rho u_z \, \mathrm{d}S = \rho^2 \pi C a^4 / 8\nu.$$

This is known as the *fourth power law.*

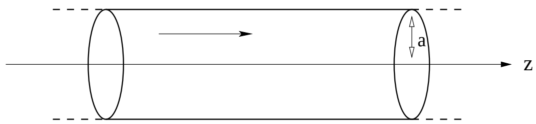

Fig. 19 Poiseuille flow: a viscous fluid flows along an infinite horizontal pipe with circular cross-section of radius a, whose centre line is aligned along the z-axis. A constant pressure gradient is assumed, as well as axisymmetry, no radial flow and no swirl.

(From Chorin and Marsden, pp. 45-6.)

Exercise (Couette flow) Let Ω be the region between two concentric cylinders of radii R_1 and R_2, where $R_1 < R_2$. Suppose the velocity field in cylindrical coordinates $\boldsymbol{u} = (u_r, u_\theta, u_z)$ of the fluid flow inside Ω, is given by

$$u_r = 0, \qquad u_z = 0, \qquad \text{and} \qquad u_\theta = \frac{A}{r} + Br,$$

where

$$A = -\frac{R_1^2 R_2^2(\omega_2 - \omega_1)}{R_2^2 - R_1^2} \qquad \text{and} \qquad B = -\frac{R_1^2\omega_1 - R_2^2\omega_2}{R_2^2 - R_1^2}.$$

This is known as a *Couette flow*—see Fig. 20. Show that the:

(a) velocity field $\boldsymbol{u} = (u_r, u_\theta, u_z)$ is a stationary solution of Euler's equations of motion for an ideal fluid with density $\rho \equiv 1$ (*hint:* you need to find a pressure field p that is consistent with the velocity field given);
(b) angular velocity of the flow (i.e. the quantity u_θ/r) is ω_1 on the cylinder $r = R_1$ and ω_2 on the cylinder $r = R_2$.
(c) the vorticity field $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = (0, 0, 2B)$.

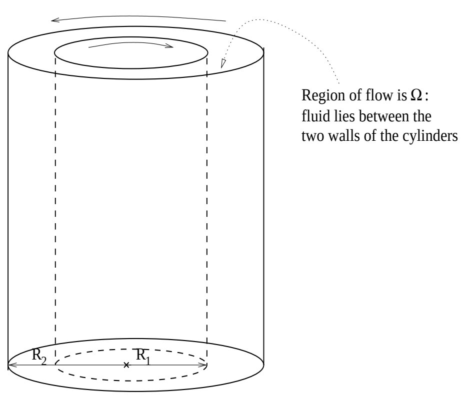

Fig. 20 Couette flow between two concentric cylinders of radii $R_1 < R_2$.

Exercise (hurricane) We devise a simple model for a hurricane.

(a) Using the Euler equations for an ideal incompressible flow in cylindrical coordinates (see the bath or sink drain problem in the main text) show that at position (r, θ, z), for a flow which is independent of θ with $u_r = u_z = 0$, we have

$$\frac{u_\theta^2}{r} = \frac{1}{\rho_0}\frac{\partial p}{\partial r},$$
$$0 = \frac{1}{\rho_0}\frac{\partial p}{\partial z} + g,$$

where $p = p(r, z)$ is the pressure and g is the acceleration due to gravity (assume this to be the body force per unit mass). Verify that any such flow is indeed incompressible.

(b) In a *simple* model for a hurricane the air is taken to have uniform constant density ρ_0 and each fluid particle traverses a horizontal circle whose centre is on the fixed vertical z-axis. The (angular) speed u_θ at a distance r from the axis is

$$u_\theta = \begin{cases} \Omega r, & \text{for } 0 \leqslant r \leqslant a, \\ \Omega \frac{a^{3/2}}{r^{1/2}}, & \text{for } r > a, \end{cases}$$

where Ω and a are known constants.

(i) Now consider the flow given above in the inner region $0 \leqslant r \leqslant a$. Using the equations in part (a) above, show that the pressure in this region is given by

$$p = P_0 + \tfrac{1}{2}\rho_0 \Omega^2 r^2 - g\rho_0 z,$$

where P_0 is a constant. A free surface of the fluid is one for which the pressure is constant. Show that the shape of a free surface for $0 \leqslant r \leqslant a$ is a paraboloid of revolution, i.e. it has the form

$$z = Ar^2 + B,$$

for some constants A and B. Specify the exact form of A and B.

(ii) Now consider the flow given above in the outer region $r > a$. Again using the equations in part (a) above, and that the pressure must be continuous at $r = a$, show that the pressure in this region is given by

$$p = P_0 - \frac{\rho_0}{r}\Omega^2 a^3 - g\rho_0 z + \tfrac{3}{2}\rho_0 \Omega^2 a^2,$$

where P_0 is the same constant (reference pressure) as that in part (i) above.

Exercise (Elliptic pipe flow) Show that for a flow of a viscous fluid through a pipe of constant cross-section under pressure gradient $-G$, the speed u satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{G}{\mu},$$

where x and y are coordinates in a plane of cross-section. State the boundary conditions for this *elliptic partial differential problem.*

Show for an elliptic pipe of semi-axes a, b, that $u = A + Bx^2 + Cy^2$ for suitable choices of A, B and C. Verify that the flux of fluid through the pipe is $\pi a^3 b^3 G/4(a^2 + b^2)\mu$. Deduce that for a given elliptical cross-sectional area, the optimal choice of a and b to maximize the discharge rate is $a = b$.

Exercise (Wind blowing across a lake) Wind blowing across the surface of a lake of uniform depth d exerts a constant and uniform tangential stress S. The water is initially at rest. Find the water velocity at the surface as a function of time for $\nu t \ll d^2$. (*Hint:* solve for the vorticity using the vorticity equation for a very deep lake.)

Suppose now that the wind has been blowing for a sufficiently long time to establish a steady state. Assuming that the water velocity can be taken to be almost uni-directional and that the horizontal dimensions of the lake are large compared with d, show that the water velocity at the surface is $Sd/4\mu$. (*Hint:* A pressure gradient would be needed to ensure no net flux (why?) in the steady state, and this pressure gradient leads to a small rise in the surface elevation of the lake in the direction of the wind.)

Exercise (Venturi tube) Consider the Venturi tube shown in Fig. 21. Assume that the ideal fluid flow through the construction is homogeneous, incompressible and steady. The flow in the wider section of cross-sectional area A_1, has velocity u_1 and pressure p_1, while that in the narrower section of cross-sectional area A_2, has velocity u_2 and pressure p_2. Separately within the uniform wide and narrow sections, we assume the velocity and pressure are uniform themselves.

(a) Why does the relation $A_1 u_1 = A_2 u_2$ hold? Why is the flow faster in the narrower region of the tube compared to the wider region of the tube?
(b) Use Bernoulli's theorem to show that

$$\tfrac{1}{2}u_1^2 + \frac{p_1}{\rho_0} = \tfrac{1}{2}u_2^2 + \frac{p_2}{\rho_0},$$

where ρ_0 is the constant uniform density of the fluid.
(c) Using the results in parts (a) and (b), compare the pressure in the narrow and wide regions of the tube.
(d) Give a practical application where the principles of the Venturi tube is used or might be useful.

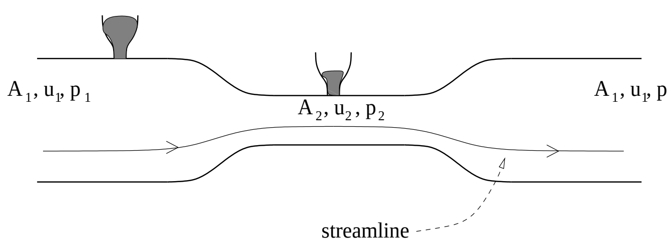

Fig. 21 *Venturi tube*: the flow in the wider section of cross-sectional area A_1 has velocity u_1 and pressure p_1, while that in the narrower section of cross-sectional area A_2 has velocity u_2 and pressure p_2. Separately within the uniform wide and narrow sections, we assume the velocity and pressure are uniform themselves.

Exercise (Clepsydra or water clock) A clepsydra has the form of a surface of revolution containing water and the level of the free surface of the water falls at a *constant* rate, as the water flows out through a small hole in the base. The basic setup is shown in Fig. 22.

(a) Apply Bernoulli's theorem to one of the typical streamlines shown in Fig. 22 to show that

$$\frac{1}{2}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = \tfrac{1}{2}U^2 - gz$$

where z is the height of the free surface above the small hole in the base, U is the velocity of the water coming out of the small hole and g is the acceleration due to gravity.

(b) Assuming that the constant rate at which the level surface is falling is very slow, explain why we can deduce that

$$U \approx \sqrt{2gz}.$$

(c) If S is the cross-sectional area of the hole in the bottom, and A is the cross-sectional area of the free surface, explain why we must have

$$A\,\frac{\mathrm{d}z}{\mathrm{d}t} = S\,U.$$

(d) Combine the results from (b) and (c) above, to show that the shape of the container that guarantees that the free surface of the water drops at a constant rate must have the form $z = C\,r^4$ in cylindrical polars, where C is a constant.

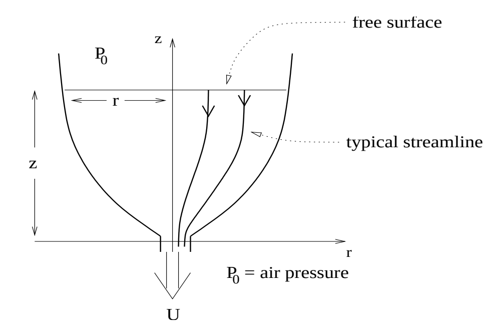

Fig. 22 Clepsydra (water clock).

Exercise (coffee in a mug) A coffee mug in the form of a right circular cylinder (diameter $2a$, height h), closed at one end, is initially filled to a depth $d > \frac{1}{2}h$ with static inviscid coffee. Suppose the coffee is then made to rotate inside the mug—see Fig. 23.

(a) Using the Euler equations for an ideal incompressible homogeneous flow in cylindrical coordinates show that at position (r, θ, z), for a flow which is independent of θ with $u_r = u_z = 0$, the Euler equations reduce to

$$\begin{aligned}\frac{u_\theta^2}{r} &= \frac{1}{\rho}\frac{\partial p}{\partial r},\\ 0 &= \frac{1}{\rho}\frac{\partial p}{\partial z} + g,\end{aligned}$$

where $p = p(r, z)$ is the pressure, ρ is the constant uniform fluid density and g is the acceleration due to gravity (assume this to be the body force per unit mass). Verify that any such flow is indeed incompressible.

(b) Assume that the coffee in the mug is rotating as a solid body with constant angular velocity Ω so that the velocity component u_θ at a distance r from the axis of symmetry for $0 \leqslant r \leqslant a$ is

$$u_\theta = \Omega r.$$

Use the equations in part (a), to show that the pressure in this region is given by

$$p = \tfrac{1}{2}\rho\Omega^2 r^2 - g\rho z + C,$$

where C is an arbitrary constant. At the free surface between the coffee and air, the pressure is constant and equal to the atmospheric pressure P_0. Use this to show that the shape of the free surface has the form

$$z = \frac{\Omega^2}{2g} r^2 + \frac{C - P_0}{\rho g}.$$

(c) Note the we are free to choose $C = P_0$ in the equation of the free surface so that it is described by $z = \Omega^2 r^2/2g$. This is equivalent to choosing the origin of our cylindrical coordinates to be the centre of the dip in the free surface. Suppose that this origin is a distance z_0 from the bottom of the mug.

(i) Explain why the total volume of coffee is $\pi a^2 d$. Then by using incompressibility, explain why the following constraint must be satisfied:

$$\pi a^2 z_0 + \int_0^a \frac{\Omega^2 r^2}{2g} \cdot 2\pi r \, \mathrm{d}r = \pi a^2 d.$$

(ii) By computing the integral in the constraint in part (i), show that some coffee will be spilled out of the mug if $\Omega^2 > 4g(h - d)/a^2$. Explain briefly why this formula does not apply when the mug is initially less than half full.

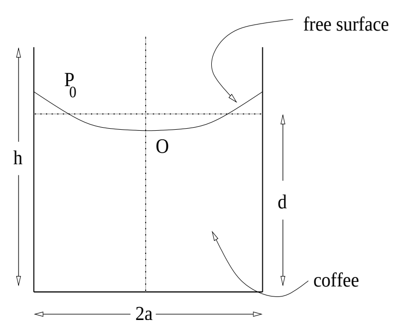

Fig. 23 Coffee mug: we consider a mug of coffee of diameter $2a$ and height h, which is initially filled with coffee to a depth d. The coffee is then made to rotate about the axis of symmetry of the mug. The free surface between the coffee and the air takes up the characteristic shape shown, dipping down towards the middle (axis of symmetry). The goal is to specify the shape of the free surface.

Exercise (Channel flow: Froude number) Recall the scenario of the steady channel flow over a gently undulating bed given in Section 9. Consider the case when the maximum permissible height y_0 compatible with the upstream conditions, and the actual maximum height $y_{\max}$ of the undulation are exactly equal, i.e. $y_{\max} = y_0$. Show that the flow becomes locally critical immediately above $y_{\max}$ and, by a local expansion about that position, show that there are subcritical and supercritical flows downstream consistent with the continuity and Bernoulli equations (friction in a real flow leads to the latter being preferred).

Exercise (Bernoulli's Theorem for irrotational unsteady flow) Consider Euler's equations of motion for an ideal homogeneous incompressible fluid, with $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ denoting the fluid velocity at position $\boldsymbol{x}$ and time t, ρ the uniform constant density, $p = p(\boldsymbol{x}, t)$ the pressure, and $\boldsymbol{f}$ denoting the body force per unit mass. Suppose that the flow is unsteady, but irrotational, i.e. we know that

$$\nabla \times \boldsymbol{u} = \boldsymbol{0}$$

throughout the flow. This means that we know there exists a scalar potential function $\varphi = \varphi(\boldsymbol{x}, t)$ such that $\boldsymbol{u} = \nabla\varphi$. Also suppose that the body force is conservative so that $\boldsymbol{f} = -\nabla\Phi$ for some potential function $\Phi = \Phi(\boldsymbol{x}, t)$.

(a) Using the identity

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \tfrac{1}{2}\nabla(|\boldsymbol{u}|^2) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}),$$

show from Euler's equations of motion that the Bernoulli quantity

$$H \coloneqq \frac{\partial \varphi}{\partial t} + \tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \Phi$$

satisfies $\nabla H = \boldsymbol{0}$ throughout the flow.

(b) From part (a) above we can deduce that H can only be a function of t throughout the flow, say $H = f(t)$ for some function f. By setting

$$V \coloneqq \varphi - \int_0^t f(\tau)\,\mathrm{d}\tau$$

throughout the flow show that the Bernoulli quantity

$$H \coloneqq \frac{\partial V}{\partial t} + \tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \Phi$$

is constant throughout the flow.

Exercise (rigid body rotation) An ideal fluid of constant uniform density ρ_0 is contained within a fixed right-circular cylinder (with symmetry axis the z-axis). The fluid moves under the influence of a body force field $\boldsymbol{f} = (\alpha x + \beta y, \gamma x + \delta y, 0)$ per unit mass, where α, β, γ and δ are independent of the space coordinates. Use Euler's equations of motion to show that a rigid body rotation of the fluid about the z-axis, with angular velocity $\omega(t)$ given by $\dot{\omega} = \frac{1}{2}(\gamma - \beta)$ is a possible solution of the equation and boundary conditions. Show that the pressure is given by

$$p = p_0 + \frac{1}{2}\rho_0\big((\omega^2 + \alpha)x^2 + (\beta + \gamma)xy + (\omega^2 + \delta)y^2\big),$$

where p_0 is the pressure at the origin.

Exercise (vorticity and streamlines) An inviscid incompressible fluid of uniform density ρ is in steady two-dimensional horizontal motion. Show that the Euler equations are equivalent to

$$\frac{\partial H}{\partial x} = v\omega \qquad \text{and} \qquad \frac{\partial H}{\partial y} = -u\omega,$$

where $H = p/\rho + \frac{1}{2}(u^2 + v^2)$, where p is the dynamical pressure, (u, v) is the velocity field and ω is the vorticity. Deduce that ω is constant along streamlines and that this is in accord with Kelvin's theorem.

Exercise (vorticity, streamlines with gravity) An incompressible inviscid fluid, under the influence of gravity, has the velocity field $\boldsymbol{u} = (2\alpha y, -\alpha x, 0)$ with the z-axis vertically upwards; and α is constant. Also the density ρ is constant. Verify that $\boldsymbol{u}$ satisfies the governing equations and find the pressure p. Show that the Bernoulli function $H = p/\rho + \frac{1}{2}|\boldsymbol{u}|^2 + \Phi$ is constant on streamlines and vortex lines, where Φ is the gravitational potential.

13 Notes

13.1 Streaklines

A *streakline* is the locus of all the fluid elements which at some time have past through a particular point, say (x_0, y_0, z_0). We can obtain the equation for a streakline through (x_0, y_0, z_0) by solving the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t),$$

assuming that at $t = t_0$ we have $\big(x(t_0), y(t_0), z(t_0)\big) = (x_0, y_0, z_0)$. Eliminating t_0 between the equations generates the streakline corresponding to (x_0, y_0, z_0). For example, ink dye injected at the point (x_0, y_0, z_0) in the flow will trace out a streakline.

13.2 Ideal fluid flow and conservation of energy

We show that an ideal fluid that conserves energy is necessarily incompressible. We have derived two conservation laws thusfar, first, conservation of mass,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0,$$

and second, balance of momentum,

$$\rho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \rho\,\boldsymbol{f}.$$

If we are in three dimensional space so $d = 3$, we have four equations, but five unknowns—namely $\boldsymbol{u}$, p and ρ. We cannot specify the fluid motion completely without specifying one more condition.

Definition 6 (Kinetic energy) The *kinetic energy* of the fluid in the region $\Omega \subseteq \mathcal{D}$ is

$$E := \tfrac{1}{2}\int_{\Omega} \rho\,|\boldsymbol{u}|^2\,\mathrm{d}V.$$

The rate of change of the kinetic energy, using the transport theorem, is given by

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\int_{\Omega_t}\rho\,|\boldsymbol{u}|^2\,\mathrm{d}V\right)\\ &= \tfrac{1}{2}\int_{\Omega_t}\rho\frac{\mathrm{D}|\boldsymbol{u}|^2}{\mathrm{D}t}\,\mathrm{d}V\\ &= \tfrac{1}{2}\int_{\Omega_t}\rho\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{u}\cdot\boldsymbol{u})\,\mathrm{d}V\\ &= \int_{\Omega_t}\rho\boldsymbol{u}\cdot\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\,\mathrm{d}V\\ &= \int_{\Omega_t}\boldsymbol{u}\cdot\left(\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)\mathrm{d}V.\end{aligned}$$

Here we assume that *all* the energy is kinetic. The principal of conservation of energy states (from Chorin and Marsden, page 13):

> *the rate of change of kinetic energy in a portion of fluid equals the rate at which the pressure and body forces do work.*

In other words we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int_{\partial\Omega_t} p\,\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S + \int_{\Omega_t}\rho\,\boldsymbol{u}\cdot\boldsymbol{f}\,\mathrm{d}V.$$

We compare this with our expression above for the rate of change of the kinetic energy. Equating the two expressions, using Euler's equation of motion, and noticing that the body force term immediately cancels, we get

$$\begin{aligned}&& \int_{\partial\Omega_t} p\,\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S &= \int_{\Omega_t}\boldsymbol{u}\cdot\nabla p\,\mathrm{d}V\\ \Leftrightarrow && \int_{\Omega_t}\nabla\cdot(\boldsymbol{u}p)\,\mathrm{d}V &= \int_{\Omega_t}\boldsymbol{u}\cdot\nabla p\,\mathrm{d}V\\ \Leftrightarrow && \int_{\Omega_t}\boldsymbol{u}\cdot\nabla p + (\nabla\cdot\boldsymbol{u})\,p\,\mathrm{d}V &= \int_{\Omega_t}\boldsymbol{u}\cdot\nabla p\,\mathrm{d}V\\ \Leftrightarrow && \int_{\Omega_t}(\nabla\cdot\boldsymbol{u})\,p\,\mathrm{d}V &= 0.\end{aligned}$$

Since Ω and therefore Ω_t is arbitrary we see that the assumption that all the energy is kinetic implies

$$\nabla\cdot\boldsymbol{u} = 0.$$

Hence our third conservation law, conservation of energy has lead to the *equation of state*, $\nabla\cdot\boldsymbol{u} = 0$, i.e. that an ideal fluid is *incompressible*.

Hence the *Euler equations* for a homogeneous incompressible flow in $\mathcal{D}$ are

$$\begin{aligned}\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} &= -\frac{1}{\rho}\nabla p + \boldsymbol{f},\\ \nabla\cdot\boldsymbol{u} &= 0,\end{aligned}$$

together with the boundary condition on $\partial\mathcal{D}$ which is $\boldsymbol{u}\cdot\boldsymbol{n} = 0$. Note, as we did for the Navier–Stokes equations, since ρ is constant, it is convenient to re-label p/ρ to be p, thus removing ρ from the equations above completely.

13.3 Isentropic flows

A compressible flow is *isentropic* if there is a function π, called the *enthalpy*, such that

$$\nabla \pi = \frac{1}{\rho} \nabla p.$$

The Euler equations for an isentropic flow are thus

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= -\nabla \pi + \boldsymbol{f} \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) &= 0, \end{aligned}$$

in $\mathcal{D}$, and on $\partial\mathcal{D}$, $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ (or matching normal velocities if the boundary is moving).

For compressible ideal gas flow, the pressure is often proportional to ρ^γ, for some constant $\gamma \geqslant 1$, i.e.

$$p = C\,\rho^\gamma,$$

for some constant C. This is a special case of an isentropic flow, and is an example of an *equation of state*. In fact we can actually compute

$$\pi = \int^\rho \frac{p'(z)}{z}\,\mathrm{d}z = \frac{\gamma\,C\,\rho^\gamma}{\gamma - 1},$$

and the *internal energy* (see Chorin and Marsden, pages 14 and 15)

$$\epsilon = \pi - (p/\rho) = \frac{C\,\rho^\gamma}{\gamma - 1}.$$

A Multivariable calculus identities

We provide here some useful multivariable calculus identities. Here ϕ and ψ are generic scalars, and $\boldsymbol{u}$ and $\boldsymbol{v}$ are generic vectors.

1. $\nabla \times \boldsymbol{u} = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}.$
2. $\nabla \cdot (\nabla \phi) = \nabla^2 \phi = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2}.$
3. $\nabla \times (\nabla \phi) \equiv \boldsymbol{0}.$
4. $\nabla \cdot (\nabla \times \boldsymbol{u}) \equiv 0.$
5. $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}.$
6. $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi.$
7. $\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u} \times (\nabla \times \boldsymbol{v}) + \boldsymbol{v} \times (\nabla \times \boldsymbol{u}).$
8. $\nabla \cdot (\phi \boldsymbol{u}) = \phi(\nabla \cdot \boldsymbol{u}) + \boldsymbol{u} \cdot \nabla\phi.$
9. $\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}).$
10. $\nabla \times (\phi \boldsymbol{u}) = \phi \nabla \times \boldsymbol{u} + \nabla\phi \times \boldsymbol{u}.$
11. $\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u}\,(\nabla \cdot \boldsymbol{v}) - \boldsymbol{v}\,(\nabla \cdot \boldsymbol{u}) + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}.$

B Navier–Stokes equations in cylindrical polar coordinates

The incompressible Navier–Stokes equations in *cylindrical polar coordinates* (r, θ, z) with the velocity field $\boldsymbol{u} = (u_r, u_\theta, u_z)$ are

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + (\boldsymbol{u} \cdot \nabla) u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) + f_r, \\
\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u} \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\Delta u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right) + f_\theta, \\
\frac{\partial u_z}{\partial t} + (\boldsymbol{u} \cdot \nabla) u_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \Delta u_z + f_z,
\end{aligned}$$

where $p = p(r, \theta, z, t)$ is the pressure, ρ is the mass density and $\boldsymbol{f} = (f_r, f_\theta, f_z)$ is the body force per unit mass. Here we also have

$$\boldsymbol{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$$

and

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Further the gradient operator and the divergence of a vector field $\boldsymbol{u}$ are given in cylindrical coordinates, respectively, by

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right)$$

and

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}.$$

Lastly in cylindrical coordinates $\nabla \times \boldsymbol{u}$ is given by

$$\nabla \times \boldsymbol{u} = \begin{pmatrix} \omega_r \\ \omega_\theta \\ \omega_z \end{pmatrix} = \begin{pmatrix} \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \end{pmatrix}.$$

C Navier–Stokes equations in spherical polar coordinates

The incompressible Navier–Stokes equations in *spherical polar coordinates* (r, θ, φ) with the velocity field $\boldsymbol{u} = (u_r, u_\theta, u_\varphi)$ are (note θ is the angle to the south-north pole axis and φ is the azimuthal angle)

$$\begin{aligned}
\frac{\partial u_r}{\partial t} &+ (\boldsymbol{u} \cdot \nabla) u_r - \frac{u_\theta^2}{r} - \frac{u_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \\
&+ \nu\left(\Delta u_r - 2\frac{u_r}{r^2} - \frac{2}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(u_\theta \sin\theta) - \frac{2}{r^2 \sin\theta}\frac{\partial u_\varphi}{\partial \varphi}\right) + f_r, \\
\frac{\partial u_\theta}{\partial t} &+ (\boldsymbol{u} \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2 \cos\theta}{r \sin\theta} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \\
&+ \nu\left(\Delta u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2\theta} - 2\frac{\cos\theta}{r^2 \sin^2\theta}\frac{\partial u_\varphi}{\partial \varphi}\right) + f_\theta, \\
\frac{\partial u_\varphi}{\partial t} &+ (\boldsymbol{u} \cdot \nabla) u_\varphi + \frac{u_r u_\varphi}{r} + \frac{u_\theta u_\varphi \cos\theta}{r \sin\theta} = -\frac{1}{\rho r \sin\theta}\frac{\partial p}{\partial \varphi} \\
&+ \nu\left(\Delta u_\varphi + \frac{2}{r^2 \sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{r^2 \sin^2\theta}\right) + f_z,
\end{aligned}$$

where $p = p(r, \theta, \varphi, t)$ is the pressure, ρ is the mass density and $\boldsymbol{f} = (f_r, f_\theta, f_\varphi)$ is the body force per unit mass. Here we also have

$$\boldsymbol{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + \frac{u_\varphi}{r \sin\theta} \frac{\partial}{\partial \varphi}$$

and

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2}.$$

Further the gradient operator and the divergence of a vector field $\boldsymbol{u}$ are given in spherical coordinates, respectively, by

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \right)$$

and

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta u_\theta) + \frac{1}{r \sin\theta} \frac{\partial u_\varphi}{\partial \varphi}.$$

Lastly in spherical coordinates $\nabla \times \boldsymbol{u}$ is given by

$$\nabla \times \boldsymbol{u} = \begin{pmatrix} \omega_r \\ \omega_\theta \\ \omega_\varphi \end{pmatrix} = \begin{pmatrix} \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta u_\varphi) - \frac{\partial u_\theta}{\partial \varphi} \\ \frac{1}{r \sin\theta} \frac{\partial u_r}{\partial \varphi} - \frac{1}{r} \frac{\partial}{\partial r} (r u_\varphi) \\ \frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \end{pmatrix}.$$

Acknowledgements These lecture notes have to a large extent grown out of a merging of, lectures on Ideal Fluid Mechanics given by Dr. Frank Berkshire [2] in the Spring of 1989, lectures on Viscous Fluid Mechanics given by Prof. Trevor Stuart [14] in the Autumn of 1989 (both at Imperial College) and the style and content of the excellent text by Chorin and Marsden [3]. They have also benefitted from lecture notes by Prof. Frank Leppington [9] on Electromagnetism. Lastly, SJAM would also like to thank Prof. Andrew Lacey for his suggestions and input.

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