Differential Equations and Linear Algebra
Exercises

Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS

CHAPTER 1

Linear second order ODEs

Exercises

1.1. (*)[1] The following differential equations represent oscillating springs.

(1) $y'' + 4y = 0, \quad y(0) = 5, \quad y'(0) = 0,$

(2) $4y'' + y = 0, \quad y(0) = 10, \quad y'(0) = 0,$

(3) $y'' + 6y = 0, \quad y(0) = 4, \quad y'(0) = 0,$

(4) $6y'' + y = 0, \quad y(0) = 20, \quad y'(0) = 0.$

Which differential equation represents

(a): the spring oscillating most quickly (with the shortest period)?

(b): the spring oscillating with the largest amplitude?

(c): the spring oscillating most slowly (with the longest period)?

(d): the spring oscillating with the largest maximum velocity?

1.2. (*)[2] (Pendulum.) A mass is suspended from the end of a light rod of length, l, the other end of which is attached to a fixed pivot so that the rod can swing freely in a vertical plane. Let $\theta(t)$ be the displacement angle (in radians) at time, t, of the rod to the vertical. Note that the arclength, $y(t)$, of the mass is given by $y = \ell\theta$. Using Newton's second law and the tangential component (to its natural motion) of the weight of the pendulum, the differential equation governing the motion of the mass is (g is the acceleration due to gravity)

$$\theta'' + \frac{g}{\ell}\sin\theta = 0\,.$$

Explain why, if we assume the pendulum bob only performs small oscillations about the equilibrium vertical position, i.e. so that $|\theta(t)| \ll 1$, then the equation governing the motion of the mass is, to a good approximation,

$$\theta'' + \frac{g}{\ell}\theta = 0\,.$$

Suppose the pendulum bob is pulled to one side and released. Solve this initial value problem explicitly and explain how you might have predicted the nature of the solution. How does the solution behave for different values of ℓ? Does this match your physical intuition?

[1] Question to be handed in on January 20th 2005

[2] Question to be handed in on January 20th 2005

CHAPTER 2

Homogeneous linear ODEs

Exercises

2.1. Find the general solution to the following differential equations:
(a) $y'' + y' - 6y = 0$;
(b) $y'' + 8y' + 16y = 0$;
(c) $y'' + 2y' + 5y = 0$;
(d) $y'' - 3y' + y = 0$.

2.2. (*)[1] For each of the following initial value problems, find the solution, and describe its behaviour:
(a) $5y'' - 3y' - 2y = 0$, with $y(0) = -1$, $y'(0) = 1$;
(b) $y'' + 6y' + 9y = 0$, with $y(0) = 1$, $y'(0) = 2$;
(c) $y'' + 5y' + 8y = 0$, with $y(0) = 1$, $y'(0) = -2$.

[1]Question to be handed in on January 27th 2005

CHAPTER 3

Non-homogeneous linear ODEs

Exercises

3.1. Find the general solution to the non-homogeneous differential equations:
(a) $y'' + 4y = \sin 3t$;
(b) $4y'' + 7y' - 2y = 1 + 2t^2$;
(c) $y'' + y' + y = 3 + 5\mathrm{e}^{2t}$;
(d) $y'' + 8y' + 16y = 50\sin 2t + 8\cos 4t$;
(e) $y'' + 2y' - 8y = t^2\mathrm{e}^{3t}$.

3.2. (*) [1] For each of the following initial value problems, find the solution:
(a) $y'' - 5y' + 6y = \cos 3t$, with $y(0) = 0$, $y'(0) = 5$;
(b) $y'' + 4y' + 4y = \mathrm{e}^{-2t}$, with $y(0) = -3$, $y'(0) = 2$.

3.3. (*) [2] Find the general solution to the non-homogeneous differential equation

$$y'' + 4y = \mathrm{e}^{-t} + \sin 2t\,.$$

How does the solution behave?

3.4. Consider the electrical circuit in Appendix A. Describe how the charge $Q(t)$ behaves for all $t > 0$, when $L = 1$ Henrys, $R = 2$ Ohms, $C = 1/5$ Farads, $Q(0) = Q_0$, $Q'(0) = 0$, and the impressed voltage is
(a) $V(t) = \mathrm{e}^{-t}\sin 3t$,
(b) $V(t) = \mathrm{e}^{-t}\cos 2t$.

3.5. Find the general solution of the following equidimensional ODEs:
(a) $x^2y'' - 2xy' + 2y = (\ln(x))^2 - \ln(x^2)$;
(b) $x^3y''' + 2xy' - 2y = x^2\ln(x) + 3x$.

3.6. (Resonance and damping.) How does damping effect the phenomenon of resonance? For example, suppose that for our frictionally damped spring system, we apply an external sinusoidal force (we might think here of a wine glass, with such a force induced by a pressure wave such as sound), i.e. suppose the equation of motion for the mass on the end of the spring system is, $my'' + Cy' + ky = f(t)$. Take the mass $m = 1$ Kg, stiffness $k = 2$ Kg/s^2, coefficient of friction $C = 2$ Kg/s and the external forcing as $f(t) = \mathrm{e}^{-t}\sin(t)$ Newtons. Assuming that the mass starts at rest at the origin, describe the subsequent behaviour of the mass for all $t > 0$.

[1] Question to be handed in on February 3rd 2005
[2] Question to be handed in on February 3rd 2005

CHAPTER 4

Laplace transforms

Exercises

For all the exercises below use the table of Laplace transforms!

4.1. Find the Laplace transforms of the following functions:

$$\text{(a)}\ \sin(2t)\cos(2t); \quad \text{(b)}\ \cosh^2(2t); \quad \text{(c)}\ \cos(at)\sinh(at); \quad \text{(d)}\ t^2 \mathrm{e}^{-3t}.$$

Hint, you will find the following identities useful:

$$\sin 2\varphi \equiv 2\sin\varphi\cos\varphi\,; \quad \sinh\varphi \equiv \frac{1}{2}\left(\mathrm{e}^{\varphi} - \mathrm{e}^{-\varphi}\right)\,; \quad \cosh\varphi \equiv \frac{1}{2}\left(\mathrm{e}^{\varphi} + \mathrm{e}^{-\varphi}\right)\,.$$

4.2. Find the inverse Laplace transforms of the following functions (you may wish to re-familiarize yourself with partial fractions first):

$$\text{(a)}\ \frac{s}{(s+3)(s+5)}\,; \quad \text{(b)}\ \frac{1}{s(s^2+k^2)}\,; \quad \text{(c)}\ \frac{1}{(s+3)^2}\,.$$

Use Laplace transforms to solve the following initial value problems:

4.3. (*)[1] $y'' + y = t$, $y(0) = 0$, $y'(0) = 2$.

4.4. $y'' + 2y' + y = 3t\mathrm{e}^{-t}$, $y(0) = 4$, $y'(0) = 2$.

4.5. $y'' + 16y = 32t$, $y(0) = 3$, $y'(0) = -2$.

4.6. $y'' - 3y' + 2y = 4$, $y(0) = 1$, $y'(0) = 0$.

4.7. $y'' + 4y' + 4y = 6\mathrm{e}^{-2t}$, $y(0) = -2$, $y'(0) = 8$.

4.8. (*)[2] Consider the damped spring system of Chapters 2&3. In particular let's suppose that the mass, initially at rest in the equilibrium position, is given a sharp hammer blow at time $t = 4\pi$, so that the equation of motion for the mass is,

$$y'' + 4y' + 5y = \delta(t - 4\pi)\,, \quad \text{with} \quad y(0) = 0\,,\ y'(0) = 3\,.$$

Use the Laplace transform to determine the solution to this initial value problem and sketch the behaviour of the solution for all $t \geq 0$.

[1]Question to be handed in on February 10th 2005
[2]Question to be handed in on February 10th 2005

CHAPTER 5

Linear algebraic equations

Exercises

State whether each of the following systems is either *determined*, *underdetermined* or *overdetermined*. Then reduce the augmented matrix to *row echelon form* using elementary row transformations—i.e. implement the method of Gaussian elimination. Hence determine which systems are *consistent* (in which case either state the unique solution if there is one, or classify the set of infinite solutions) or *not consistent* (there is no solution).

5.1.

$$\begin{aligned} x - y + 2z &= -2\,, \\ 3x - 2y + 4z &= -5\,, \\ 2y - 3z &= 2\,. \end{aligned}$$

5.2. (*)[1]

$$\begin{aligned} x - 2y + 2z &= -3\,, \\ 2x + y - 3z &= 8\,, \\ -x + 3y + 2z &= -5\,. \end{aligned}$$

5.3. (*)[2]

$$\begin{aligned} 3x + y + z &= 8\,, \\ -x + y - 2z &= -5\,, \\ x + y + z &= 6\,, \\ -2x + 2y - 3z &= -7\,. \end{aligned}$$

5.4. (*)[3]

$$\begin{aligned} 3x - y + 2z &= 3\,, \\ 2x + 2y + z &= 2\,, \\ x - 3y + z &= 4\,. \end{aligned}$$

5.5. (*)[4]

$$\begin{aligned} 3x - 7y + 35z &= 18\,, \\ 5x + 4y - 20z &= -17\,. \end{aligned}$$

[1]Question to be handed in on February 17th 2005
[2]Question to be handed in on February 17th 2005
[3]Question to be handed in on February 17th 2005
[4]Question to be handed in on February 17th 2005

5.6.

$$\begin{aligned} -3w + x - 2y + 13z &= -3\,, \\ 2w - 3x + y - 8z &= 2\,, \\ w + 4x + 3y - 9z &= 1\,. \end{aligned}$$

5.7. Solve the following systems of equations using Gaussian elimination. For what values of α are these systems consistent?

$$(a)\quad \begin{aligned} x - 2y + 2z &= -3\,, \\ 2x + y - 3z &= 8\,, \\ 9x - 3y - 3z &= \alpha\,. \end{aligned} \qquad (b)\quad \begin{aligned} 2x + 3y &= 7\,, \\ x - y &= 1\,, \\ \alpha x + 2y &= 8\,. \end{aligned}$$

5.8. Using that the rank of a matrix is invariant to EROs, find the rank of the following matrices.

$$(a)\quad \begin{pmatrix} -4 & 1 & 3 \\ 2 & 2 & 0 \end{pmatrix}, \qquad (b)\quad \begin{pmatrix} 2 & 2 & 1 \\ 1 & -1 & 3 \\ 0 & 0 & 1 \\ 4 & 0 & 7 \end{pmatrix},$$

$$(c)\quad \begin{pmatrix} 7 & -2 & 1 & -2 \\ 0 & 2 & 6 & 3 \\ 7 & 2 & 13 & 4 \\ 7 & 0 & 7 & 1 \end{pmatrix}, \qquad (d)\quad \begin{pmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{pmatrix},$$

$$(e)\quad \begin{pmatrix} 1 & 2 & -1 \\ 4 & 3 & 1 \\ 2 & 0 & -3 \end{pmatrix}, \qquad (f)\quad \begin{pmatrix} -1 & 3 & -2 & 4 \\ -1 & 4 & -3 & 5 \\ -1 & 5 & -4 & 6 \end{pmatrix}.$$

CHAPTER 6

Gaussian elimination in practice

Exercises

6.1. (*)[1] Consider the following system:

$$\begin{aligned} 10^{-3}x - y &= 1\,, \\ x + y &= 0\,. \end{aligned}$$

(a) Use 3-digit arithmetic without pivoting to solve this system.

(b) Find a system that is exactly satisfied by your solution from part (a), and note how close the system is to the original system.

(c) Now use partial pivoting and 3-digit arithmetic to solve the original system.

(d) Find a system that is exactly satisfied by your solution from part (c), and note how close this system is to the original system.

(e) Use exact arithmetic to obtain the solution to the original system, and compare the exact solution with the results of parts (a) and (c).

(f) Round the exact solution to three significant digits, and compare the result with those of parts (a) and (c).

6.2. (*)[2] Determine the exact solution of the following system:

$$\begin{aligned} 8x + 5y + 2z &= 15\,, \\ 21x + 19y + 16z &= 56\,, \\ 39x + 48y + 53z &= 140\,. \end{aligned}$$

Now change 15 to 14 in the first equation and again solve the system with exact arithmetic. Is the system ill-conditioned?

6.3. Using geometric considerations, rank the following three systems according to their condition.

(a) $$\begin{aligned} 1.001x - y &= .235\,, \\ x + .0001y &= .765\,, \end{aligned}$$ (b) $$\begin{aligned} 1.001x - y &= .235\,, \\ x + .9999y &= .765\,, \end{aligned}$$ (c) $$\begin{aligned} 1.001x + y &= .235\,, \\ x + .9999y &= .765\,. \end{aligned}$$

6.4. Use Gaussian elimination, with partial pivoting, to solve the following systems of equations. Work to five decimal places.

(a) $$\begin{aligned} x - y + 2z &= -2\,, \\ 3x - 2y + 4z &= -5\,, \\ 2y - 3z &= 2\,, \end{aligned}$$ (b) $$\begin{aligned} 33x + 16y + 72z &= 359\,, \\ -24x - 10y - 57z &= 281\,, \\ -8x - 4y - 17z &= 85\,. \end{aligned}$$

6.5. Calculate the inverse of $A = \begin{pmatrix} 2 & 1 & -4 \\ 0 & -2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$. Check $AA^{-1} = I$.

[1] Question to be handed in on February 24th 2005

[2] Question to be handed in on February 24th 2005

CHAPTER 7

Linear algebraic eigenvalue problems

Exercises

7.1. (*)[1] Two particles of equal mass $m = 1$ move in one dimension at the junction of three springs. The springs each have unstretched length $a = 1$ and have spring stiffness constants, k, $3k$ and k (with $k \equiv 1$) respectively—see Figure 7.1. Applying Newton's second law and Hooke's law, this mass-spring system gives rise to the differential equation system

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = A\,\boldsymbol{x}\,, \tag{7.1}$$

where A is the stiffness matrix given by

$$A = \begin{pmatrix} -4 & 3 \\ 3 & -4 \end{pmatrix}\,, \tag{7.2}$$

and $\boldsymbol{x} = (x_1, x_2)$ is the vector of unknown position displacements for each of the masses shown in Figure 7.1.

Find the eigenvalues and associated eigenvectors of the stiffness matrix A. Hence enumerate the possible modes of vibration of the masses corresponding to each eigenvalue-eigenvector pair. Finally, write down the general solution of the system of equations (7.1).

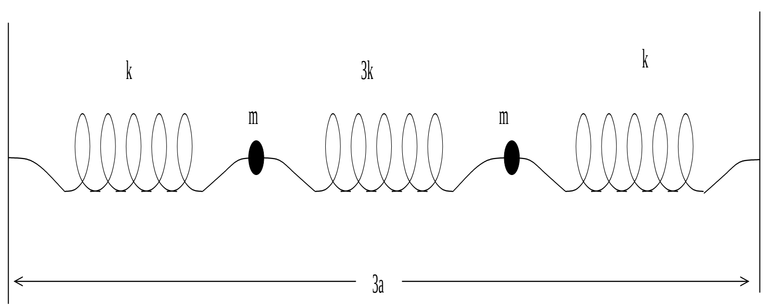

FIGURE 7.1. Simple mass-spring two-particle system.

7.2. Consider the simple model for a tri-atomic shown in Figure 7.2. The molecule consists of three atoms of the same mass m, constrained so that only longitudinal motion is possible. Molecular bonds are modelled by the springs shown, each with stiffness $k = 1$.

Applying Newton's second law and Hooke's law, this mass-spring system gives rise to the differential equation system

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = A\,\boldsymbol{x}\,, \tag{7.3}$$

where A is the stiffness matrix given by

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\,,$$

[1] Question to be handed in on March 3rd 2005

and $\boldsymbol{x} = (x_1, x_2, x_3)$ is the vector of unknown position displacements for each of the three masses shown in Figure 7.2.

Find the eigenvalues and associated eigenvectors of the stiffness matrix, A, and enumerate the possible modes of vibration of the masses corresponding to each eigenvalue-eigenvector pair. Hence write down the general solution of the system of equations (7.3).

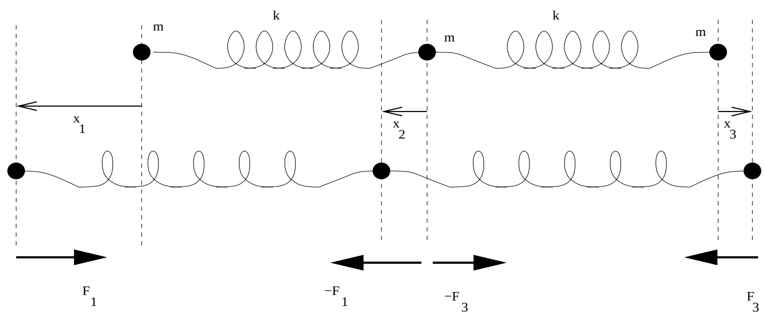

FIGURE 7.2. Simple model for a tri-atomic molecule.

Find the eigenvalues and corresponding eigenvectors of the following matrices. Also find the matrix X that diagonalizes the given matrix via a similarity transformation. Verify your calculated eigenvalues.

7.3.
$$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}.$$

7.4.
$$\begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}.$$

7.5.
$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -2 & 3 \end{pmatrix}.$$

7.6.
$$\begin{pmatrix} 26 & -2 & 2 \\ 2 & 21 & 4 \\ 4 & 2 & 28 \end{pmatrix}.$$

7.7.
$$\begin{pmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{pmatrix}.$$

7.8.
$$\begin{pmatrix} 5 & -3 & 13 \\ 0 & 4 & 0 \\ -7 & 9 & -15 \end{pmatrix}.$$

7.9.

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix}.$$