SECOND ORDER ODES: SUMMARY

1. Springs and Hooke's Law

The general solution to

$$\boxed{\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \omega^2 s = 0 \quad \text{is} \quad s(t) = C_1 \sin \omega t + C_2 \cos \omega t,}$$

where C_1 and C_2 are arbitrary constants.

2. Solving homogeneous linear ODEs

In general, we wish to solve the linear, second order, homogeneous, constant coefficient ordinary differential equation,

$$\boxed{a\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = 0,} \tag{1}$$

where a, b and c are constants (not necessarily positive).

Theorem. The Principle of Superposition for linear, homogeneous differential equations. *If $y_1(t)$ and $y_2(t)$ satisfy (1), then for any two constants C_1 and C_2, the Principle of Superposition says that*

$$\boxed{y(t) = C_1 y_1(t) + C_2 y_2(t)} \tag{2}$$

is a solution also. If $y_1(t)$ is not a multiple of $y_2(t)$, then the general solution *of* (1) *takes the form* (2).

3. The auxiliary equation

Trying a solution to (1) of the form $y(t) = \mathrm{e}^{\lambda t}$ generates the *auxiliary equation (AE)*:

$$\boxed{a\lambda^2 + b\lambda + c = 0.}$$

Hence $\lambda_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\lambda_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Case I: $b^2 - 4ac > 0$, "The overdamped case". There are two real, distinct solutions to the auxiliary equation; the general solution is

$$\boxed{y(t) = C_1 \mathrm{e}^{\lambda_1 t} + C_2 \mathrm{e}^{\lambda_2 t}.}$$

Case II: $b^2 - 4ac = 0$**, "The critically damped case".** There is one real, repeated root to the auxillary equation, $\lambda_1 = \lambda_2 = -\dfrac{b}{2a}$; the general solution is

$$\boxed{y(t) = (C_1 + C_2 t)\mathrm{e}^{-\frac{b}{2a}t}.}$$

Case III: $b^2 - 4ac < 0$**, "The underdamped case".** There are two complex roots to the auxillary equation, namely

$$\begin{aligned} \lambda_1 &= p + \mathrm{i}q, \quad \text{where} \quad & p &= -b/2a, \\ \lambda_2 &= p - \mathrm{i}q, & q &= \sqrt{|b^2 - 4ac|}/2a. \end{aligned}$$

The general solution is

$$\boxed{y(t) = \mathrm{e}^{pt}(C_1 \cos qt + C_2 \sin qt).}$$

4. Solving non-homogeneous linear ODEs

A 2^{nd} order ODE is *linear* if it has the form

$$\boxed{a(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}y}{\mathrm{d}x} + c(x)y = f(x),} \tag{3}$$

where the *coefficients* $a(x)$, $b(x)$ and $c(x)$ can, in general, be functions of x. The equation (3), is said to be *homogeneous* if $f(x) \equiv 0$, otherwise it is said to be *non-homogeneous.*

If L is the *differential operator* $L = a(x)\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\dfrac{\mathrm{d}}{\mathrm{d}x} + c(x)$, we can write (3) as

$$Ly(x) = f(x).$$

To solve the non-homogeneous ODE (3), we:

(1) find the general solution to the associated homogeneous ODE (called the *complementary function*):

$$\boxed{Ly_{\mathrm{CF}}(x) = 0;} \tag{4}$$

(2) then find a *particular solution* or *integral*, $y(x) = y_{\mathrm{PI}}(x)$, of the full non-homogeneous equation (3):

$$\boxed{Ly_{\mathrm{PI}}(x) = f(x).}$$

Then the complete, *general solution* of (3) is

$$\boxed{y(x) = y_{\mathrm{CF}}(x) + y_{\mathrm{PI}}(x).}$$

5. Method of undetermined coefficients to find $y_{\text{PI}}(x)$

When the inhomogeneity $f(x)$ on the right-hand side has the form (or is any constant multiplied by this form):

$f(x)$	Try $y_{\text{PI}}(x)$
$e^{\alpha x}$	$Ae^{\alpha x}$
$\sin\alpha x$ or $\cos\alpha x$	$A\sin\alpha x + B\cos\alpha x$
$b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$	$A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n$
$e^{\alpha x}\sin\beta x$ or $e^{\alpha x}\cos\beta x$	$Ae^{\alpha x}\sin\beta x + Be^{\alpha x}\cos\beta x$.

Occasionally a modification, will be necessary (such as multiply your initial choice by x) to remove all duplications of the solutions of the homogeneous equation that appear in your guess for $y_{\text{PI}}(x)$. It should not be necessary to multiply by x more than a second time (for second order equations).

6. Initial Value Problems

When presented with an initial-value problem, i.e. with conditions $y(0) = \alpha$ and $y'(0) = \beta$ given, for (3), proceed as follows. Find the general solution to the equation by the procedure above, i.e. suppose that the general solution is

$$y(x) = y_{\text{CF}}(x) + y_{\text{PI}}(x) = C_1 y_1(x) + C_2 y_2(x) + y_{\text{PI}}(x). \tag{5}$$

After finding the general solution, substitute in the first initial condition so that

$$\alpha = C_1 y_1(0) + C_2 y_2(0) + y_{\text{PI}}(0). \tag{6}$$

Then differentiate the general solution (5) and substitute in the second initial condition:

$$\beta = C_1 y_1'(0) + C_2 y_2'(0) + y_{\text{PI}}'(0). \tag{7}$$

Then solve the simultaneous equations (6) and (7) for C_1 and C_2 and substitute these values into (5).