Review notes

Introductory Schubert calculus

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Abstract We review here the Grassmann variety, Schubert cells, Schubert subvarieties and then Schubert cycles and how to compute their product. We then review vector bundles over the sphere, clutching functions and the universal bundle.

Keywords Grassmann variety, Schubert calculus

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1 Grassmannian

This is the more formal review from Ledoux, Malham and Thümmler [11]. It has been been adapted slightly to conform with the more traditional exposition in the literature, which is to use row-span rather than column-span for the k-planes. Our main references are Chern [2], Griffiths and Harris [6], Milnor and Stasheff [12], Montgomery [13], Steenrod [17] and Warner [18, p. 130].

1.1 Stiefel and Grassmann manifolds

A k-frame is a k-tuple of $k \leqslant n$ linearly independent vectors in $\mathbb{C}^n$. The *Stiefel manifold* $\mathbb{V}(n,k)$ of k-frames is the open subset of $\mathbb{C}^{n\times k}$ of all k-frames centred at the origin. The set of k-dimensional subspaces of $\mathbb{C}^n$ forms a complex manifold $\mathrm{Gr}(n,k)$ called the *Grassmann manifold* of k-planes in $\mathbb{C}^n$ (see Steenrod [17, p. 35] or Griffiths and Harris [6, p. 193]).

The fibre bundle $\pi\colon \mathbb{V}(n,k) \to \mathrm{Gr}(n,k)$ is a *principal fibre bundle*. For each y in the base space $\mathrm{Gr}(n,k)$, the inverse image $\pi^{-1}(y)$ is homeomorphic to the fibre space $\mathrm{GL}(k)$ which is a Lie group; see Montgomery [13, p. 151]. The projection map π is the natural quotient map sending each k-frame centered at the origin to the k-plane it spans; see Milnor and Stasheff [12, p. 56].

1.2 Representation

Following the exposition in Griffiths and Harris [6], any k-plane in $\mathbb{C}^n$ can be represented by an $k \times n$ matrix of rank k, say $Y \in \mathbb{C}^{k\times n}$. Any two such matrices Y and Y' represent the same k-plane element of $\mathrm{Gr}(n,k)$ if and only if $Y' = uY$ for some $u \in \mathrm{GL}(k)$ (the k-dimensional subspace elements are invariant to rank k closed transformations mapping k-planes to k-planes).

Let $\mathbf{j} = \{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}$ denote a multi-index of cardinality k. Let $Y_{\mathbf{j}^\circ} \subset \mathbb{C}^n$ denote the $(n-k)$-plane in $\mathbb{C}^n$ spanned by the vectors $\{e_j : j \notin \mathbf{j}\}$ and

$$\mathbb{U}_{\mathbf{j}} = \big\{ Y \in \mathrm{Gr}(n,k) \colon Y \cap Y_{\mathbf{j}^\circ} = \{0\} \big\}\,.$$

In other words, $\mathbb{U}_{\mathbf{j}}$ is the set of k-planes $Y \in \mathrm{Gr}(n,k)$ such that the $k \times k$ submatrix of one, and hence any, matrix representation of Y is nonsingular (representing a coordinate patch labelled by $\mathbf{j}$).

Any element of $\mathbb{U}_{\mathbf{j}}$ has a unique matrix representation $y_{\mathbf{j}^\circ}$ whose $\mathbf{j}$th $k\times k$ submatrix is the identity matrix. For example, if $\mathbf{j} = \{1,\ldots,k\}$, then any element of $\mathbb{U}_{\{1,\ldots,k\}}$ can be uniquely represented by a matrix of the form

$$y_{\mathbf{j}^\circ} = \begin{pmatrix} 1 & 0 & \cdots & 0 & \hat{y}_{1,k+1} & \hat{y}_{1,k+2} & \cdots & \hat{y}_{1,n} \\ 0 & 1 & \cdots & 0 & \hat{y}_{2,k+1} & \hat{y}_{2,k+2} & \cdots & \hat{y}_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \hat{y}_{k,k+1} & \hat{y}_{k,k+2} & \cdots & \hat{y}_{k,n} \end{pmatrix},$$

where $\hat{y}_{i,j} \in \mathbb{C}$ for $i = 1,\ldots,k$ and $j = k+1,\ldots,n$. Conversely, an $k \times n$ matrix of this form represents a k-plane in $\mathbb{U}_{\mathbf{j}}$. Each coordinate patch $\mathbb{U}_{\mathbf{j}}$ is an open, dense subset of $\mathrm{Gr}(n,k)$, and the union of all such patches covers $\mathrm{Gr}(n,k)$. For each $\mathbf{j}$, there is a bijective map $\varphi_{\mathbf{j}}\colon \mathbb{U}_{\mathbf{j}} \to \mathbb{C}^{k(n-k)}$ given by

$$\varphi_{\mathbf{j}}\colon y_{\mathbf{j}^\circ} \mapsto \hat{y}\,.$$

Each $\varphi_{\mathbf{j}}$ is thus a local coordinate chart for the coordinate patch $\mathbb{U}_{\mathbf{j}}$ of $\mathrm{Gr}(n,k)$. For all $\mathbf{j},\mathbf{j}'$, if $Y \in \mathbb{U}_{\mathbf{j}} \cap \mathbb{U}_{\mathbf{j}'}$ and $u_{\mathbf{j}',\mathbf{j}}$ is the $\mathbf{j}'$th $k\times k$ submatrix of $y_{\mathbf{j}^\circ}$, then $y_{(\mathbf{j}')^\circ} = (u_{\mathbf{j}',\mathbf{j}})^{-1} y_{\mathbf{j}^\circ}$. Since $u_{\mathbf{j}',\mathbf{j}}$ represents the transformation between representative patches and depends holomorphically on $y_{\mathbf{j}^\circ}$, we deduce that $\varphi_{\mathbf{j}} \circ \varphi_{\mathbf{j}'}^{-1}$ is holomorphic. Note that $\mathrm{Gr}(n,k)$ has a structure of a complex manifold (see Griffiths and Harris [6, p. 194]). Further the unitary group $\mathbb{U}(n)$ acts continuously and surjectively on $\mathrm{Gr}(n,k)$. Hence $\mathrm{Gr}(n,k)$ is compact and connected. Lastly, the general linear group $\mathrm{GL}(n)$ acts transitively on $\mathrm{Gr}(n,k)$, and it is a homogeneous manifold isomorphic to $\mathrm{GL}(n)/\mathrm{GL}(n-k)\times \mathrm{GL}(k)$ (see Chern [2, p. 65] or Warner [18, p. 130]).

Example. This example is taken from Billey [1]. Suppose Y is the 3-plane in $\mathbb{C}^4$ given by $Y = \mathrm{span}\{6e_1+3e_2, 4e_1+2e_3, 9e_1+e_3+e_4\}$. We can use the following matrix representation for this plane

$$M_Y = \begin{pmatrix} 6&3&0&0\\ 4&0&2&0\\ 9&0&1&1 \end{pmatrix}.$$

Then $Y \in \mathrm{Gr}(4,3)$ iff the rows of M_Y are independent vectors in $\mathbb{C}^n$, which is true iff some 3×3 minor of M_Y is not zero. Further, every subspace in $\mathrm{Gr}(n,k)$ can represented by a unique matrix in row echelon form:

$$M_Y = \begin{pmatrix} 6&3&0&0\\ 4&0&2&0\\ 9&0&1&1 \end{pmatrix} = \begin{pmatrix} 3&0&0\\ 0&2&0\\ 0&1&1 \end{pmatrix} \begin{pmatrix} 2&1&0&0\\ 2&0&1&0\\ 7&0&0&1 \end{pmatrix}$$

where the matrix on the right represents $\mathrm{span}\{2e_1+e_2, 2e_1+e_3, 7e_1+e_4\}$.

2 Projective spaces

We introduce projective space. We will be much less formal here and in the following sections; and driven more by example. Our main references here are Kleiman and Laksov [9] and Hatcher [7].

2.1 Affine and projective space

We quote from Kleiman and Laksov [9]. A set of n-tuples $\big(a_1,\ldots,a_n\big)$ of complex numbers is called *affine n-space* and denoted $\mathbb{C}^n$. A point P of *projective n-space* $\mathbb{P}^n$ is defined by an $(n+1)$-tuple $\big(p_0,\ldots,p_n\big)$ of complex numbers not all zero. The p_i are called the *coordinates* of P. Another $(n+1)$-tuple $\big(\hat{p}_0,\ldots,\hat{p}_n\big)$ also defines P if and only if there is a number c satisfying $p_i = c\,\hat{p}_i$ for all $i=0,\ldots,n$. Identifying a point $\big(a_1,\ldots,a_n\big)$ of $\mathbb{C}^n$ with the point $\big(1,a_1,\ldots,a_n\big)$ of $\mathbb{P}^n$, we may think of $\mathbb{P}^n$ as $\mathbb{C}^n$ completed by the points $\big(0,b_1,\ldots,b_n\big)$ "at infinity". We will use $\mathbb{RP}^n$ to distinguish real projective n-space from complex projective n-space $\mathbb{P}^n$. Let $\mathbb{D}^n$ denote the unit ball in $\mathbb{R}^n$. Note that we can regard the n-sphere $\mathbb{S}^n$ as the quotient space $\mathbb{D}^n/\partial\mathbb{D}^n$.

Example (stereographic projection). There are several different versions of stereographic projection of the plane which depend on where you take the plane to go through the north pole, equator or south pole of the unit sphere and whether you project from the poles or centre of the unit sphere (obviously taking the projection point distinct from the said intersection). Let us take the plane to intersect the unit sphere at the equator and the north pole as the projection point. Then $\mathbb{RP}^2 = \mathbb{R}^2 \cup \mathbb{R}^1 \cup \mathbb{R}^0 \cong \mathbb{H}_+$ where $\mathbb{H}_+ \cong \mathbb{D}^1$ is the northern hemisphere. Here in this cell decomposition of $\mathbb{RP}^2$, we have that $\mathbb{RP}^2$ is the union of the real plane $\mathbb{R}^2$ with the one dimensional projective real space $\mathbb{RP}^1 = \mathbb{R}^1 \cup \mathbb{R}^0$ representing the "ray" directions at "infinity" which itself has the decomposition into the real line plus a point at "infinity". In coordinates $(1, x, y)$ parameterizes the plane $\mathbb{R}^2$ which corresponds to the northen hemisphere without the equator, while $(0, 1, y)$ parameterizes the equator without the final point $e_3 := (0, 0, 1)$.

Example (Real projective n-space). This example comes from Hatcher [7, p. 6]. $\mathbb{RP}^n$ is the space of all lines through the origin in $\mathbb{R}^{n+1}$. Each such line is determined by a nonzero vector in $\mathbb{R}^{n+1}$ which will be unique up to nonzero scalar multiplication; hence we can restrict ourselves to vectors of unit length in $\mathbb{R}^{n+1}$. Thus $\mathbb{RP}^n$ can be topologized as $\mathbb{R}^{n+1}\backslash\{0\}$ quotiented by the equivalence relation $v \sim \lambda v$ for scalars $\lambda \neq 0$. We can also regard $\mathbb{RP}^n$ as the quotient space $\mathbb{S}^n/\{v \sim -v\}$, i.e. the sphere with antipodal points identified. Equivalently we can regard $\mathbb{RP}^n$ as a hemisphere $\mathbb{D}^n$ with antipodal points of $\partial\mathbb{D}^n$ identified. Note that $\partial\mathbb{D}^n$ with antipodal points identified is (by our definition) simply $\mathbb{RP}^{n-1}$; and this reveals how we can develop the cell-complex structure of $\mathbb{RP}^n$.

Example (Complex projective n-space). This also comes from Hatcher [7, p. 6–7]. As for real projective n-space, $\mathbb{P}^n$ or $\mathbb{CP}^n$ can be topologized as the quotient space of $\mathbb{C}^{n+1}\backslash\{0\}$ under the equivalence relation $v \sim \lambda v$ for complex scalars $\lambda \neq 0$. Equivalently this can be thought of as the quotient space of the unit sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ with $v \sim \lambda v$ for $|\lambda| = 1$.

The vectors in $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ with last coordinate real and non-negative are the vectors of the form $(w, \sqrt{1-w^2})$ in $\mathbb{C}^n \times \mathbb{C}$ with $|w| \leqslant 1$. These vectors form the graph of the function $w \mapsto \sqrt{1-w^2}$. This is a disk $\mathbb{D}^{2n}_+$ bounded by the sphere $\mathbb{S}^{2n-1} \subset \mathbb{S}^{2n+1}$ consisting of vectors $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| = 1$. Each vector in $\mathbb{S}^{2n+1}$ is equivalent under the identification $v \sim \lambda v$ to a vector in $\mathbb{D}^{2n}_+$. Such a vector in $\mathbb{D}^{2n}_+$ is unique if its last coordinate is nonzero (if the last coordinate is zero, we have the identifications $v \sim \lambda v$ for $v \in \mathbb{S}^{2n-1}$). Hence $\mathbb{P}^n$ is the quotient space of $\mathbb{D}^{2n}_+$ under the identifications $v \sim \lambda v$ for $v \in \mathbb{S}^{2n-1}$; and this reveals how we can develop the cell-complex structure of $\mathbb{P}^n$ with cells of even (real) dimension.

2.2 Projective linear spaces

Again we quote from Kleiman and Laksov [9]. A *projective linear space* L in $\mathbb{P}^n$ is defined as the set of points $P = \big(p_0, \ldots, p_n\big) \in \mathbb{P}^n$ whose coordinates p_j satisfy a system of linear equations $B \cdot P = 0$ for some constant matrix $B \in \mathbb{C}^{(n-k)\times(n+1)}$. We say that L is k-dimensional if these $(n-k)$ equations are independent, i.e. if B has a non-zero $(n-k) \times (n-k)$ minor. There are then $(k+1)$ points P_i in L, with $i = 0, 1, \ldots, k$, which span L. We call L a *projective line* if $k = 1$, a *projective plane* if $k = 2$, etc.

Example (Grassmannian of projective planes). We can think of the Grassmannian $\mathrm{Gr}(n,k)$, of affine k-planes in affine n-space, as the parameter space of $(k-1)$-dimensional projective linear spaces in $\mathbb{P}^{n-1}$. When this point of view is being considered, we denote the Grassmannian by $\mathbb{G}(k-1,n-1)$. See Coskun [3, p. 3].

3 Schubert cells

We summarize well known facts about Schubert cells. Our main references are Billey [1], Coskun [3], Fulton [4], Griffiths and Harris [6], Hatcher [7,8] Kleiman and Laksov [9] and Kresch [10].

3.1 Flag manifolds

Let $\mathbb{V}$ be an n-dimensional vector space. A *flag* for $\mathbb{V}$ is a nested sequence of vector subspaces $\mathbb{V}_i$ of $\mathbb{V}$ where the difference in dimension of two consecutive vector spaces is one. Usually we denote a flag by

$$\mathbb{F}_\bullet : \mathbb{F}_1 \subset \ldots \subset \mathbb{F}_n = \mathbb{V}.$$

For example the *standard flag* $\mathbb{E}_\bullet$ for $\mathbb{C}^n$ would have $\mathbb{E}_i = \mathrm{span}\{e_1,\ldots,e_i\} = \mathbb{C}^i$ where e_i are the normalized coordinate (complex) vectors for $i = 1,\ldots,n$ in $\mathbb{C}^n$. See for example Kresch [10, p. 5].

3.2 Schubert cell decomposition

We have already seen an example of a cell decomposition, that of $\mathbb{P}^n = \mathrm{Gr}(n+1,1)$ which is given by

$$\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \ldots \cup \mathbb{C}^1 \cup \mathbb{C}^0.$$

Let $\mathrm{Gr}(n,k)$ denote the classical/affine Grassmannian that parameterizes k-dimensional linear subspaces of a fixed n-dimensional vector space $\mathbb{V}$. Fix a flag, say the standard flag $\mathbb{E}_\bullet$ of $\mathbb{V} = \mathbb{C}^n$. The Grassmannian has a decomposition as the disjoint union of Schubert cells:

$$\mathrm{Gr}(n,k) = \bigsqcup_{\mathsf{j}\in[n]} \mathcal{C}_{\mathsf{j}},$$

where for each index $\mathsf{j} = \{j_1,\ldots,j_k\}$ the Schubert cell $\mathcal{C}_{\mathsf{j}}$ has a unique representation as a $k \times n$ matrix in row echelon form, where the (ℓ, j_ℓ) position ($\ell = 1,\ldots,k$ and $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n$) contains 1 with zeros above, below and to the right of that position. Using matrix representation, any $Y \in \mathrm{Gr}(n,k)$ by Gaussian elimination, lies in one such cell. Hence each cell denotes a set of k-planes that satisfy a common set of conditions—they all intersect say the standard flag $\mathbb{E}_\bullet$ with the same *attitude*—see Kleiman and Laksov [9]. Note that for $\mathsf{j} = \{j_1,\ldots,j_k\}$, we have $\dim(\mathcal{C}_{\mathsf{j}}) = \sum j_\ell - \ell$. To cement the importance of the Schubert cells we quote the following proposition from Hatcher [8, p. 33].

Proposition 1 *The cells $\mathcal{C}_j$ are the cells of a CW structure on* $\mathrm{Gr}(n,k)$.

Moreover, $\mathcal{C}_{\mathsf{j}} \cong \mathbb{C}^{\sum j_\ell - \ell}$ and the cells are even dimensional (in terms of real dimensions); see Coskun [3, p. 4] and Kresch [10, p. 3].

Example. This comes from Postnikov [14] and Billey [1]; also see Hatcher [8, p. 32]. Consider an element $Y \in \mathrm{Gr}(10,3)$, which after we perform Gaussian elimination where we strive to make zeros in the upper right corner, we arrive at

$$\begin{pmatrix} 6&5&1&0&0&0&0&0&0&0\\ 2&7&0&9&3&2&1&0&0&0\\ 7&5&0&8&4&4&0&3&1&0 \end{pmatrix}.$$

The row-span of this matrix is an equivalent prescription for our original 3-plane Y in $\mathbb{C}^{10}$. Note that the pivot elements occur in columns $3,7,9$, respectively, as we run down rows $1,2,3$. Hence in this example $\mathsf{j}=\{3,7,9\}$. This k-subset j, determines the *position* of Y with respect to the fixed basis; see Billey [1].

Since for a given multi-index $\mathsf{j} \in [n]$ a Schubert cell

$$\mathcal{C}_{\mathsf{j}} = \big\{ Y \in \mathrm{Gr}(n,k)\colon \mathrm{position}(Y) = \{j_1,\ldots,j_k\} \big\}$$

then the Schubert cell $\mathcal{C}_{\{3,7,9\}}$ is the set of all planes whose attitude/position, with respect to a fixed flag, is $\{3,7,9\}$. Hence $\mathcal{C}_{\{3,7,9\}}$ is parameterized by the set of all row echelon matrices of the form

$$\begin{pmatrix} *&*&1&0&0&0&0&0&0&0\\ *&*&0&*&*&*&1&0&0&0\\ *&*&0&*&*&*&0&*&1&0 \end{pmatrix},$$

where the entries '$*$' are arbitrary. Note that $\dim(\mathcal{C}_{\{3,7,9\}}) = 13$. We call j the *Schubert symbol*.

Remarks. Several indexing conventions exist, each bijectively mapped to j, for example, for $\ell = 1,\ldots,k$, two common indexing labels which we will use later are

$$a_\ell := j_\ell - \ell \qquad \text{or} \qquad \lambda_\ell := n-k-a_\ell.$$

Here a_ℓ counts the number of non-zero elements to the left of (ℓ, j_ℓ). The index $\lambda = \{\lambda_1,\ldots,\lambda_k\}$ counts the number of zero elements to the right of (ℓ, j_ℓ) minus the number of remaining pivots below row ℓ. It gives a partition of $k\cdot(n-k)$ associated with a *Young diagram*; see Section 4.3 below. These bijections mean that $\dim(\mathcal{C}_{\mathsf{j}}) = \sum j_\ell - \ell = \sum a_\ell = k(n-k) - \sum \lambda_\ell$; see Billey [1] or Fulton [4]. We have fixed the flag to be the standard flag $\mathbb{E}_\bullet$. With respect to this flag, or indeed any other flag $\mathbb{F}_\bullet$ of $\mathbb{C}^n$, another prescription for the Schubert cell $\mathcal{C}_{\mathsf{j}}(\mathbb{F}_\bullet)$ is

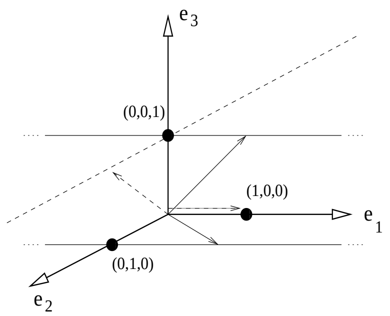
$$\mathcal{C}_{\mathsf{j}}(\mathbb{F}_\bullet) = \big\{ Y \in \mathrm{Gr}(n,k)\colon\ \dim(Y \cap \mathbb{F}_{j_\ell}) = \ell \big\}.$$

See for example Coskun [3, p. 3].

Example. The Grassmannian $\mathrm{Gr}(3,2)$ has three cells corresponding to the Schubert symbols $\mathsf{j}=\{1,2\}$, $\mathsf{j}=\{1,3\}$ and $\mathsf{j}=\{2,3\}$. If we in this case specify this as the Grassmannian of affine real 2-planes in $\mathbb{R}^3$, then these cells have real dimensions 0, 1 and 2, respectively. With respect to the flag $\mathbb{E}_\bullet$ they are parameterized as follows:

$$\mathcal{C}_{\{1,2\}}\colon \begin{pmatrix} 1&0&0\\ 0&1&0 \end{pmatrix}; \qquad \mathcal{C}_{\{1,3\}}\colon \begin{pmatrix} 1&0&0\\ 0&*&1 \end{pmatrix}; \qquad \mathcal{C}_{\{2,3\}}\colon \begin{pmatrix} *&1&0\\ *&0&1 \end{pmatrix}.$$

Hence we see that $\mathcal{C}_{\{2,3\}}$ parameterizes the planes in $\mathbb{R}^3$ through the origin and intersecting the lines through $(0,1,0)$ and $(0,0,1)$ parallel to e_1. The cell $\mathcal{C}_{\{1,3\}}$ parameterizes the planes in $\mathbb{R}^3$ through the origin, the point $(1,0,0)$ and that intersect the line through $(0,0,1)$ parallel to e_2 (indicated by the dashed lines in figure below). Lastly the cell $\mathcal{C}_{\{1,2\}}$ is the single plane through the origin and the points $(1,0,0)$ and $(0,1,0)$, i.e. the plane of the e_1 and e_2 axes.



Example. The Grassmannian $\mathrm{Gr}(4,2)$ has six cells corresponding to the Schubert symbols $\mathbf{j}=\{1,2\}$, $\mathbf{j}=\{1,3\}$, $\mathbf{j}=\{1,4\}$, $\mathbf{j}=\{2,3\}$, $\mathbf{j}=\{2,4\}$ and $\mathbf{j}=\{3,4\}$, and these cells have complex dimensions 0, 1, 2, 2, 3 and 4, respectively. With respect to the flag $\mathbb{E}_\bullet$ they are parameterized as follows:

$$\mathcal{C}_{\{1,2\}}\colon \begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix}; \qquad \mathcal{C}_{\{1,3\}}\colon \begin{pmatrix}1&0&0&0\\0&*&1&0\end{pmatrix}; \qquad \mathcal{C}_{\{1,4\}}\colon \begin{pmatrix}1&0&0&0\\0&*&*&1\end{pmatrix};$$
$$\mathcal{C}_{\{2,3\}}\colon \begin{pmatrix}*&1&0&0\\*&0&1&0\end{pmatrix}; \qquad \mathcal{C}_{\{2,4\}}\colon \begin{pmatrix}*&1&0&0\\*&0&*&1\end{pmatrix}; \qquad \mathcal{C}_{\{3,4\}}\colon \begin{pmatrix}*&*&1&0\\*&*&0&1\end{pmatrix}.$$

In terms of the other two common indexing conventions, these cells correspond to a given by: $\{0,0\}$; $\{0,1\}$; $\{0,2\}$; $\{1,1\}$; $\{1,2\}$ and $\{2,2\}$; and λ given by: $\{2,2\}$; $\{2,1\}$; $\{2\}$; $\{1,1\}$; $\{1\}$ and $\{0\}$ (it is usual to drop superfluous ending zeros).

4 Schubert varieties

4.1 Plücker embedding

There is a natural map, the *Plücker map*,

$$p\colon \mathrm{Gr}(n,k) \to \mathbb{P}\big(\textstyle\bigwedge^k \mathbb{C}^n\big)$$

that sends each k-plane with basis $Y=[Y_1,\ldots,Y_k]$ to $Y_1\wedge\ldots\wedge Y_k$; see Griffiths and Harris [6] or Coskun [3]. If we change the basis, the basis for the image changes by the determinant of the transformation matrix. Hence the map is a point in $\mathbb{P}\big(\bigwedge^k \mathbb{C}^n\big)$. We can recover Y from its image $Y_1\wedge\ldots\wedge Y_k$ as the set of all vectors v such that $v\wedge Y_1\wedge\ldots\wedge Y_k=0$. Further, a point of $\mathbb{P}\big(\bigwedge^k \mathbb{C}^n\big)$ is in the image of p if and only if its representation as a linear combination of the basis elements of $\bigwedge^k \mathbb{C}^n$, consisting of all possible distinct wedge products of a k-dimensional basis in $\mathbb{C}^n$, is completely decomposable. Hence the image of p is a subvariety of $\mathbb{P}\big(\bigwedge^k \mathbb{C}^n\big)$ of completely decomposable elements. It can also be realized as follows. A natural coordinatization of $\mathbb{P}\big(\bigwedge^k \mathbb{C}^n\big)$ is through the determinants of all the $k \times k$ submatrices of Y, normalized by a chosen minor characterized by an index $\mathbf{j}$; hence $\mathbb{P}\big(\bigwedge^k \mathbb{C}^n\big) \cong \mathbb{P}^{\binom{n}{k}-1}$. These minor determinants (the Plücker coordinates) are not all independent; indeed, they satisfy quadratic relations known as the *Plücker relations* (which may themselves not all be independent). The image of the Plücker map p is thus the subspace of $\mathbb{P}^{\binom{n}{k}-1}$ cut out by the quadratic Plücker relations.

Example. For the Grassmannian $\mathrm{Gr}(4,2)$ there is a unique Plücker relation given by

$$f_{12}f_{34} - f_{13}f_{24} + f_{14}f_{23} = 0,$$

where the $f_{ij} = f_{ij}(Y)$ are the 2×2 determinants of the submatrix consisting of columns i and j from Y. Hence the Plücker map embeds $\mathrm{Gr}(4,2)$ in $\mathbb{P}^5$ as a smooth quadric hypersurface; see Coskun [3, p. 8].

4.2 **Schubert varieties**

As we have just seen, $\mathrm{Gr}(n,k)$ is itself a variety; it is the subvariety of $\mathbb{P}\big(\bigwedge^k \mathbb{C}^n\big)$ consisting of the subspace cut out by the quadratic Plücker relations. For $\mathbf{j} = \{j_1,\ldots,j_k\}$ and $Y \in \mathbb{C}^{k\times n}$, define $f_{\mathbf{j}}(Y)$ to be the homogeneous polynomial of degree k given by

$$f_{\mathbf{j}}(Y) := \det \begin{pmatrix} y_{1,j_1} & y_{1,j_2} & \cdots & y_{1,j_k} \\ y_{2,j_1} & y_{2,j_2} & \cdots & y_{2,j_k} \\ \vdots & \vdots & \vdots & \vdots \\ y_{k,j_1} & y_{k,j_2} & \cdots & y_{k,j_k} \end{pmatrix}.$$

The *Zariski topology* on $\mathbb{C}^{k\times n}$ is the topology of *closed sets* given by

$$V(f_{\mathbf{j}}) = \{Y \in \mathbb{C}^{k\times n} \colon f_{\mathbf{j}}(Y) = 0 \text{ for all } \mathbf{j} \in [n]\}.$$

Definition 1 The *Schubert varieties* $\mathcal{X}_{\mathbf{j}}$ for each $\mathbf{j} \in [n]$, are defined as the closure in the Zariski topology of the corresponding Schubert cells, i.e. $\mathcal{X}_{\mathbf{j}} := \overline{\mathcal{C}_{\mathbf{j}}}$. Equivalently we could define the Schubert variety, with respect to any complete flag $\mathbb{F}_\bullet$, corresponding to $\mathbf{j}$ by $\mathcal{X}_{\mathbf{j}}(\mathbb{F}_\bullet) = \big\{Y \in \mathrm{Gr}(n,k) \colon \dim(Y \cap \mathbb{F}_{j_\ell}) \geqslant \ell\big\}$.

See Griffiths and Harris [6, p. 195] and Coskun [3, p. 3]. For reasons that will become apparently presently, it is more convenient to use the λ indexing convention for Schubert varieties.

Example. This comes from Billey [1] and Sottile [16, p. 50]. Practically to determine a variety $\mathcal{X}_{\mathbf{j}}(\mathbb{E}_\bullet)$, with say $\mathbf{j} = \{3,7,9\}$, we should ask ourselves, which linear spaces are in the closure of the corresponding cell $\mathcal{C}_{\{3,7,9\}}$? Consider the set of matrices with full rank 3 where the entries in row ℓ are undetermined up to and including the column j_ℓ and zero thereafter. For $\mathbf{j} = \{3,7,9\}$, these matrices have the form

$$\begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & 0 \end{pmatrix}.$$

The answer, i.e. the Schubert variety $\mathcal{X}_{\mathbf{j}}$ with $\mathbf{j} = \{3, 7, 9\}$, is determined by all the minors $f_{j_1 j_2 j_3}$ which vanish on this set of matrices. These are the minors with either: $4 \leqslant j_1 \leqslant 8$; or with $j_1 = 3$ and $8 \leqslant j_2 \leqslant 9$; or with $j_1 = 3$, $j_2 = 7$ and $j_3 = 10$. Note that the pivots $\mathbf{j}'$ of any matrix from this set will occur weakly to the left of the columns $\mathbf{j}$. Further, for $Y \in \mathcal{X}_{\mathbf{j}}$ we have that $f_{\mathbf{j}'}(Y) = 0$ unless $\mathbf{j}' \leqslant \mathbf{j}$.

Example. Recall our decomposition of $\mathrm{Gr}(4, 2)$ into six cells of complex dimensions 0,1,2,2,3 and 4, corresponding to the Schubert symbols $\mathbf{j} = \{1, 2\}$, $\mathbf{j} = \{1, 3\}$, $\mathbf{j} = \{1, 4\}$, $\mathbf{j} = \{2, 3\}$, $\mathbf{j} = \{2, 4\}$ and $\mathbf{j} = \{3, 4\}$, respectively. Also recalling that for any $Y \in \mathrm{Gr}(4, 2)$ we know that $f_{12}f_{34} - f_{13}f_{24} + f_{14}f_{23} = 0$, the corresponding Schubert varieties, using the λ index, are respectively:

$$\begin{aligned}
\mathcal{X}_{2,2} &= \{Y \in \mathrm{Gr}(4, 2) \colon f_{13} = f_{14} = f_{23} = f_{24} = f_{34} = 0\}\\
\mathcal{X}_{2,1} &= \{Y \in \mathrm{Gr}(4, 2) \colon f_{14} = f_{23} = f_{24} = f_{34} = 0\}\\
\mathcal{X}_{2} &= \{Y \in \mathrm{Gr}(4, 2) \colon f_{23} = f_{24} = f_{34} = 0\}\\
\mathcal{X}_{1,1} &= \{Y \in \mathrm{Gr}(4, 2) \colon f_{14} = f_{24} = f_{34} = 0\}\\
\mathcal{X}_{1} &= \{Y \in \mathrm{Gr}(4, 2) \colon f_{34} = 0\}.
\end{aligned}$$

Note that the Schubert variety $\mathcal{X}_0$ of dimension 4 is simply $\mathrm{Gr}(4, 2)$.

4.3 Partitions and Young tableaux

Definition 2 A *partition* of a number n into k parts, is an additive decomposition of n into a weakly decreasing sequence of k non-negative integers, i.e. we can write $n = |\lambda| = \lambda_1 + \cdots \lambda_k$ with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k$.

Partitions can be visualized using *Young diagrams* where we stack boxes with λ_1 boxes stacked on top of λ_2 boxes and so forth, and by convention we align them to the left. As we have already pointed out, there is a bijection between k-subsets of $[n]$ and partitions whose Young diagram is contained in a $k \times (n - k)$ rectangle, given by

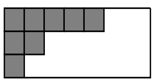
$$\mathbf{j} \mapsto \lambda = \{n - k - j_1 + 1, n - k - j_2 + 2, \ldots, n - j_k\}.$$

The bijection $\mathrm{sh} \colon \mathbf{j} \mapsto a = \{j_1 - 1, j_2 - 2, \ldots, j_k - k\}$ gives the *shape* of a variety $\mathcal{X}_{\mathbf{j}}$. See Billey [1], Fulton [4] and Fulton and Anderson [5] for more details. For a given λ or $\mathbf{j}$ we have the following decomposition of a Schubert variety into Schubert cells:

$$\mathcal{X}_{\mathbf{j}} = \bigcup_{\mathrm{sh}(\mathbf{j}') \subseteq \mathrm{sh}(\mathbf{j})} \mathcal{C}_{\mathbf{j}'} \qquad \Leftrightarrow \qquad \mathcal{X}_{\lambda} = \bigcup_{\lambda \subseteq \mu} \mathcal{C}_{\mu}.$$

We already hinted at this result above in the example with $\mathbf{j} = \{3, 7, 9\}$. Further note that $\dim(\mathcal{X}_\lambda) = n - k - |\lambda| = |\mathrm{sh}(\mathbf{j})| = |a|$.

Example (Young diagram). For the case $n = 10$ and $k = 3$ with $\mathbf{j} = \{3, 7, 9\}$, we see that $a = \{2, 5, 6\}$ and $\lambda = \{5, 2, 1\}$. The corresponding Young diagram is



5 Schubert cycles and cohomology

5.1 **Schubert classes**

We begin by remarking that Schubert varieties can be defined with respect to any flag $\mathbb{F}_\bullet$ of $\mathbb{C}^n$ (let us focus on the affine case for the moment, we return to the projective case in Section 5.3). If $\mathbb{E}_\bullet$ is the standard flag, then since $\dim(\mathbb{F}_i) = \dim(\mathbb{E}_i)$ for each $i = 1, \ldots, n$ there is an element in $\mathrm{GL}(n)$ carrying each $\mathbb{E}_i$ onto $\mathbb{F}_i$ for each i. Hence $\mathcal{X}_\lambda(\mathbb{F}_\bullet)$ is the translate of $\mathcal{X}_\lambda(\mathbb{E}_\bullet)$ by a suitable element of $\mathrm{GL}(n)$. Indeed any k-plane Y in $\mathbb{C}^n$ is carried by, say $T \in \mathrm{GL}(n)$, to $T(Y)$. If Y satisfies the Schubert conditions $\dim(Y \cap \mathbb{E}_{j_\ell}) \geqslant \ell$ for all ℓ, then $T(Y)$ satisfies the Schubert condition $\dim(T(Y) \cap \mathbb{F}_{j_\ell}) \geqslant \ell$ for all ℓ, since $T(\mathbb{E}_i) = \mathbb{F}_i$. Further the Plücker coordinates with respect to the flag $\mathbb{F}_\bullet$ are simply linear combinations of the Plücker coordinates with respect to $\mathbb{E}_\bullet$. See Kleiman and Laksov [9, p. 1067–8] and Coskun [3, p. 5].

Indeed, the general linear group $\mathrm{GL}(n)$ acts transitively on the flags in $\mathbb{C}^n$, and as we have demonstrated, acts transitively on the subvarieties $\mathcal{X}_\lambda$. If two subvarieties belong to the same continuous system of subvarieties, by translation/deformation through elements in $\mathrm{GL}(n)$ (i.e. they are homotopic), then both are assigned the same *cohomology class*. The cohomology classes $[\mathcal{X}_\lambda]$ of the Schubert varieties $\mathcal{X}_\lambda$ are called *Schubert cycles* σ_λ. They depend *only* on the Schubert symbols λ, or equivalently $\mathbf{j}$ or a.

Example. The cohomology class of the Schubert subvariety $\mathcal{X}_\lambda(\mathbb{F}_\bullet)$ is by definition independent of the chosen flag $\mathbb{F}_\bullet$. Recalling the Schubert cell structure for $\mathrm{Gr}(4,2)$ from above, when the Schubert cycles σ_λ are realized with respect to a given flag $\mathbb{F}_\bullet$ as affine 2-planes in $\mathbb{C}^4$, they are prescribed by

$$\begin{aligned}
&\text{codim } 1\colon && \sigma_1(\mathbb{F}_2) = \{Y\colon\ \dim(Y \cap \mathbb{F}_2) \geqslant 1\}\\
&\text{codim } 2\colon && \sigma_{1,1}(\mathbb{F}_3) = \{Y : Y \subset \mathbb{F}_3\}\\
&\text{codim } 2\colon && \sigma_2(\mathbb{F}_1) = \{Y : \mathbb{F}_1 \subset Y\}\\
&\text{codim } 3\colon && \sigma_{2,1}(\mathbb{F}_1, \mathbb{F}_3) = \{Y : \mathbb{F}_1 \subset Y \subset \mathbb{F}_3\}.
\end{aligned}$$

Suppose we now think of $\mathbb{G}(3,1) = \mathrm{Gr}(4,2)$ as the set of projective lines l in $\mathbb{P}^3$. Fix the projective flag of $\mathbb{P}^3$ consisting of a point $\hat{p}$ contained in a line $\hat{l}$ contained in a hyperplane $\hat{h}$ in $\mathbb{P}^3$, then

$$\begin{aligned}
\sigma_1(\hat{l}) &= \{l\colon l \cap \hat{l} \neq \phi\}\\
\sigma_{1,1}(\hat{h}) &= \{l\colon l \in \hat{h}\}\\
\sigma_2(\hat{p}) &= \{l\colon \hat{p} \in l\}\\
\sigma_{2,1}(\hat{p}, \hat{h}) &= \{l\colon \hat{p} \in l \subset \hat{h}\}.
\end{aligned}$$

In other words, in terms of varieties with respect to the projective flag $\hat{p} \subset \hat{l} \subset \hat{h}$, then: $\mathcal{X}_1$ parameterizes projective lines that intersect $\hat{l}$; $\mathcal{X}_{1,1}$ parameterizes projective lines that are contained in $\hat{h}$; $\mathcal{X}_2$ parameterizes projective lines that contain $\hat{p}$; and $\mathcal{X}_{2,1}$ parameterizes projective lines that are contained in $\hat{h}$ and that contain $\hat{p}$. See Griffiths and Harris [6, p. 197] and Coskun [3, p. 4].

5.2 **Cohomology**

Multiplication of Schubert classes, the *cup-product*, corresponds to intersecting the corresponding Schubert varieties with respect to different flags (bases):

$$\sigma_\lambda \cdot \sigma_\mu = [\mathcal{X}_\lambda] \smile [\mathcal{X}_\mu] := [\mathcal{X}_\lambda(\mathbb{F}_\bullet) \cap \mathcal{X}_\mu(\mathbb{F}'_\bullet)],$$

where $\mathbb{F}_\bullet$ and $\mathbb{F}'_\bullet$ are two distinct flags. However, any general pair of distinct flags $\mathbb{F}_\bullet$ and $\mathbb{F}'_\bullet$ can be mapped by a suitable element of $\mathrm{GL}(n)$ to a specific pair consisting of the standard flag $\mathbb{E}_\bullet$ with $\mathbb{E}_i = \mathrm{span}\{e_1, \dots, e_i\}$ and the opposite flag $\mathbb{E}'_\bullet$ with $\mathbb{E}'_i = \mathrm{span}\{e_{n-i+1}, \dots, e_n\}$. Hence to compute $\sigma_\lambda \cdot \sigma_\mu$ we examine the intersection between the realizations $\mathcal{X}_\lambda(\mathbb{E}_\bullet)$ and $\mathcal{X}_\mu(\mathbb{E}'_\bullet)$, and then determine the class of the intersection, i.e. in practice we compute

$$\sigma_\lambda \cdot \sigma_\mu := [\mathcal{X}_\lambda(\mathbb{E}_\bullet) \cap \mathcal{X}_\mu(\mathbb{E}'_\bullet)].$$

This computation is neatly performed using Young tableaux as can be found in Fulton [4, pp. 145–153]; see Section 6.

Importantly, the Schubert cycles σ_λ give a basis for the *cohomology ring* of the Grassmannian $\mathrm{Gr}(n,k)$ under the cup-product. Indeed from algebraic topology we know that the cohomology group with integer coefficients $H^i(\mathrm{Gr}(n,k),\mathbb{Z})$ is zero when i is not in the interval $[0, 2k(n-k)]$, and these groups grade the cohomology ring of $\mathrm{Gr}(n,k)$:

$$H^*\big(\mathrm{Gr}(n,k);\mathbb{Z}\big) = \bigoplus_i H^i(\mathrm{Gr}(n,k),\mathbb{Z}).$$

Further, the product of two Schubert cycles σ_λ and σ_μ can be expressed as a linear combination of Schubert cycles

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu}\, \sigma_\nu,$$

where the structure constants $c^\nu_{\lambda,\mu}$ are known as the Littlewood–Richardson coefficients. The algebra of the Schubert cycles mirrors that of *Schur functions*. See Kleiman and Laksov [9], Coskun [3] and Billey [1].

Example. For the Grassmannian $\mathrm{Gr}(4,2) = \mathbb{G}(3,1)$ it is possible to compute the Littlewood–Richardson coefficients explicitly relatively easily. Of course, in practice, we work with the Schubert subvarieties. To calculate $\mathcal{X}_2 \cap \mathcal{X}_2$, where $\mathcal{X}_2$ is the class of projective lines that pass through a point, we take two points, and realize that there is a unique line containing them both. Hence we must have $\mathcal{X}_2 \cap \mathcal{X}_2 = \mathcal{X}_{2,2}$. Similarly $\mathcal{X}_{1,1} \cap \mathcal{X}_{1,1} = \mathcal{X}_{2,2}$ because there is a unique line of intersection between any two distinct planes in $\mathbb{P}^3$. Further $\mathcal{X}_{1,1} \cap \mathcal{X}_2 = \phi$, since there is not a line which is contained in a given plane that simultaneously passes through a point not in the plane. We consider the case $\mathcal{X}_1 \cap \mathcal{X}_1$ in detail in the next section.

We will consider the explicit construction of the homology and cohomology groups later; they can be found in Hatcher [7].

5.3 **Enumerative geometry**

To illustrate the power of thinking in terms of the Schubert classes and their cup product, consider Schubert's original question:

How many lines intersect four given lines in $\mathbb{R}^3$?

Recall that for $\mathrm{Gr}(4,2) = \mathbb{G}(3,1)$, the Schubert variety $\mathcal{X}_1$ parameterizes the set of projective lines which intersect a given line $\hat{l}$. The lines that intersect four given lines $\hat{l}_1$, $\hat{l}_2$, $\hat{l}_3$ and $\hat{l}_4$, are represented by the product $\sigma_1 \cdot \sigma_1 \cdot \sigma_1 \cdot \sigma_1 = (\sigma_1)^4$. With a slight abuse of notation this is $\sigma_1(\hat{l}_1) \cap \sigma_1(\hat{l}_2) \cap \sigma_1(\hat{l}_3) \cap \sigma_1(\hat{l}_4)$. More correctly, we determine how many subspaces in $\mathrm{Gr}(4,2)$ lie in

$$\mathcal{X}_1(\mathbb{F}_\bullet^{(1)}) \cap \mathcal{X}_1(\mathbb{F}_\bullet^{(2)}) \cap \mathcal{X}_1(\mathbb{F}_\bullet^{(3)}) \cap \mathcal{X}_1(\mathbb{F}_\bullet^{(4)}).$$

Here the *projective* flags $\mathbb{F}_\bullet^{(i)}$ for $i = 1, \ldots, 4$ are distinct and chosen so that $\mathbb{F}_1^{(i)} = \hat{l}_i$. Formally the computation follows a recipe given by the multiplication of Schur functions or using Young diagrams; we get

$$[\mathcal{X}_1(\mathbb{F}_\bullet^{(1)}) \cap \mathcal{X}_1(\mathbb{F}_\bullet^{(2)}) \cap \mathcal{X}_1(\mathbb{F}_\bullet^{(3)}) \cap \mathcal{X}_1(\mathbb{F}_\bullet^{(4)})] = (\sigma_1)^4 = 2\sigma_{2,2} + \cdots .$$

The coefficient of the zero dimensional cycle $\sigma_{2,2}$ is 2, representing two lines meeting four given lines in general position. See Billey [1] and Kleiman and Laksov [9, pp. 1068–9]. Lastly, we quote from Kleiman & Laksov [9, pp. 1070–1]:

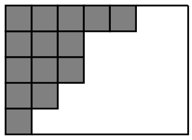
> Perhaps the most important result in the theory of cohomological classes is this: When several subvarieties intersect properly in a finite set of points, then the number of points counted with multiplicity, is equal to the degree of the product of the corresponding cohomology classes. Roughly put, the theorem holds because passing to cohomology classes turns intersection into cup-product. For example suppose that each subvariety represents the k-planes in $\mathbb{P}^n$ which satisfy certain geometric conditions. The the number of k-planes which simultaneously satisfy all the conditions, multiplicities being taken into account, can be determined by formally computing with the corresponding cohomology classes. Since the cohomology classes all remain the same when the subvarieties vary in a continuous system, this number will remain constant when the geometric conditions are varied (or specialized) in a continuous way.

6 **Computing the product of Schubert cycles**

6.1 **Using Young diagrams**

As we have seen, the product of two Schubert cycles can be computed as $\sigma_\lambda \cdot \sigma_\mu := [\mathcal{X}_\lambda(\mathbb{E}_\bullet) \cap \mathcal{X}_\mu(\mathbb{E}'_\bullet)]$. Hence in practice we need to compute the intersection between the realizations $\mathcal{X}_\lambda(\mathbb{E}_\bullet)$ and $\mathcal{X}_\mu(\mathbb{E}'_\bullet)$.

Example. Note that if $n = 12$, $k = 5$ and $\lambda = \{5, 3, 3, 2, 1\}$, then the corresponding Young diagram of $\mathcal{X}_\lambda$ is



and Schubert cell $\mathcal{C}_\lambda(\mathbb{E}_\bullet)$ is

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & 0 & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & 0 & * & 0 & * & 1 & 0 \end{pmatrix}.$$

With respect to the opposite flag $\mathbb{E}'_\bullet$ the Schubert cell $\mathcal{C}_\lambda(\mathbb{E}'_\bullet)$ becomes

$$\begin{pmatrix} 0 & 1 & * & 0 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & 0 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{pmatrix}.$$

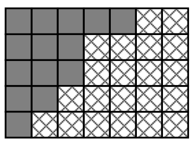
The corresponding Young diagram for $\mathcal{X}_\lambda(\mathbb{E}'_\bullet)$ is the Young diagram above for $\mathcal{X}_\lambda(\mathbb{E}_\bullet)$ rotated by 180°.

Our first formal result on the product of two Schubert cycles σ_λ and σ_μ is as follows; see Fulton [4, p. 148].

Lemma 1 *If $\mathcal{X}_\lambda(\mathbb{E}_\bullet) \cap \mathcal{X}_\lambda(\mathbb{E}'_\bullet) \neq \phi$ then necessarily $\lambda_\ell + \mu_{k+1-\ell} \leqslant n-k$ for all $1 \leqslant \ell \leqslant n-k$.*

The condition for a non-empty intersection embodied in this lemma can be visualized as follows. If in the $k \times (n-k)$ rectangle we fit the Young diagram for λ and at the same time fit the Young diagram for μ rotated by 180°, then the two diagrams must not overlap. If in particular $|\lambda|+|\mu| = k(n-k)$, then the intersection can be non-empty if these diagrams exactly fit together.

Example. If $n = 12$, $k = 5$, $\lambda = \{5,3,3,2,1\}$ and $\mu = \{6,5,4,4,2\}$ we get



Here the Schubert cell $\mathcal{C}_\mu(\mathbb{E}'_\bullet)$ is parameterized by

$$\begin{pmatrix} 0 & 0 & 1 & * & * & 0 & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}.$$

In fact $\mathcal{X}_\lambda(\mathbb{E}_\bullet)$ and $\mathcal{X}_\lambda(\mathbb{E}'_\bullet)$ meet in exactly one point, which is spanned by the basis vectors corresponding to the pivots in $\mathcal{C}_\lambda(\mathbb{E}_\bullet)$ and $\mathcal{C}_\mu(\mathbb{E}'_\bullet)$.

Our last example establishes the following corollary or *duality theorem*:

$$\sigma_\lambda \cdot \sigma_\mu = \begin{cases} 1, & \text{if } \lambda_\ell + \mu_{k+1-\ell} = n-k \text{ for all } 1 \leqslant \ell \leqslant k, \\ 0, & \text{if } \lambda_\ell + \mu_{k+1-\ell} > n-k \text{ for any } 1 \leqslant \ell \leqslant k. \end{cases}$$

6.2 **Pieri's and Giambelli's formulae**

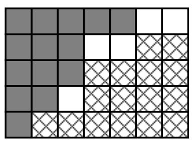
Consider the partition $\lambda = i = \{i, 0, 0, \ldots\}$, i.e. where all parts of the partition except the first are zero. The corresponding variety $\mathcal{X}_\lambda = \mathcal{X}_i$ is known as a *special Schubert variety* and the corresponding classes $\sigma_i = [\mathcal{X}_i]$ are known as *special Schubert cycles. Pieri's rule* gives an algorithm for computing the product of a special Schubert cycle with any Schubert cycle. The formal result, we quote from Coskun [3, p. 7], is as follows.

Theorem 1 (Pieri's formula) *Let σ_λ be a special Schubert cycle and suppose σ_μ is any Schubert cycle with $\mu = \{\mu_1, \ldots, \mu_k\}$. Then we have*

$$\sigma_\lambda \cdot \sigma_\mu = \sum \sigma_\nu$$

where the sum is over all ν such that $\mu_\ell \leqslant \nu_\ell \leqslant \mu_{\ell-1}$ and $\sum \nu_\ell - \mu_\ell = \lambda$.

Example/proof. We can use the duality theorem to establish Pieri's formula; see Fulton [4, p. 150]. We must show that both sides of Pieri's formula have the same intersection number with all cycles $\sigma_{\mu'}$ with $|\mu'| = k(n-k) - |\mu| - i$. In terms of Young diagrams, we put μ in the top left corner of the $k \times (n-k)$ rectangle, and μ' rotated by 180° into the lower right. Pieri's formula is then equivalent to the assertion that $\sigma_{\mu'} \cdot \sigma_\mu \cdot \sigma_i$ is 1 when the two diagrams do not overlap and no two of the i boxes between the two diagrams are in the same column; and $\sigma_{\mu'} \cdot \sigma_\mu \cdot \sigma_i$ is 0 otherwise. For example with $n = 12$, $k = 5$, $\mu = \{5, 3, 3, 2, 1\}$ and $\mu' = \{6, 4, 4, 2, 0\}$ we get



We see first that the two diagrams do not overlap, and second that no two intervening boxes are in the same column.

Theorem 2 (Giambelli's formula) *Any Schubert cycle can be expressed as a linear combination of products of special Schubert cycles as follows:*

$$\sigma_{\{\lambda_1,\ldots,\lambda_k\}} = \det \begin{pmatrix} \sigma_{\lambda_1} & \sigma_{\lambda_1+1} & \sigma_{\lambda_1+2} & \cdots & \sigma_{\lambda_1+k-1} \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & \sigma_{\lambda_2+1} & \cdots & \sigma_{\lambda_2+k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{\lambda_k-k+1} & \sigma_{\lambda_k-k+2} & \sigma_{\lambda_k-k+3} & \cdots & \sigma_{\lambda_k} \end{pmatrix}.$$

This follows directly from Pieri's formula. As a consequence, we see that the special Schubert cycles generate the cohomological ring of the Grassmannian. Pieri's and Giambelli's formulae give us an algorithm for computing the cup product of any two Schubert cycles.

6.3 Littlewood–Richardson rule

Pieri's formula is just a special case of the general product rule for the product of two Schubert cycles σ_λ and σ_μ:

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu}\,\sigma_\nu,$$

where the $c^\nu_{\lambda,\mu}$ are the Littlewood–Richardson coefficients. There are many rules for determining the Littlewood–Richardson coefficients; see Coskun [3, p. 9].

7 Vector bundles over the sphere

We review some basic facts about vector bundles over $\mathbb{S}^k$ and fix some notation. Most of the material in this section comes directly from Hatcher [8, Ch. 1]. Henceforth we will also be much more results focused and include fewer examples (which can be found in the cited literature). Before we proceed further though, let us first fix our main object.

Definition 3 An *n-dimensional vector bundle* is a map $p\colon E \to B$ together with a vector space structure on the *fibres* $p^{-1}(b)$ for each b in the *base space* B, such that the bundle is locally trivial. By this we mean that there is a cover $\{U_\alpha\}$ of B such that for each open set U_α, there exists a homeomorphism $g_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$. We call E the *total space.*

There is a natural procedure to construct vector bundles $E \to \mathbb{S}^k$. We start by thinking of the base space $\mathbb{S}^k$ as the union of two hemispheres $\mathbb{D}^k_+$ and $\mathbb{D}^k_-$ and note that $\mathbb{D}^k_+ \cap \mathbb{D}^k_- = \mathbb{S}^{k-1}$. Given a map $g\colon \mathbb{S}^{k-1} \to \mathrm{GL}(n)$, let E_g be the quotient space consisting of $\mathbb{D}^k_+ \times \mathbb{R}^n \sqcup \mathbb{D}^k_- \times \mathbb{R}^n$ obtained by identifying $(x, v) \in \partial\mathbb{D}^k_- \times \mathbb{R}^n$ with $\big(x, g(x)(v)\big) \in \partial\mathbb{D}^k_+ \times \mathbb{R}^n$. There is a natural projection $E_g \to \mathbb{S}^k$ and this is an n-dimensional vector bundle. The map g is called the *clutching or gluing function* for E_g; see Hatcher [8, p. 22].

We shall use $[X, Y]$ to denote the set of homotopy classes of maps $X \to Y$. We denote the set of isomorphism classes of n-dimensional vector bundles over B by $\mathrm{Vect}(B, n)$. The we have the following result for complex vector bundles; see Hatcher [8, p. 23].

Proposition 2 *The map* $[\mathbb{S}^{k-1}, \mathrm{GL}(n)] \to \mathrm{Vect}(\mathbb{S}^k, n)$ *which sends a clutching function* g *to the vector bundle* E_g *is a bijection.*

8 Universal bundles

8.1 Pullback bundles

We quote a proposition from Hatcher [8, p. 18].

Proposition 3 *Suppose we are given the vector bundle* $p\colon E \to B$ *and a map* $f\colon A \to B$. *Then there exists a vector bundle* $p'\colon E' \to A$ *and a map* $f'\colon E' \to E$*, that takes each fibre of* E' *over a point* $a \in A$ *isomorphically onto a fibre of* E *over* $f(a) \in B$. *The vector bundle* E' *is unique up to isomorphism.*

The uniqueness of E' up to isomorphism means there exists a function $f^*\colon \mathrm{Vect}(B) \to \mathrm{Vect}(A)$ taking the isomorphism class of E to the isomorphism class of E'. Indeed we will denote E' by $f^*(E)$—the bundle *induced* by f, i.e. the *pullback* of E by f.

8.2 **Universal vector bundle**

There is a canonical n-dimensional vector bundle over $\mathrm{Gr}(n,k)$, the Grassmannian of k-planes in $\mathbb{C}^n$. Define

$$E_k(\mathbb{C}^n) = \big\{(Y,v) \in \mathrm{Gr}(n,k) \times \mathbb{C}^n \colon v \in Y\big\}.$$

Then the projection $p\colon E_k(\mathbb{C}^n) \to \mathrm{Gr}(n,k)$ given by $p\colon (Y,v) \mapsto Y$, is a vector bundle. We are now in a position to quote the following theorem; see Milnor and Stasheff [12, Sec. 5,14] or Hatcher [8, p. 29].

Theorem 3 *For a paracompact base space* X*, the map*

$$\begin{aligned} [X, \mathrm{Gr}(n,k)] &\to \mathrm{Vect}(X,n), \\ [f] &\mapsto f^*(E_k), \end{aligned}$$

is a bijection, provided n *is sufficiently large.*

Hence vector bundles over a fixed base space are classified by homotopy classes of maps into $\mathrm{Gr}(n,k)$. We thus call $\mathrm{Gr}(n,k)$ the *classifying space* for n-dimensional vector bundles and $E_k \to \mathrm{Gr}(n,k)$ is called the *universal bundle*.

Remarks. Some important comments are:

1. Thus, any n-dimensional bundle over a paracompact base space, is obtainable as a pullback of $E_k \to \mathrm{Gr}(n,k)$; for n sufficiently large.
2. Every CW complex is paracompact; see Hatcher [8, p. 36].
3. Explicit calculation of $[X, \mathrm{Gr}(n,k)]$ is usually technically very difficult; and the usefulness of the theorem is its theoretical implications. However, as we shall see, for *linear spectral problems* what we construct is precisely $[\mathbb{S}^2, \mathrm{Gr}(n,k)]$. The bundle of interest though, has base space $\mathbb{S}^2$ with the *fibres* $\mathrm{Gr}(n,k)$.

Example (tangent bundles). This example comes from Milnor and Stasheff [12, pp. 60-1]. Given a smooth compact k-dimensional manifold $\mathcal{M} \subset \mathbb{R}^n$, the *generalized Gauss map* $G\colon \mathcal{M} \to \mathrm{Gr}(n,k)$, assigns to each $x \in \mathcal{M}$ the tangent space $\mathrm{T}_x\mathcal{M} \in \mathrm{Gr}(n,k)$. Hence, up to isomorphism, we have the map $[\mathcal{M}, \mathrm{Gr}(n,k)]$, and the tangent bundle $\mathrm{T}\mathcal{M} = G^*(E_k)$.

9 Chern characteristic classes

Here we will be extremely brief due to the anticipated connection to Schubert cycles. Our main references are Hatcher [8, Ch. 3] and Fulton [4, Ch. 9].

9.1 **Chern classes**

The Chern classes $c_i(E) \in H^{2i}(B,\mathbb{Z})$ for a complex vector bundle $E \to B$, measure successively more sophisticated obstructions to trivality—applied successively to higher dimensional skeletal components of the CW complex for B. They have the following properties; we quote from Hatcher [8, p. 78].

Theorem 4 *There is a unique sequence of functions $c_1, c_2, \dots$ assigning to each complex vector bundle $E \to B$ a class $c_i(E) \in H^{2i}(B,\mathbb{Z})$, depending only on the isomorphism class of E, such that:*

1. $c_i\big(f^*(E)\big) = f^*\big(c_i(E)\big)$ *for a pullback* $f^*(E)$*;*
2. $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$ *for* $c = 1 + c_1 + c_2 + \cdots \in H^*(B,\mathbb{Z})$*;*
3. $c_i(E) = 0$ *if* $i > \dim(E)$.

9.2 **Cohomology of Grassmannians**

The cohomological ring of the Grassmannian is generated by the Chern classes of the universal bundle $E_k \to \mathrm{Gr}(n,k)$. Indeed, we have $H^*\big(\mathrm{Gr}(n,k);\mathbb{Z}\big) \cong \mathbb{Z}[c_1, c_2, \dots, c_N]$. The ring of symmetric functions is generated by the Schur polynomials. There is an additive homomorphism from the ring of Schur polynomials to the ring $H^*\big(\mathrm{Gr}(n,k);\mathbb{Z}\big)$, that sends each Schur polynomial to the corresponding Schubert cycle; see Fulton [4, p. 152]. Indeed we already know that (see for example Kresch [10, p. 4])

$$H^*\big(\mathrm{Gr}(n,k);\mathbb{Z}\big) = \bigoplus_{\lambda \subseteq (n-k)^k} \mathbb{Z} \cdot \sigma_\lambda.$$

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