Lie group stochastic integrators

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Introduction

$$\mathrm{d}y_t = V_0(y_t)\,\mathrm{d}t + V_1(y_t)\,\mathrm{d}W_t^1 + V_2(y_t)\,\mathrm{d}W_t^2$$

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1. Stochastic Taylor series
2. Exponential Lie series
3. Castell–Gaines method
4. Quadrature & efficiency
5. Lie group stochastic integrators
6. Conclusions

Introduction

Take home messages:

For numerical SDE methods that are conditioned on two driving Wiener paths, order 1 methods provide the same accuracy as higher order methods for a given computational effort.

Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.

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$$y_t = y_0 + \int_0^t V_0(y_\tau)\,\mathrm{d}\tau + \int_0^t V_1(y_\tau)\,\mathrm{d}W_\tau^1 + \int_0^t V_2(y_\tau)\,\mathrm{d}W_\tau^2$$

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$$W_t - W_s \sim \mathrm{N}(0, \sqrt{t-s})$$

Abstract SDE

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Flow-map:

$$y_t = \varphi_t \circ y_0$$

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$$\mathsf{K}_i \mathsf{K}_j \circ V_j V_i \circ y_0 \equiv \underbrace{\int_0^t \int_0^{\tau_1} \mathrm{d}W^i_{\tau_2} \mathrm{d}W^j_{\tau_1}}_{J_{ij}} V_j V_i \circ y_0$$

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- *Magnus* 1954, *Kunita* 1980, *Ben Arous* 1989, *Castell* 1993, *Burrage* 1999.

Numerical SDE schemes

General integrators

- Euler-Maruyama and Milstein methods
- Runge–Kutta type methods (*Kloeden & Platen* 1999)
- Exponential Lie series methods (*Burrage* 1999; *Misawa* 2001; *P-C. Moan* 2004)
- Castell–Gaines method

Castell–Gaines method

Truncated exponential Lie series across $[t_n, t_{n+1}]$:

$$\hat{\psi}_{t_n,t_{n+1}} = J_1 V_1 + J_2 V_2 + J_0 V_0 + \tfrac{1}{2}(J_{21} - J_{12})[V_1, V_2]\,.$$

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$$u'(\tau) = \hat{\psi}_{t_n,t_{n+1}} \circ u(\tau)$$

across $\tau \in [0, 1]$ **with** $u(0) = y_{t_n}$ **gives** $u(1) \approx y_{t_{n+1}}$**.**

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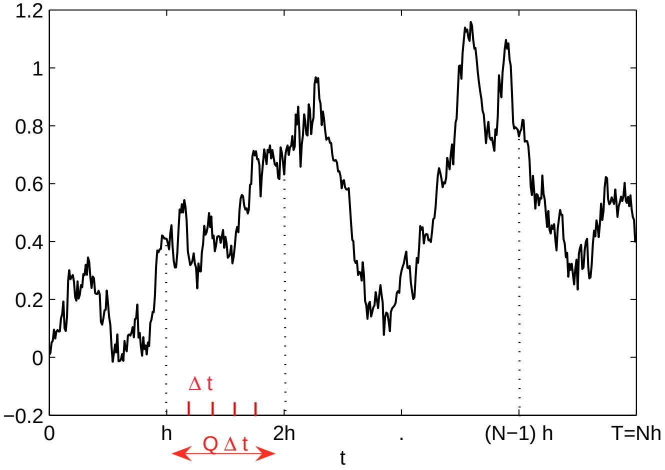
Need approximations for iterated integrals.

Accuracy

Uniformly accurate exponential Lie integrator:

$$\begin{aligned}\hat{\psi}^* = {} & J_1 V_1 + J_2 V_2 + J_0 V_0 + \tfrac{1}{2}(J_{21} - J_{12})[V_1, V_2] \\ & + \tfrac{h^2}{12}\big([V_1,[V_1,V_0]] + [V_2,[V_2,V_0]]\big) \\ & + \tfrac{h^2}{12}\big(V_1[V_2,[V_2,V_1]] + V_2[V_1,[V_1,V_2]]\big)\,.\end{aligned}$$

Quadrature approximation



Quadrature approximation

Two inherent scales:

- Quadrature scale Δt—the *smallest scale* on which the Wiener paths W^1 and W^2 are generated;
- Time-step scale $h = Q\Delta t$—on which the SDE is stepped forward.

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- Polygonal volumes.
- **Wiktorsson 2001; Stump & Hill 2005.**

Quadrature approximation II

$$\begin{aligned}
\mathbb{E}\left(\left|J_{12}(t_n,t_{n+1}) - \hat{J}_{12}(t_n, t_{n+1})\right|^2\right) \\
&= \sum_{q=0}^{Q-1} \mathbb{E}\left(\mathrm{Var}\left[J_{12}(\tau_q, \tau_{q+1})\middle|\mathcal{F}_Q\right]\right) \\
&= \mathcal{O}\left((\Delta t)^2\, Q\right) \\
&= \mathcal{O}\left(h^2/Q\right).
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\end{aligned}
$$

Clark & Cameron 1980

Global error vs computational effort

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Proof.

$$\mathcal{E} = K_Q(M)\,\frac{h^{\frac{1}{2}}}{\sqrt{Q}}\,.$$

$$Q = \frac{1}{h^{2M-1}} \quad \Rightarrow \quad \mathcal{E} = K_{\mathcal{E}}\,h^M\,.$$

$$\mathcal{U} = \left(k_{\mathcal{U}}\,Q + \text{eval}\right)N = k_{\mathcal{U}}\,\frac{1}{h^{2M-1}}\,\frac{T}{h} = \mathcal{O}\left(\mathcal{E}^{-2}\right)\,.$$

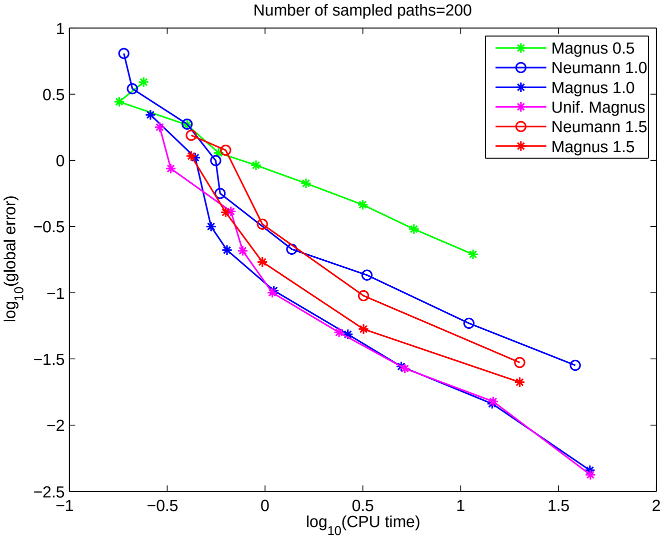
□

Linear system

$$V_i(y) = a_i y \,.$$

$$a_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{51}{200} \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix},$$

Linear system



Lie group action

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Lie group action

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- $\Lambda(\mathrm{Id}, y) = y$ and $\Lambda\big(R, \Lambda(S, y)\big) = \Lambda(RS, y)$.
- $R(\tau) \in \mathcal{G}$: $R(0) = S$ and $R'(0) = a$. If $y = \Lambda(S, y_0) \Leftrightarrow$ $y_0 = \Lambda(S^{-1}, y)$:

$$
\begin{aligned}
(\partial_1 \Lambda)(S, y_0) \circ a \equiv\ & \partial_\tau \Lambda\big(R(\tau), y_0\big)\Big|_{\tau=0} \\
=\ & \partial_\tau \Lambda(R(\tau), \Lambda(S^{-1}, y))\Big|_{\tau=0} \\
=\ & \partial_\tau \Lambda(R(\tau)S^{-1}, y)\Big|_{\tau=0} \\
=\ & (\partial_1 \Lambda)(\mathrm{Id}, y) \circ (aS^{-1})\,.
\end{aligned}
$$

Pullback to the Lie group I

$$y_t = y_0 + \sum_{i=0}^{d} \int_0^t V_i(\tau, y_\tau)\,\mathrm{d}W_\tau^i$$

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$$S_t = \mathrm{Id} + \sum_{i=0}^{d} \int_0^t \big((\partial_1\Lambda)(\mathrm{Id}, y_\tau)\big)^{-1} \circ \big(V_i(\tau, y_\tau)\big)\, S_\tau\,\mathrm{d}W_\tau^i \,,$$

Pullback to the Lie group II

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$$V_i(t, y) = (\partial_1\Lambda)(\mathrm{Id}, y) \circ a_i(t, y)\,,$$

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Examples

$$\mathcal{G} = \mathrm{Diff}(\mathcal{M})\,, \quad \Lambda(S, y) = S \circ y$$

$$\Longrightarrow \quad y_t = S_t \circ y_0 \quad \text{with} \quad S_t \equiv \varphi_t\,.$$

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$$V_i(t, y) = a_i(t, y)\, y \quad \text{with} \quad a_i \in \mathfrak{so}(n), \quad \mathcal{M} = \mathcal{G} = \mathrm{SO}(n):$$
$$\Lambda(S, y) = Sy \quad \Longrightarrow \quad (\partial_1 \Lambda)(\mathrm{Id}, y) \circ a = ay \quad \Longrightarrow \quad y_t = S_t y_0\,.$$

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$$\exp(\sigma_t) = \mathsf{Id} + \int_0^t \mathsf{dexp}_{\sigma_\tau} \circ \mathrm{d}\sigma_\tau \ \exp(\sigma_\tau)\,.$$

$$\sigma_t = \sum_{i=0}^{d} \int_0^t \mathrm{dexp}_{\sigma_\tau}^{-1} \circ a_i\big(\tau, \Lambda(\exp\sigma_\tau, y_0)\big)\, \mathrm{d}W_\tau^i \,,$$

Stochastic Taylor Munthe-Kaas method

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All operations closed in the Lie algebra.

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Algorithm:

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All operations closed in the Lie algebra.

Algorithm:

$$\hat{\sigma}_{t_n,t_{n+1}} = (J_0 v_0 + J_1 v_1 + J_2 v_2 + \tfrac{1}{2} J_1^2 v_1^2 + J_{12} v_2 v_1 + J_{21} v_1 v_2 + \tfrac{1}{2} J_2^2 v_2^2) \circ O \,.$$

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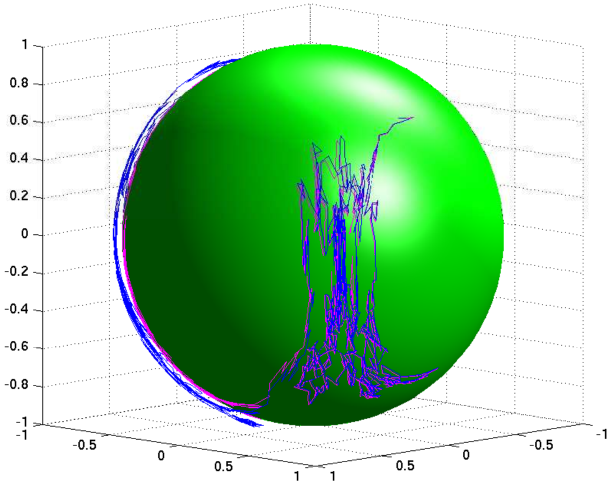
$$y_{t_{n+1}} \approx \Lambda(\exp \hat{\sigma}_{t_n,t_{n+1}}, y_{t_n})$$

Rigid body



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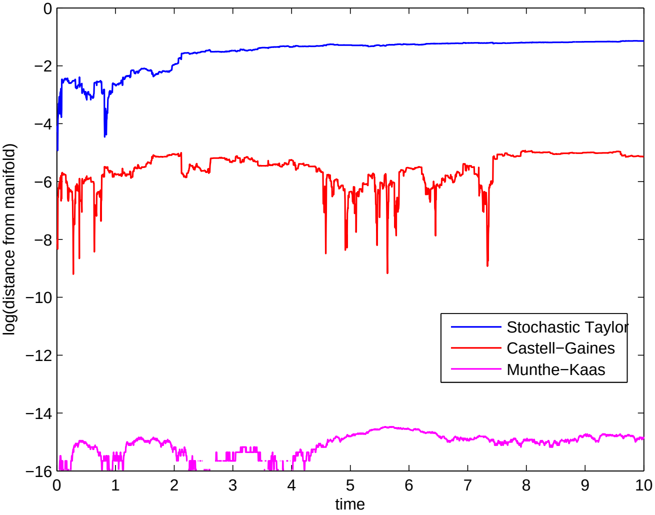

0
−2
−4
−6
−8
−10
−12
−14
−16
0
1
2
3
4
5
6
7
8
9
10
time
log(distance from manifold)
Stochastic Taylor
Castell−Gaines
Munthe−Kaas

Conclusions

Take home messages:

For numerical SDE methods that are conditioned on two driving Wiener paths, order 1 methods provide the same accuracy as higher order methods for a given computational effort.

Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.

Conclusions

Take home messages:

For numerical SDE methods that are conditioned on two driving Wiener paths, order 1 methods provide the same accuracy as higher order methods for a given computational effort.

Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.

Future directions:

- Variable step scheme (*Gaines & Lyons* 1997).
- Zakai equation: Markov chain filters.
- Backwards SDEs

Accuracy I

For order $1/2$ schemes:

$$R^{\text{els}} = \frac{1}{2}(J_{21} - J_{12})[V_1, V_2] \circ y_0\,;$$

$$R^{\text{st}} = R^{\text{els}} + \underbrace{\frac{1}{2}(J_{21} + J_{12})}_{\hat{R}}(V_1 V_2 + V_2 V_1) \circ y_0\,.$$

Local error:

$$\begin{aligned}\mathbb{E}\big((R^{\text{st}})^{\mathsf{T}} R^{\text{st}}\big) &= \mathbb{E}\big((R^{\text{els}} + \hat{R})^{\mathsf{T}} (R^{\text{els}} + \hat{R})\big)\\ &= \mathbb{E}\big((R^{\text{els}})^{T} R^{\text{els}}\big) + \mathbb{E}\big(\hat{R}^{T}\hat{R}\big)\\ &\quad + \underbrace{\mathbb{E}\big(\hat{R}^{T} R^{\text{els}}\big)}_{=0} + \underbrace{\mathbb{E}\big((R^{\text{els}})^{T}\hat{R}\big)}_{=0}\,.\end{aligned}$$

Quadrature approximation

Quadrature		$\hat{J}_{12}$	$\hat{J}_{112}$	$\hat{J}_{120}$	$\hat{J}_{1112}$
$\mathcal{E}^{\text{loc}}(n)$		$h/Q^{1/2}$	$h^{3/2}/Q^{1/2}$	$h^2/Q^{1/2}$	$h^2/Q^{1/2}$
$\mathcal{U}$	$\mathcal{O}(h^{3/2})$	$\mathbf{h^{-1}}$	$\cdots$	$\cdots$	$\cdots$
	$\mathcal{O}(h^2)$	h^{-2}	h^{-1}	$\cdots$	$\cdots$
	$\mathcal{O}(h^{5/2})$	$\mathbf{h^{-3}}$	h^{-2}	h^{-1}	h^{-1}

Global error I

Global interval: $[0, T] = \cup_{n=0}^{N-1} [t_n, t_{n+1}]\,, \quad t_n = nh.$

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$$R = \exp(\psi) - \exp(\hat{\psi}) = \exp(\hat{\psi} + \rho) - \exp(\hat{\psi}) = \rho + \mathcal{O}(\psi\rho)\,.$$

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Stochastic Taylor or exponential Lie series:

$$y_{t_n,t_{n+1}} = \hat{y}_{t_n,t_{n+1}} + R_{t_n,t_{n+1}}\,.$$

Global error II

$$\mathcal{E}^2 \equiv \mathbb{E}\left\|\left(\prod_{n=N-1}^{0} \varphi_{t_n,t_{n+1}} - \prod_{n=N-1}^{0} \hat{\varphi}_{t_n,t_{n+1}}\right) \circ y_0\right\|^2$$

$$= \mathbb{E}\left\|\underbrace{\left(\sum_{n=0}^{N-1} \varphi_{t_{n+1},t_N} R_{t_n,t_{n+1}} \hat{\varphi}_{t_0,t_n}\right)}_{\mathcal{R}} \circ y_0\right\|^2 .$$

Recall: exponential Lie series (order 1)

$$y_t = \exp(\psi_t) \circ y_0 \,, \qquad \psi_t = \psi_{1/2} + \psi_1 + \psi_{3/2} + \cdots$$

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$$\psi_{1/2} = J_1 V_1 + J_2 V_2 + J_0 V_0 \, ,$$

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$$\begin{aligned} \psi_{1/2} &= J_1 V_1 + J_2 V_2 + J_0 V_0 \,, \\ \psi_1 &= \tfrac{1}{2}(J_{21} - J_{12})[V_1, V_2] \,, \end{aligned}$$

Recall: exponential Lie series (order 1)

$$y_t = \exp(\psi_t) \circ y_0 \,, \qquad \psi_t = \psi_{1/2} + \psi_1 + \psi_{3/2} + \cdots$$

$$\begin{aligned}
\psi_{1/2} &= J_1 V_1 + J_2 V_2 + J_0 V_0 \,, \\
\psi_1 &= \tfrac{1}{2}(J_{21} - J_{12})[V_1, V_2] \,, \\
\psi_{3/2} &= \tfrac{1}{2}(J_{10} - J_{01})[V_0, V_1] + \tfrac{1}{2}(J_{20} - J_{02})[V_0, V_2] \\
&\quad + \left(J_{112} - \tfrac{1}{2}J_1 J_{12} + \tfrac{1}{12}J_1^2 J_2\right)[V_1, [V_1, V_2]] \\
&\quad + \left(J_{221} - \tfrac{1}{2}J_2 J_{21} + \tfrac{1}{12}J_2^2 J_1\right)[V_2, [V_2, V_1]] \\
&\quad + \left(J_{110} - \tfrac{1}{2}J_1 J_{10} + \tfrac{1}{12}J_1^2 J_0\right)[V_1, [V_1, V_0]] \\
&\quad + \left(J_{220} - \tfrac{1}{2}J_2 J_{20} + \tfrac{1}{12}J_2^2 J_0\right)[V_2, [V_2, V_0]] \,.
\end{aligned}$$

Global error III

$$\mathcal{R} = \sum_{n=0}^{N-1} \sum_{\alpha} \left(\varphi t_{n+1},\, t_N V_\alpha \varphi_{t_0,t_n} \circ y_0 \right) J_\alpha(t_n, t_{n+1}) \,.$$

$$\begin{aligned}
&\mathbb{E}\left(\mathcal{R}^T \mathcal{R}\right) \\
&= \sum_{n=0}^{N-1} \sum_{\alpha,\beta} \mathbb{E}\Big(\left(\varphi_{t_{n+1},t_N} V_\alpha \varphi_{t_0,t_n} \circ y_0\right)^T \left(\varphi_{t_{n+1},t_N} V_\beta \varphi_{t_0,t_n} \circ y_0\right) \Big) \cdot \\
&\qquad \cdot \mathbb{E}\left(J_\alpha(t_n, t_{n+1})\, J_\beta(t_n, t_{n+1}) \right) \\
&+ \sum_{n \neq m} \sum_{\alpha,\beta} \mathbb{E}\Big(\left(\varphi_{t_{n+1},t_N} V_\alpha \varphi_{t_0,t_n} \circ y_0\right)^T \left(\varphi_{t_{m+1},t_N} V_\beta \varphi_{t_0,t_m} \circ y_0\right) \Big) \cdot \\
&\qquad \cdot \mathbb{E}\left(J_\alpha(t_n, t_{n+1}) \right) \mathbb{E}\left(J_\beta(t_m, t_{m+1}) \right) .
\end{aligned}$$

Stochastic Riccati system

$$S(t) = I + \sum_{i=0}^{d} \int_0^t f_i(\tau, S(\tau))\,\mathrm{d}W_i(\tau)\,.$$

$$f_i(t,S) = S(t)A_i(t)S(t) + B_i(t)S(t) + S(t)C_i(t) + D_i(t)\,.$$

- Stochastic linear-quadratic optimal control.
- eg. mean-variance hedging in finance (*Bobrovnytska & Schweizer* 2004; *Kohlmann & Tang* 2003).

Riccati II

If $\mathbb{A}_i(t) \equiv \begin{pmatrix} B_i(t) & D_i(t) \\ -A_i(t) & -C_i(t) \end{pmatrix}$ and $\mathbb{U} = \begin{pmatrix} U \\ V \end{pmatrix}$ satisfies

$$\mathbb{U}(t) = \mathbb{I} + \sum_{i=0}^{d} \int_0^t \mathbb{A}_i(\tau)\mathbb{U}(\tau)\,\mathrm{d}W_i(\tau)\,,$$

then $S = UV^{-1}$ solves the Riccati system.

$$A_0 = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}, \quad C_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad D_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

$D_1 = a_1$ and $D_2 = a_2$.

Riccati III

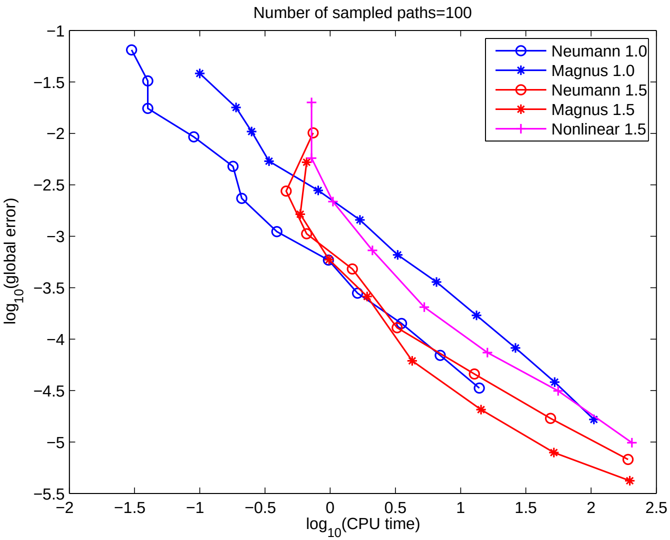
Kloeden & Platen:

$$
\begin{aligned}
S_{n+1} = {} & S_n + f(S_n)h + D_1 J_1 + D_2 J_2 \\
& + \tfrac{h}{4}\big(f(Y_1^+) + f(Y_1^-) + f(Y_2^+) + f(Y_2^-) - 4f(S_n)\big) \\
& + \tfrac{1}{2\sqrt{h}}\left(\big(f(Y_1^+) - f(Y_1^-)\big)J_{10} + \big(f(Y_2^+) - f(Y_2^-)\big)J_{20}\right) ,
\end{aligned}
$$

$$
Y_j^{\pm} = S_n + \tfrac{h}{2} f(S_n) \pm D_j \sqrt{h}
$$

$$
f(S) = SA_0 S + B_0 S + SC_0 + D_0 \,.
$$

Riccati IV



Number of sampled paths=100
Neumann 1.0
Magnus 1.0
Neumann 1.5
Magnus 1.5
Nonlinear 1.5
log10(global error)
log10(CPU time)
−1
−1.5
−2
−2.5
−3
−3.5
−4
−4.5
−5
−5.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5