Efficient strong integrators for linear stochastic systems

Simon J.A. Malham

Work in collaboration with:

Anke Wiese & Gabriel Lord

Heriot–Watt University, Edinburgh, UK

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Introduction

$$\mathrm{d}y_t = V_0(y_t)\,\mathrm{d}t + V_1(y_t)\,\mathrm{d}W_t^1 + V_2(y_t)\,\mathrm{d}W_t^2$$

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1. Stochastic Taylor series
2. Exponential Lie series
3. Castell–Gaines method
4. Quadrature & efficiency
5. Lie group stochastic integrators
6. Conclusions

Introduction

Take home messages:

For numerical SDE methods that are conditioned on two or more driving Wiener paths, order 1 numerical schemes provide the most accuracy for a given computational effort.

Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.

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$$V_i = \sum_{j=1}^{n} V_i^j(y)\partial_y$$

Flow map and abstract SDE

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Stochastic Taylor series

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$$y_t = y_0 + K \circ V \circ y_0 + (K \circ V)^2 \circ y_0 + (K \circ V)^3 \circ y_0 + \cdots$$

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$$\varphi_t = \mathrm{Id} + K \circ V + (K \circ V)^2 + (K \circ V)^3 + \cdots$$

Integral operators

$$
\begin{aligned}
\mathsf{K} \circ V \circ y_0 &= \mathsf{K}_0 \circ V_0 \circ y_0 + \mathsf{K}_1 \circ V_1 \circ y_0 + \mathsf{K}_2 \circ V_2 \circ y_0 \\
&= t V_0 \circ y_0 + \underbrace{\int_0^t \mathrm{d}W_\tau^1}_{J_1} V_1 \circ y_0 + \underbrace{\int_0^t \mathrm{d}W_\tau^2}_{J_2} V_2 \circ y_0
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\end{aligned}
$$

$$
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(\mathsf{K} \circ V)^2 \circ y_0 &= (\mathsf{K}_0 \circ V_0 + \mathsf{K}_1 \circ V_1 + \mathsf{K}_2 \circ V_2)^2 \circ y_0 \\
&= \big((\mathsf{K}_0 V_0)^2 + \mathsf{K}_0 V_0 \mathsf{K}_1 V_1 + \mathsf{K}_1 V_1 \mathsf{K}_0 V_0 \\
&\quad + (\mathsf{K}_1 V_1)^2 + \mathsf{K}_1 V_1 \mathsf{K}_2 V_2 + \mathsf{K}_2 V_2 \mathsf{K}_1 V_1 + \cdots \big) \circ y_0
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$$\begin{aligned}(\mathsf{K} \circ V)^2 \circ y_0 &= (\mathsf{K}_0 \circ V_0 + \mathsf{K}_1 \circ V_1 + \mathsf{K}_2 \circ V_2)^2 \circ y_0 \\ &= \big((\mathsf{K}_0 V_0)^2 + \mathsf{K}_0 V_0 \mathsf{K}_1 V_1 + \mathsf{K}_1 V_1 \mathsf{K}_0 V_0 \\ &\quad + (\mathsf{K}_1 V_1)^2 + \mathsf{K}_1 V_1 \mathsf{K}_2 V_2 + \mathsf{K}_2 V_2 \mathsf{K}_1 V_1 + \cdots\big) \circ y_0\end{aligned}$$

$$\mathsf{K}_j \mathsf{K}_i \circ V_j V_i \circ y_0 \equiv \underbrace{\int_0^t \int_0^{\tau_1} \mathrm{d}W_{\tau_2}^i \mathrm{d}W_{\tau_1}^j}_{J_{ij}} V_j V_i \circ y_0$$

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\varphi_{3/2} &= J_{10} V_0 V_1 + J_{01} V_1 V_0 + J_{20} V_0 V_2 + J_{02} V_0 V_2 \\
&\quad + J_{111} V_1^3 + J_{112} V_2 V_1^2 + J_{121} V_1 V_2 V_1 + J_{122} V_2^2 V_1 \\
&\quad + J_{211} V_1^2 V_2 + J_{212} V_2 V_1 V_2 + J_{221} V_1 V_2^2 + J_{222} V_2^3 \\
&\quad + J_{00} V_0^2 + J_{011} V_1^2 V_0 + J_{101} V_1 V_0 V_1 + J_{110} V_0 V_1^2 + J_{022} V_2^2 V_0 \\
&\quad + J_{202} V_2 V_0 V_2 + J_{220} V_0 V_2^2 + J_{1111} V_1^4 + J_{1122} V_2^2 V_1^2 \\
&\quad + J_{1212} V_2 V_1 V_2 V_1 + J_{2121} V_1 V_2 V_1 V_2 + J_{2211} V_1^2 V_2^2 + J_{2222} V_2^4 \,.
\end{aligned}$$

Exponential Lie series

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\psi_{3/2} &= \tfrac{1}{2}(J_{10} - J_{01})[V_0, V_1] + \tfrac{1}{2}(J_{20} - J_{02})[V_0, V_2] \\
&\quad + \left(J_{112} - \tfrac{1}{2} J_1 J_{12} + \tfrac{1}{12} J_1^2 J_2\right) [V_1, [V_1, V_2]] \\
&\quad + \left(J_{221} - \tfrac{1}{2} J_2 J_{21} + \tfrac{1}{12} J_2^2 J_1\right) [V_2, [V_2, V_1]] \\
&\quad + \left(J_{110} - \tfrac{1}{2} J_1 J_{10} + \tfrac{1}{12} J_1^2 J_0\right) [V_1, [V_1, V_0]] \\
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\end{aligned}$$

- *Magnus* 1954, *Kunita* 1980, *Ben Arous* 1989, *Castell* 1993, *Burrage* 1999.

Numerical SDE schemes

General integrators

- Euler-Maruyama and Milstein methods
- Runge–Kutta type methods (*Kloeden & Platen* 1999)
- Magnus (*Burrage* 1999; *Misawa* 2001; *P-C. Moan* 2004)
- Linear systems: Neumann $\equiv$ stochastic Taylor $\equiv$ Runge–Kutta

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Exponential Lie series integrators

- **Castell–Gaines method: more accurate.**

Castell–Gaines method

Truncated exponential Lie series across $[t_n, t_{n+1}]$:

$$\hat{\psi}_{t_n,t_{n+1}} = J_1 V_1 + J_2 V_2 + J_0 V_0 + \tfrac{1}{2}(J_{21} - J_{12})[V_1, V_2]\,.$$

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Solution:

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$$u'(\tau) = \hat{\psi}_{t_n,t_{n+1}} \circ u(\tau)$$

across $\tau \in [0,1]$ **with** $u(0) = y_{t_n}$ **gives** $u(1) \approx y_{t_{n+1}}$**.**

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Need approximations for iterated integrals.

Accuracy I

For order $1/2$ schemes:

$$R^{\text{els}} = \tfrac{1}{2}(J_{21} - J_{12})[V_1, V_2] \circ y_0\,;$$

$$R^{\text{st}} = R^{\text{els}} + \underbrace{\tfrac{1}{2}(J_{21} + J_{12})}_{\hat{R}}(V_1 V_2 + V_2 V_1) \circ y_0\,.$$

Local error:

$$\begin{aligned}
\mathbb{E}\big((R^{\text{st}})^{\mathsf{T}} R^{\text{st}}\big) &= \mathbb{E}\big((R^{\text{els}} + \hat{R})^{\mathsf{T}} (R^{\text{els}} + \hat{R})\big) \\
&= \mathbb{E}\big((R^{\text{els}})^T R^{\text{els}}\big) + \mathbb{E}\big(\hat{R}^T \hat{R}\big) \\
&\quad + \underbrace{\mathbb{E}\big(\hat{R}^T R^{\text{els}}\big)}_{=0} + \underbrace{\mathbb{E}\big((R^{\text{els}})^T \hat{R}\big)}_{=0}\,.
\end{aligned}$$

Accuracy II

For order 1 schemes:

$$\mathbb{E}\left((R^{\mathrm{st}})^{\mathsf{T}} R^{\mathrm{st}}\right) = \mathbb{E}\left((R^{\mathrm{els}})^{T} R^{\mathrm{els}}\right) + \underbrace{\mathbb{E}\left(\hat{R}^{T}\hat{R}\right) + \mathbb{E}\left(\hat{R}^{T} R^{\mathrm{els}}\right) + \mathbb{E}\left((R^{\mathrm{els}})^{T}\hat{R}\right)}_{h^3 X^T B X + \mathcal{O}(h^4)} .$$

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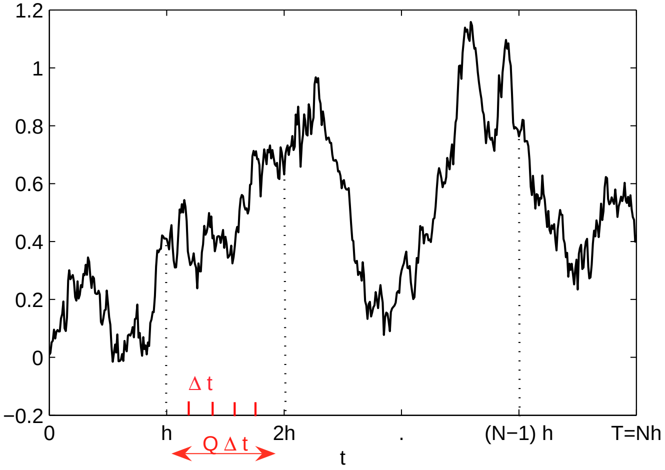
Uniformly accurate exponential Lie integrator:

$$\begin{aligned}\hat{\psi}^{*} &= J_1 V_1 + J_2 V_2 + J_0 V_0 + \tfrac{1}{2}(J_{21} - J_{12})[V_1, V_2] \\ &+ \tfrac{h^2}{12}\big([V_1,[V_1, V_0]] + [V_2,[V_2, V_0]]\big) \\ &+ \tfrac{h^2}{12}\big(V_1[V_2,[V_2, V_1]] + V_2[V_1,[V_1, V_2]]\big) .\end{aligned}$$

Quadrature approximation



Quadrature approximation

Two inherent scales:

- Wiener-discrete-path scale Δt—the *smallest scale* on which the Wiener paths W^1 and W^2 are generated;
- Time-step scale $h = Q\Delta t$—on which the SDE is stepped forward.

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Practical approach:

Quadrature approximation

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Practical approach:

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- Polygonal volumes.
- **Wiktorsson 2001; Stump & Hill 2005.**

Quadrature approximation II

$$\mathbb{E}\left(\left|J_{12}(t_n,t_{n+1}) - \hat{J}_{12}(t_n,t_{n+1})\right|^2\right)$$
$$= \sum_{q=0}^{Q-1} \mathbb{E}\left(\mathrm{Var}\left[J_{12}(\tau_q,\tau_{q+1})\middle|\mathcal{F}_Q\right]\right)$$
$$= \mathcal{O}\left((\Delta t)^2\, Q\right)$$
$$= \mathcal{O}\left(h^2/Q\right).$$

Quadrature approximation III

Quadrature		$\hat{J}_{12}$	$\hat{J}_{112}$	$\hat{J}_{120}$	$\hat{J}_{1112}$
$\mathcal{E}^{\text{loc}}(n)$		$h/Q^{1/2}$	$h^{3/2}/Q^{1/2}$	$h^2/Q^{1/2}$	$h^2/Q^{1/2}$
$\mathcal{U}$	$\mathcal{O}(h^{3/2})$	$\mathbf{h^{-1}}$	$\dots$	$\dots$	$\dots$
	$\mathcal{O}(h^2)$	h^{-2}	h^{-1}	$\dots$	$\dots$
	$\mathcal{O}(h^{5/2})$	$\mathbf{h^{-3}}$	h^{-2}	h^{-1}	h^{-1}

Global error vs computational effort

Theorem

$$\mathcal{U} = \underbrace{\left(k_{\mathcal{U}}\, K_{\mathcal{E}}^{2}\right)\mathcal{E}^{-2}}_{\mathcal{U}^{\text{quad}}} + \underbrace{\left((c_M\, n^2 + c_E) K_{\mathcal{E}}^{1/M}\right)\mathcal{E}^{-1/M}}_{\mathcal{U}^{\text{eval}}} \,.$$

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Proof.

$$\mathcal{E} = K_Q(M)\, \frac{h^{\frac{1}{2}}}{\sqrt{Q}} + K_T(M)\, h^M .$$

$$Q = \frac{1}{h^{2M-1}} \quad \Rightarrow \quad \mathcal{E} = K_{\mathcal{E}}\, h^M .$$

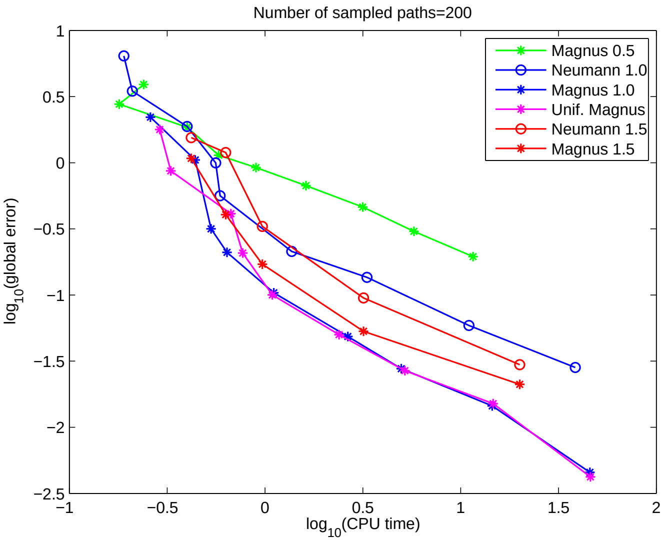
$$\mathcal{U} = \left(k_{\mathcal{U}}\, Q + c_M\, n^2 + c_E\right) N .$$

□

Linear system

$$a_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{51}{200} \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix},$$

Linear system



Lie group action

Munthe-Kaas (1999): $y_t = \Lambda(S_t, y_0)$

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- $\Lambda(\mathrm{Id}, y) = y$ and $\Lambda\big(R, \Lambda(S, y)\big) = \Lambda(RS, y)$.
- $R(\tau) \in \mathcal{G}$: $R(0) = S$ and $R'(0) = a$

$$(\partial_1 \Lambda)(S, y) \circ a \equiv \partial_\tau \Lambda\big(R(\tau), y\big)\Big|_{\tau=0} .$$

$\partial_1 \Lambda\big|_{\mathfrak{g}}$ linear.

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$\partial_1\Lambda|_{\mathfrak{g}}$ linear.

- **If** $y = \Lambda(S, y_0) \Leftrightarrow y_0 = \Lambda(S^{-1}, y)$**:**

$$\begin{aligned}(\partial_1\Lambda)(S, y_0) \circ a &= \partial_\tau \Lambda(R(\tau), \Lambda(S^{-1}, y))\big|_{\tau=0} \\ &= \partial_\tau \Lambda(R(\tau)S^{-1}, y)\big|_{\tau=0} \\ &= (\partial_1\Lambda)(\mathrm{Id}, y) \circ (aS^{-1}) .\end{aligned}$$

Pullback to the Lie group I

$$y_t = y_0 + \sum_{i=0}^{d} \int_0^t V_i(\tau, y_\tau) \, \mathrm{d}W_\tau^i$$

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\end{aligned}$$

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$$\int_0^t (\partial_1 \Lambda)(\mathrm{Id}, y_\tau) \circ (\mathrm{d}S_\tau \cdot S_\tau^{-1}) = \sum_{i=0}^{d} \int_0^t V_i(\tau, y_\tau)\,\mathrm{d}W_\tau^i \,.$$

$$S_t = \mathrm{Id} + \sum_{i=0}^{d} \int_0^t \big((\partial_1 \Lambda)(\mathrm{Id}, y_\tau)\big)^{-1} \circ \big(V_i(\tau, y_\tau)\big)\, S_\tau\,\mathrm{d}W_\tau^i \,,$$

Pullback to the Lie group II

$$S_t = \mathsf{Id} + \sum_{i=0}^{d} \int_0^t \big((\partial_1 \Lambda)(\mathsf{Id}, y_\tau)\big)^{-1} \circ \big(V_i(\tau, y_\tau)\big)\, S_\tau \, \mathrm{d}W_\tau^i \,,$$

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Examples:
$\mathcal{G} = \mathrm{Diff}(\mathcal{M})$ & $\Lambda(S, y) = S \circ y \implies y_t = S_t \circ y_0$ with $S_t \equiv \varphi_t$.

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$V_i(t,y) = a_i(t,y)\,y$ with $a_i \in \mathfrak{so}(n)$, $\mathcal{M} = \mathcal{G} = \mathrm{SO}(n)$:
$\Lambda(S,y) = Sy \implies (\partial_1 \Lambda)(\mathrm{Id}, y) \circ a = ay \implies y_t = S_t y_0$

Pullback to the Lie algebra

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$$\exp(\sigma_t) = \mathrm{Id} + \sum_{i=0}^{d} \int_0^t a_i\big(\tau, \Lambda(\exp(\sigma_\tau), y_0)\big)\, \exp(\sigma_\tau) \,\mathrm{d}W_\tau^i \,.$$

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Stochastic Taylor Munthe-Kaas method

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Algorithm:

Stochastic Taylor Munthe-Kaas method

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Algorithm:

$$\hat{\sigma}_{t_n, t_{n+1}} = (J_0 v_0 + J_1 v_1 + J_2 v_2 + \tfrac{1}{2} J_1^2 v_1^2 + J_{12} v_2 v_1 + J_{21} v_1 v_2 + \tfrac{1}{2} J_2^2 v_2^2) \circ O \,.$$

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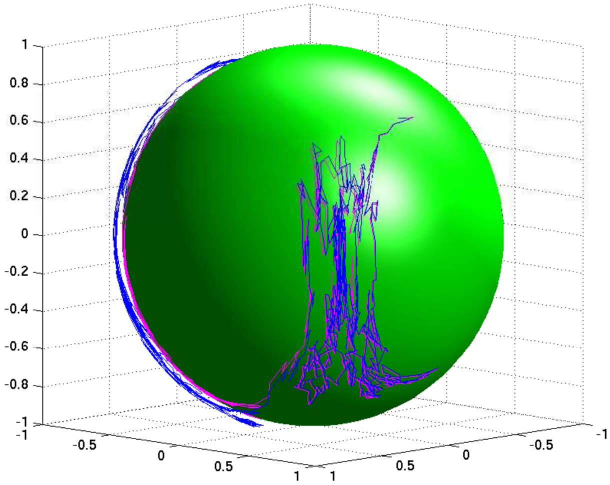
$$y_{t_{n+1}} \approx \Lambda\big(\exp \hat{\sigma}_{t_n,t_{n+1}}, y_{t_n}\big)$$

Rigid body



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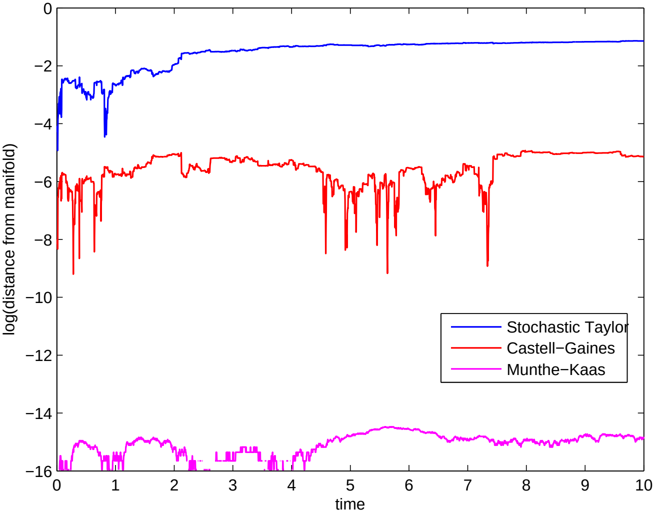

log(distance from manifold)
time
0
−2
−4
−6
−8
−10
−12
−14
−16
0 1 2 3 4 5 6 7 8 9 10
Stochastic Taylor
Castell−Gaines
Munthe−Kaas

Conclusions

Take home messages:

For numerical SDE methods that are conditioned on two or more driving Wiener paths, order 1 numerical schemes provide the most accuracy for a given computational effort.

Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.

Conclusions

Take home messages:

For numerical SDE methods that are conditioned on two or more driving Wiener paths, order 1 numerical schemes provide the most accuracy for a given computational effort.

Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.

Future directions:

- Variable step scheme (*Gaines & Lyons* 1997).
- Zakai equation: Markov chain filters.
- Backwards SDEs

Global error I

Global interval: $[0, T] = \cup_{n=0}^{N-1} [t_n, t_{n+1}]\,, \quad t_n = nh.$

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Local remainder across $[t_n, t_{n+1}]$:

$$R = \exp(\psi) - \exp(\hat{\psi}) = \exp(\hat{\psi} + \rho) - \exp(\hat{\psi}) = \rho + \mathcal{O}(\psi\rho)\,.$$

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Stochastic Taylor or exponential Lie series:

$$y_{t_n,t_{n+1}} = \hat{y}_{t_n,t_{n+1}} + R_{t_n,t_{n+1}}\,.$$

Global error II

$$\mathcal{E}^2 \equiv \mathbb{E}\left\| \left(\prod_{n=N-1}^{0} \varphi_{t_n,t_{n+1}} - \prod_{n=N-1}^{0} \hat{\varphi}_{t_n,t_{n+1}} \right) \circ y_0 \right\|^2$$

$$= \mathbb{E}\left\| \underbrace{\left(\sum_{n=0}^{N-1} \varphi_{t_{n+1},t_N} R_{t_n,t_{n+1}} \hat{\varphi}_{t_0,t_n} \right)}_{\mathcal{R}} \circ y_0 \right\|^2 .$$

Recall: exponential Lie series (order 1)

$$y_t = \exp(\psi_t) \circ y_0 \,, \qquad \psi_t = \psi_{1/2} + \psi_1 + \psi_{3/2} + \cdots$$

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$$\begin{aligned}\psi_{1/2} &= J_1 V_1 + J_2 V_2 + J_0 V_0\,,\\ \psi_1 &= \tfrac{1}{2}(J_{21} - J_{12})[V_1, V_2]\,,\end{aligned}$$

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\psi_{1/2} &= J_1 V_1 + J_2 V_2 + J_0 V_0\,,\\
\psi_1 &= \tfrac{1}{2}(J_{21} - J_{12})[V_1, V_2]\,,\\
\psi_{3/2} &= \tfrac{1}{2}(J_{10} - J_{01})[V_0, V_1] + \tfrac{1}{2}(J_{20} - J_{02})[V_0, V_2]\\
&\quad + \left(J_{112} - \tfrac{1}{2}J_1 J_{12} + \tfrac{1}{12}J_1^2 J_2\right)[V_1, [V_1, V_2]]\\
&\quad + \left(J_{221} - \tfrac{1}{2}J_2 J_{21} + \tfrac{1}{12}J_2^2 J_1\right)[V_2, [V_2, V_1]]\\
&\quad + \left(J_{110} - \tfrac{1}{2}J_1 J_{10} + \tfrac{1}{12}J_1^2 J_0\right)[V_1, [V_1, V_0]]\\
&\quad + \left(J_{220} - \tfrac{1}{2}J_2 J_{20} + \tfrac{1}{12}J_2^2 J_0\right)[V_2, [V_2, V_0]]\,.
\end{aligned}$$

Global error III

$$\mathcal{R} = \sum_{n=0}^{N-1} \sum_{\alpha} \left(\varphi t_{n+1},\, t_N V_\alpha \varphi_{t_0,t_n} \circ y_0 \right) J_\alpha(t_n, t_{n+1}) \,.$$

$$\begin{aligned}
&\mathbb{E}\big(\mathcal{R}^T \mathcal{R}\big) \\
&= \sum_{n=0}^{N-1} \sum_{\alpha,\beta} \mathbb{E}\Big(\big(\varphi_{t_{n+1},t_N} V_\alpha \varphi_{t_0,t_n} \circ y_0\big)^T \big(\varphi_{t_{n+1},t_N} V_\beta \varphi_{t_0,t_n} \circ y_0\big)\Big) \cdot \\
&\qquad \cdot \mathbb{E}\big(J_\alpha(t_n, t_{n+1})\, J_\beta(t_n, t_{n+1})\big) \\
&+ \sum_{n \neq m} \sum_{\alpha,\beta} \mathbb{E}\Big(\big(\varphi_{t_{n+1},t_N} V_\alpha \varphi_{t_0,t_n} \circ y_0\big)^T \big(\varphi_{t_{m+1},t_N} V_\beta \varphi_{t_0,t_m} \circ y_0\big)\Big) \cdot \\
&\qquad \cdot \mathbb{E}\big(J_\alpha(t_n, t_{n+1})\big)\, \mathbb{E}\big(J_\beta(t_m, t_{m+1})\big) \,.
\end{aligned}$$

Stochastic Riccati system

$$S(t) = I + \sum_{i=0}^{d} \int_0^t f_i(\tau, S(\tau))\,\mathrm{d}W_i(\tau)\,.$$

$$f_i(t, S) = S(t)A_i(t)S(t) + B_i(t)S(t) + S(t)C_i(t) + D_i(t)\,.$$

- Stochastic linear-quadratic optimal control.
- eg. mean-variance hedging in finance (*Bobrovnytska & Schweizer* 2004; *Kohlmann & Tang* 2003).

Riccati II

If $\mathbb{A}_i(t) \equiv \begin{pmatrix} B_i(t) & D_i(t) \\ -A_i(t) & -C_i(t) \end{pmatrix}$ and $\mathbb{U} = \begin{pmatrix} U \\ V \end{pmatrix}$ satisfies

$$\mathbb{U}(t) = \mathbb{I} + \sum_{i=0}^{d} \int_0^t \mathbb{A}_i(\tau)\mathbb{U}(\tau)\,\mathrm{d}W_i(\tau)\,,$$

then $S = UV^{-1}$ solves the Riccati system.

$$A_0 = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}, \quad C_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad D_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

$D_1 = a_1$ and $D_2 = a_2$.

Riccati III

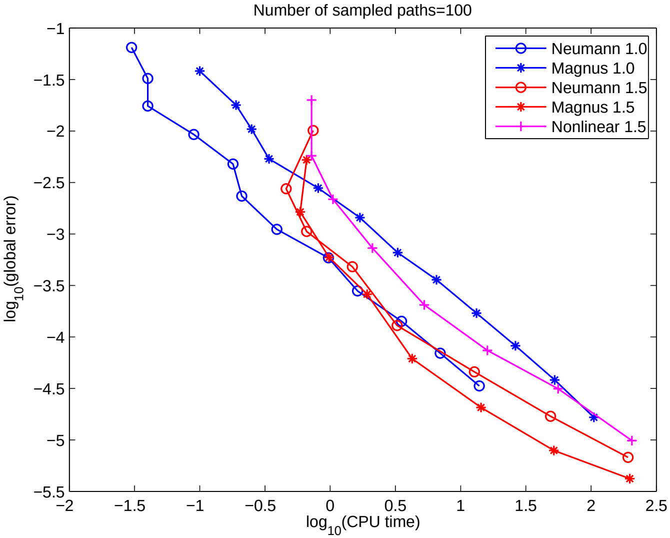
Kloeden & Platen:

$$\begin{aligned} S_{n+1} = {} & S_n + f(S_n)h + D_1 J_1 + D_2 J_2 \\ & + \tfrac{h}{4}\big(f(Y_1^+) + f(Y_1^-) + f(Y_2^+) + f(Y_2^-) - 4f(S_n)\big) \\ & + \tfrac{1}{2\sqrt{h}}\left(\big(f(Y_1^+) - f(Y_1^-)\big)J_{10} + \big(f(Y_2^+) - f(Y_2^-)\big)J_{20}\right) \,, \end{aligned}$$

$$Y_j^{\pm} = S_n + \tfrac{h}{2} f(S_n) \pm D_j \sqrt{h}$$

$$f(S) = S A_0 S + B_0 S + S C_0 + D_0 \,.$$

Riccati IV



Number of sampled paths=100
Neumann 1.0
Magnus 1.0
Neumann 1.5
Magnus 1.5
Nonlinear 1.5
log10(global error)
log10(CPU time)
−1
−1.5
−2
−2.5
−3
−3.5
−4
−4.5
−5
−5.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5