Numerical evaluation of the Evans function

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Outline

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Take home message:

For non-selfadjoint stiff problems, the Evans function method, which is a shooting & matching technique, is the most accurate or even the only approach.

Miss distance function

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$$y' = A(x,\lambda)\,y\,, \qquad \text{where} \quad A(x,\lambda) = \begin{pmatrix} 0 & 1 \\ q(x)-\lambda & 0 \end{pmatrix}, \qquad \textbf{(*)}$$

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Denote by $y^-(x)$ the solution of **(*)** with $y^-(a) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The miss-distance function is

$$D(\lambda) = y_1^-(b).$$

Eigenvalues correspond to zeros of the miss-distance function.

The matching point ξ

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The miss-distance function, evaluated at $\xi \in [a, b]$ is

$$D(\lambda) = \det \begin{pmatrix} y_1^-(\xi) & y_1^+(\xi) \\ y_2^-(\xi) & y_2^+(\xi) \end{pmatrix}\,.$$

For $\xi = b$, we get $D(\lambda) = y_1^-(b)$, as before.

General spectral problems

For $x \in \mathbb{R}$ consider

$$y' = A(x, \lambda)\, y\,,$$

with $y(x) \in \mathbb{C}^n$.

We assume there is a region $\Omega \subset \mathbb{C}$ such that for all $\lambda \in \Omega$:

- $A(x, \lambda) \to A^{\pm}(\lambda)$ as $x \to \pm\infty$;
- $A^{\pm}(\lambda)$ are hyperbolic;
- $A^{-}(\lambda)$ has k unstable eigenvalues $\mu_1^-, \ldots, \mu_k^-$, with corresponding eigenvectors $v_1^-, \ldots, v_k^-$;
- $A^{+}(\lambda)$ has $n-k$ unstable eigenvalues $\mu_1^+, \ldots, \mu_{n-k}^+$, with corresponding eigenvectors $v_1^+, \ldots, v_{n-k}^+$.

General spectral problems II

Hence there exist:

- k solutions: $y_i^-(x) \sim \mathrm{e}^{\mu_i^- x} v_i^-$ as $x \to -\infty$.
- $n-k$ solutions: $y_i^+(x) \sim \mathrm{e}^{\mu_i^+ x} v_i^+$ as $x \to +\infty$.

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The Evans function is defined by

$$D(\lambda) = \det\bigl(y_1^-(\xi) \cdots y_k^-(\xi)\, y_1^+(\xi) \cdots y_{n-k}^+(\xi)\bigr)\,.$$

It is analytic in Ω and its zeros correspond to eigenvalues.
(Evans '75; Alexander, Gardner & Jones '90; Sandstede '02)

Outline

Computing the Evans function numerically

$$D(\lambda) = \det\big(y_1^-(\xi)\ \cdots\ y_k^-(\xi)\, y_1^+(\xi)\ \cdots\ y_{n-k}^+(\xi)\big)\,.$$

Basic numerical computation:

- Compute the unstable eigenvectors $v_1^-, \ldots, v_k^-$ of A^-.
- For $i = 1, \ldots, k$, solve $y' = A(x, \lambda)\, y$ with initial condition $y(-L) = v_i^-$ (where L is large) to get $y_i^-(\xi)$.
- Compute $y_i^+(\xi)$ similarly, and calculate the determinant.

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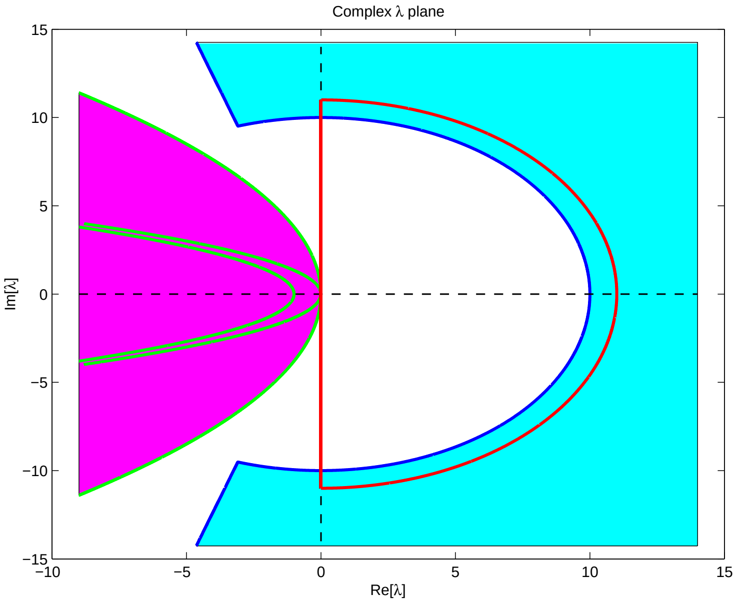
The Evans function is analytic, so we can use the argument principle to count the number of eigenvalues in a given region. **(Evans & Faroe '77)**

Use Newton's method to solve $D(\lambda) = 0$ and locate eigenvalues. **(Pego, Smereka & Weinstein '93)**

Spectrum structure



Problems

- Unstable space $\Rightarrow$ for example y_i^- grow with rate μ_i^-.

  Solution: Rescale: write $y_i^- = \mathrm{e}^{\mu_i^- x} u_i^-$ and solve

  $$u' = \big(A(x,\lambda) - \mu_i^- I\big)\, u\,.$$

  with $u_i(x) \to v_i^-$ as $x \to -\infty$.

- Eigenvectors $v_i^\pm$ must be analytic functions of λ.

  Solution: Kato's algorithm.

- If $\operatorname{Re}\mu_1^- > \operatorname{Re}\mu_2^-$, then u_1^- can be computed accurately, but when computing u_2^- any errors in the u_1^- direction dominate the u_2^- solution.

  Solution: Do not look at the y_i^- individually, but look at the subspace $S = \operatorname{span}\{y_1^-, \ldots, y_k^-\}$ and lift the equation $y' = A(x,\lambda)\, y$ to $S' = \ell\big(A(x,\lambda)\big)\, S$.

Outline

Evans function

For $v_i \in \mathbb{C}^n$:

$$\det\big(v_1 \ \cdots \ v_n\big) \equiv v_1 \wedge \cdots \wedge v_n \, .$$

Hence Evans function

$$D(\lambda) \equiv \mathrm{e}^{-\int_0^\xi \mathrm{Tr}\, A(x,\lambda)\mathrm{d}x} \big(\underbrace{y_1^- \wedge \cdots \wedge y_k^-}_{\mathbf{w}^-(\xi,\lambda)} \wedge \underbrace{y_1^+ \wedge \cdots \wedge y_{n-k}^+}_{\mathbf{w}^+(\xi,\lambda)}\big) \, .$$

Prefactor ensures ξ-independence, from Abel's theorem.

Exterior product of a vector space

Let V be a vector space with basis $e_1, \ldots, e_n$.

The exterior product space $\Lambda^k(V)$ is a vector space with basis

$$\{e_{i_1} \wedge \ldots \wedge e_{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq n\}.$$

For example, $\Lambda^2(\mathbb{C}^4)$ is six dimensional with basis

$$e_1 \wedge e_2,\; e_1 \wedge e_3,\; e_1 \wedge e_4,\; e_2 \wedge e_3,\; e_2 \wedge e_4,\; e_3 \wedge e_4 .$$

The Grassmannian manifold $G_k(V)$ is the set of k-dimensional subspaces of V. We consider the identification

$$\mathrm{span}\{v_1, \ldots, v_k\} \in G_k(V) \leftrightarrow v_1 \wedge \ldots \wedge v_k \in \Lambda^k(V).$$

The Grassmannian manifold

We can embed $G_k(V)$ in $\Lambda^k(V)$, or to be precise, $\mathbb{P}\big(\Lambda^k(V)\big)$.

A form $w \in \Lambda^k(V)$ is decomposable if it can be written as $w = v_1 \wedge \ldots \wedge v_k$ with $v_i \in V$.

Only decomposable forms correspond to subspaces.

Consider for example $\Lambda^2(\mathbb{C}^4)$. The form

$$S_1 e_1 \wedge e_2 + S_2 e_1 \wedge e_3 + S_3 e_1 \wedge e_4 + S_4 e_2 \wedge e_3 + S_5 e_2 \wedge e_4 + S_6 e_3 \wedge e_4$$

is decomposable iff

$$S_1 S_6 - S_2 S_5 + S_3 S_4 = 0 \,.$$

Lifting the differential equation

A linear differential equation on V

$$y' = A(x)\,y \qquad \textbf{(*)}$$

induces an equation on $\Lambda^k(V)$:

$$w' = \ell\big(A(x)\big)\,w \qquad \textbf{(}\dagger\textbf{)}\,.$$

For example with $k = 2$ and $V = \mathbb{C}^n$.

If y_1 and y_2 solve **(*)** then $w = y_1 \wedge y_2$ solves **(†)**.

Outline

Computing the Evans function II

Now basic numerical computation:

- Compute the unstable eigenvectors $v_1^-, \dots, v_k^-$ of A^-.
- Lift the equation to $\Lambda^k(\mathbb{C}^n)$: $w' = \ell\big(A(x,\lambda)\big)\,w$; and rescale $w(x) = \mathrm{e}^{(\mu_1^- + \cdots + \mu_k^-)x}\,u(x)$ to solve

$$u' = \Big(\ell\big(A(x,\lambda)\big) - (\mu_1^- + \cdots + \mu_k^-)\,I\Big)\,u\,,$$

  with initial condition $u(-L) = v_1^- \wedge \cdots \wedge v_k^-$ (where L is large) to get $w^-(\xi)$.
- Compute $w^+(\xi)$ similarly, and evaluate

$$D(\lambda) = w^- \wedge w^+\,.$$

(Bridges '99; Brin '00; Afendikov & Bridges '01)

Same as Compound matrix method.
(Davey '79; Ng & Reid '79)

Magnus series

We need to solve the linear differential equation

$$y' = A(x)\, y\,.$$

Solution

$$y(x) = \exp\big(\sigma(x)\big)\, y(0)\,,$$

where

$$\sigma(x) = \int_0^x A(\xi)\,\mathrm{d}\xi + \frac{1}{2}\int_0^x \int_0^{\xi_1} [A(\xi_1), A(\xi_2)]\,\mathrm{d}\xi_2 \mathrm{d}\xi_1 + \cdots\,.$$

Converges if $\int_0^x \|A(\xi)\|\mathrm{d}\xi < \pi$. **(Moan & Niesen '06)**

Magnus numerical method

Truncate Magnus series and replace $A(x)$ by interpolant at Gauss–Legendre points $x_{1,2} = \left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right)x$:

$$y(x) \approx \exp\left(\tfrac{1}{2}x\big(A(x_1) + A(x_2)\big) - \tfrac{\sqrt{3}}{12}\big[A(x_1), A(x_2)\big] \right) y(0) \,.$$

This is a method of order four. **(Iserles & Norsett '99)**

Preserves Grassmannian invariants (decomposability conditions). **(Niesen '05)**

Other numerical methods

- In the case $k = 2$ and $n = 4$, the Grassmannian is attractive, if we replace $w' = \ell\big(A(x,\lambda)\big)\, w$ by $w' = \ell\big(A(x,\lambda - \sigma I)\big)\, w$, where σ is the largest eigenvalue, provided the spectrum of A changes not too much as x varies.

  (Bridges, Derks & Gottwald '02)

- If $\operatorname{tr} A = 0$, the Grassmannian is a strong quadratic invariant. Gauss–Legendre methods (eg. implicit midpoint rule) conserve these. **(Allen & Bridges '02)**

- In practice, with proof when $n = 2$, fourth order Gauss–Legendre is globally most accurate.

  (Aparicio, Malham & Oliver '05; Malham & Niesen '06)

Outline

Autocatalytic model

The autocatalytic reaction $U + mV \to (m+1)V$ is modeled by

$$u_t = \delta u_{xx} - uv^m \,,$$
$$v_t = v_{xx} + uv^m \,.$$

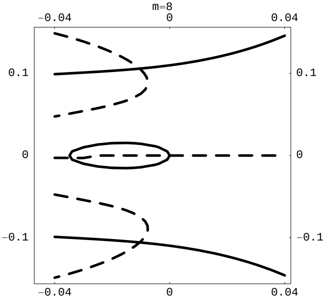
There is a unique travelling wave solution with $(u, v) \to (0, 1)$ as $x \to -\infty$ and $(u, v) \to (1, 0)$ as $x \to +\infty$ for any speed $c \in [c_*, \infty)$.

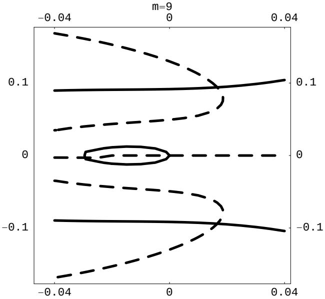
Evans function evaluated with the Magnus and Gauss–Legendre methods to assess the stability of travelling wave with $c = c_*$.

Precomputation:: $w' = A(x, \lambda)\, w$ with $w(-L) = w_0$.

$$A(x, \lambda) = A_0(x) + \lambda\, A_1(x) + \mu\, A_2(x)\,.$$

Autocatalytic model II





Stability of rotating Ekman layer

Linearization of the 3d Navier–Stokes equation in a rotating frame about the Ekman layer coupled to a compliant surface leads to

$$\begin{aligned} \phi'''' - b(x)\phi'' - a(x)\phi + 2\psi' &= 0\,, \\ \psi'' + \big(\gamma^2 - b(x)\big)\psi - \mathrm{i}\gamma R V'(x)\phi - 2\phi' &= 0\,, \end{aligned}$$

for $0 \leq x < \infty$ with compliant surface BCs at $x = 0$.

Lifting yields an ODE on $\Lambda^3(\mathbb{C}^6)$, which has dimension 20. However the dimension of the Grassmannian is only 9.

(Allen & Bridges '03)

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- **(+)** Continuous orthogonalization.

Continuous orthogonalization

(Humphreys & Zumbrun '05)

With

$$Y = \left(y_1^- \ \cdots \ y_k^-\right),$$

consider polar decomposition

$$Y = \Omega\,\alpha\,, \qquad \det\alpha = \gamma\,.$$

Then

$$\Omega' = \left(I - \Omega\Omega^*\right) A(x,\lambda)\,\Omega\,,$$
$$\gamma' = \operatorname{Tr}\left(\Omega^* A(x,\lambda)\Omega\right)\gamma\,.$$

Integrate: need to preserve Stiefel manifold: $\Omega^*\Omega = I$.