# Numerical evaluation of the Evans function

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March 1, 2006

# Outline

1 Evans function: definition and properties
2 Numerical computation
3 Exterior product spaces
4 Numerical methods
5 Examples

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5. Examples

**Take home message:**

*For non-selfadjoint stiff problems, the Evans function method, which is a shooting & matching technique, is the most accurate or even the only approach.*

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$$y' = A(x, \lambda)\, y\,, \qquad \text{where} \quad A(x, \lambda) = \begin{pmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{pmatrix}\,, \qquad (*)$$

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with boundary conditions $y_1(a) = y_1(b) = 0$.

Denote by $y^-(x)$ the solution of (*) with $y^-(a) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The miss-distance function is

$$D(\lambda) = y_1^+(b).$$

Eigenvalues correspond to zeros of the miss-distance function.

# The matching point $\xi$

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Denote by $y^+(x)$ the solution of **(*)** with $y^+(b) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

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The miss-distance function, evaluated at $\xi \in [a,b]$ is

$$D(\lambda) = \det \begin{pmatrix} y_1^-(\xi) & y_1^+(\xi) \\ y_2^-(\xi) & y_2^+(\xi) \end{pmatrix}\,.$$

For $\xi = b$, we get $D(\lambda) = y_1^-(b)$, as before.

# General spectral problems

Consider $y' = A(x,\lambda)\, y$ with $y(x) \in \mathbb{C}^n$ for $x \in \mathbb{R}$.

We assume there is a region $\Omega \subset \mathbb{C}$ such that for all $\lambda \in \Omega$:

- $A^{\pm}(\lambda) = \lim_{x \to \pm\infty} A(x,\lambda)$ exist and are hyperbolic;
- $A^{-}(\lambda)$ has $k$ unstable eigenvalues $\mu_1^-, \ldots, \mu_k^-$;
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Let $y_i^-$ be the solution $y_i^-(x) \sim \mathrm{e}^{\mu_i^- x} v_i^-$ as $x \to -\infty$.

Let $y_i^+$ be the solution $y_i^+(x) \sim \mathrm{e}^{\mu_i^+ x} v_i^+$ as $x \to +\infty$.

The Evans function is defined by

$$D(\lambda) = \det\left(y_1^-(\xi) \cdots y_k^-(\xi)\, y_1^+(\xi) \cdots y_{n-k}^+(\xi)\right).$$

It is analytic in $\Omega$ and its zeros correspond to eigenvalues.

**(Evans '75; Alexander, Gardner & Jones '90; Sandstede '02)**

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# Computing the Evans function numerically

$$D(\lambda) = \det\bigl(y_1^-(\xi) \cdots y_k^-(\xi)\, y_1^+(\xi) \cdots y_{n-k}^+(\xi)\bigr)\,.$$

Basic numerical computation:

- Compute the unstable eigenvectors $v_1^-, \ldots, v_k^-$ of $A^-$.
- For $i = 1, \ldots, k$, solve $y' = A(x, \lambda)\, y$ with initial condition $y(-L) = v_i^-$ (where $L$ is large) to get $y_i^-(\xi)$.
- Compute $y_i^+(\xi)$ similarly, and calculate the determinant.

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$$D(\lambda) = \det\bigl(y_1^-(\xi) \cdots y_k^-(\xi)\, y_1^+(\xi) \cdots y_{n-k}^+(\xi)\bigr)\,.$$

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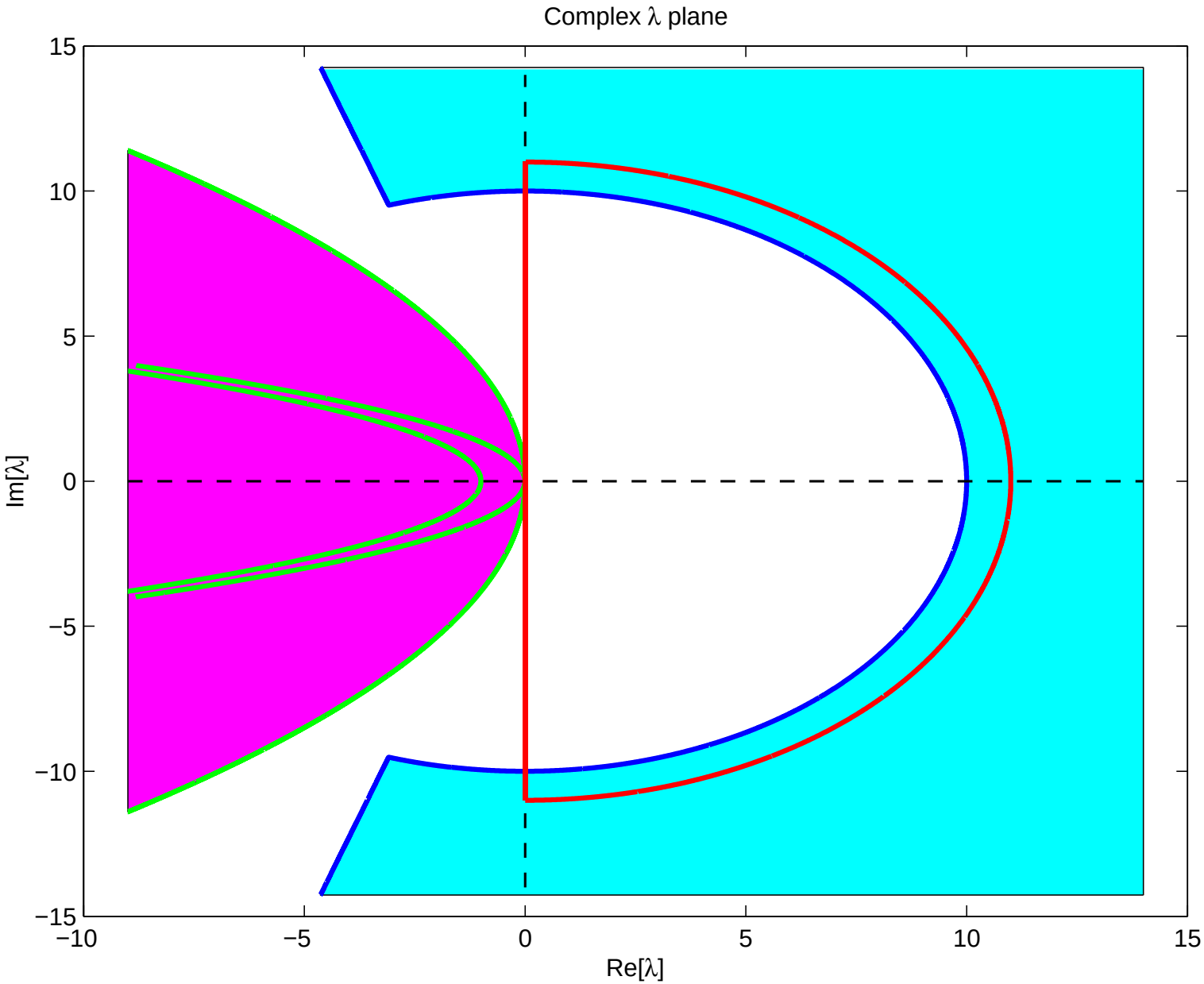
The Evans function is analytic, so we can use the argument principle to count the number of eigenvalues in a given region.
**(Evans & Faroe '77)**

Alternatively, we can use Newton's method to solve $D(\lambda) = 0$ and locate eigenvalues.
**(Pego, Smereka & Weinstein '93)**

# Spectrum structure



# Problems

- Unstable space $\Rightarrow$ for example $y_i^-$ grow with rate $\mu_i^-$.

  **Solution:** Rescale $y_i^- \to \mathrm{e}^{\mu_i^- x} y_i^-$ and solving

  $$y' = \big(A(x,\lambda) - \mu_i^- I\big)\, y\,.$$

- Eigenvectors $v_i^\pm$ must be analytic functions of $\lambda$.

  **Solution:** Kato's algorithm.

- If $\operatorname{Re}\mu_1^- > \operatorname{Re}\mu_2^-$, then $y_1^-$ grows faster than $y_2^-$, and so any errors in the $y_1^-$ direction will dominate the $y_2^-$ solution.

  **Solution:** Do not look at the $Y_i^-$ individually, but look at the subspace $S = \operatorname{span}\{y_1^-, \ldots, y_k^-\}$ and lift the equation $y' = A(x,\lambda)\, y$ to $S' = \ell\big(A(x,\lambda)\big)\, S$.

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# Evans function

For $v_i \in \mathbb{C}^n$:

$$\det\big(v_1 \cdots v_n\big) \equiv v_1 \wedge \cdots \wedge v_n \,.$$

Hence Evans function

$$D(\lambda) \equiv \mathrm{e}^{-\int_0^\xi \mathrm{Tr}\, A(x,\lambda)\mathrm{d}x} \big(\underbrace{y_1^- \wedge \cdots \wedge y_k^-}_{\mathbf{w}^-(\xi,\lambda)} \wedge \underbrace{y_1^+ \wedge \cdots \wedge y_{n-k}^+}_{\mathbf{w}^+(\xi,\lambda)}\big) \,.$$

Prefactor ensures $\xi$-independence, from Abel's theorem.

# Exterior product of a vector space

Let $V$ be a vector space with basis $e_1, \ldots, e_n$.

The exterior product space $\Lambda^k(V)$ is a vector space with basis

$$\{e_{i_1} \wedge \ldots \wedge e_{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq n\}.$$

For example, $\Lambda^2(\mathbb{C}^4)$ is six dimensional with basis

$$e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_1 \wedge e_4,\ e_2 \wedge e_3,\ e_2 \wedge e_4,\ e_3 \wedge e_4\,.$$

The Grassmannian manifold $G_k(V)$ is the set of $k$-dimensional subspaces of $V$. We consider the identification

$$\operatorname{span}\{v_1, \ldots, v_k\} \in G_k(V) \leftrightarrow v_1 \wedge \ldots \wedge v_k \in \Lambda^k(V)\,.$$

# The Grassmannian manifold

We can embed $G_k(V)$ in $\Lambda^k(V)$, or to be precise, $\mathbb{P}\big(\Lambda^k(V)\big)$.

A form $w \in \Lambda^k(V)$ is decomposable if it can be written as $w = v_1 \wedge \ldots \wedge v_k$ with $v_i \in V$.

Only decomposable forms correspond to subspaces.

Consider for example $\Lambda^2(\mathbb{C}^4)$. The form

$$S_1 e_1 \wedge e_2 + S_2 e_1 \wedge e_3 + S_3 e_1 \wedge e_4 + S_4 e_2 \wedge e_3 + S_5 e_2 \wedge e_4 + S_6 e_3 \wedge e_4$$

is decomposable iff

$$S_1 S_6 - S_2 S_5 + S_3 S_4 = 0\,.$$

# Lifting the differential equation

A linear differential equation on $V$

$$y' = A(x)\, y \qquad (*)$$

induces an equation on $\Lambda^k(V)$:

$$w' = \ell\big(A(x)\big)\, w \qquad (\dagger)\,.$$

For example with $k = 2$ and $V = \mathbb{C}^n$.

If $y_1$ and $y_2$ solve $(*)$ then $w = y_1 \wedge y_2$ solves $(\dagger)$.

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# Computing the Evans function II

Now basic numerical computation:

- Compute the unstable eigenvectors $v_1^-, \ldots, v_k^-$ of $A^-$.
- Lift the equation to $\Lambda^k(\mathbb{C}^n)$:

$$w' = \ell\big(A(x,\lambda)\big)\, w\,,$$

with initial condition $w(-L) = v_1^- \wedge \cdots \wedge v_k^-$ (where $L$ is large) to get $w^-(\xi)$.

- Compute $w^+(\xi)$ similarly, and evaluate

$$D(\lambda) = w^- \wedge w^+\,.$$

**(Bridges '99; Brin '00; Afendikov & Bridges '01)**

Same as Compound matrix method.
**(Davey '79; Ng & Reid '79)**

# Magnus series

We need to solve the linear differential equation

$$y' = A(x)\, y\,.$$

Solution

$$y(x) = \exp\bigl(\sigma(x)\bigr)\, y(0)\,,$$

where

$$\sigma(x) = \int_0^x A(\xi)\,\mathrm{d}\xi + \frac{1}{2}\int_0^x \int_0^{\xi_1} [A(\xi_1), A(\xi_2)]\,\mathrm{d}\xi_2 \mathrm{d}\xi_1 + \cdots\,.$$

Converges if $\int_0^x \|A(\xi)\|\mathrm{d}\xi < \pi$. **(Moan & Niesen '06)**

# Magnus numerical method

Truncate Magnus series and replace $A(x)$ by interpolant at Gauss–Legendre points $x_{1,2} = \left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right)x$:

$$y(x) \approx \exp\left(\tfrac{1}{2}x\big(A(x_1) + A(x_2)\big) - \tfrac{\sqrt{3}}{12}\big[A(x_1), A(x_2)\big]\right) y(0)\,.$$

This is a method of order four. **(Iserles & Norsett '99)**

Preserves Grassmannian invariants (decomposability conditions). **(Niesen '05)**

# Other numerical methods

- In the case $k = 2$ and $n = 4$, the Grassmannian is attractive, if we replace $w' = \ell\big(A(x, \lambda)\big)\, w$ by $w' = \ell\big(A(x, \lambda - \sigma I)\big)\, w$, where $\sigma$ is the largest eigenvalue, provided the spectrum of $A$ changes not too much as $x$ varies.
  **(Bridges, Derks & Gottwald '02)**
- If $\operatorname{tr} A = 0$, the Grassmannian is a strong quadratic invariant. Gauss–Legendre methods (eg. implicit midpoint rule) conserve these. **(Allen & Bridges '02)**
- In practice, with proof when $n = 2$, fourth order Gauss–Legendre is globally most accurate.
  **(Aparicio, Malham & Oliver '05; Malham & Niesen '06)**

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# Autocatalytic model

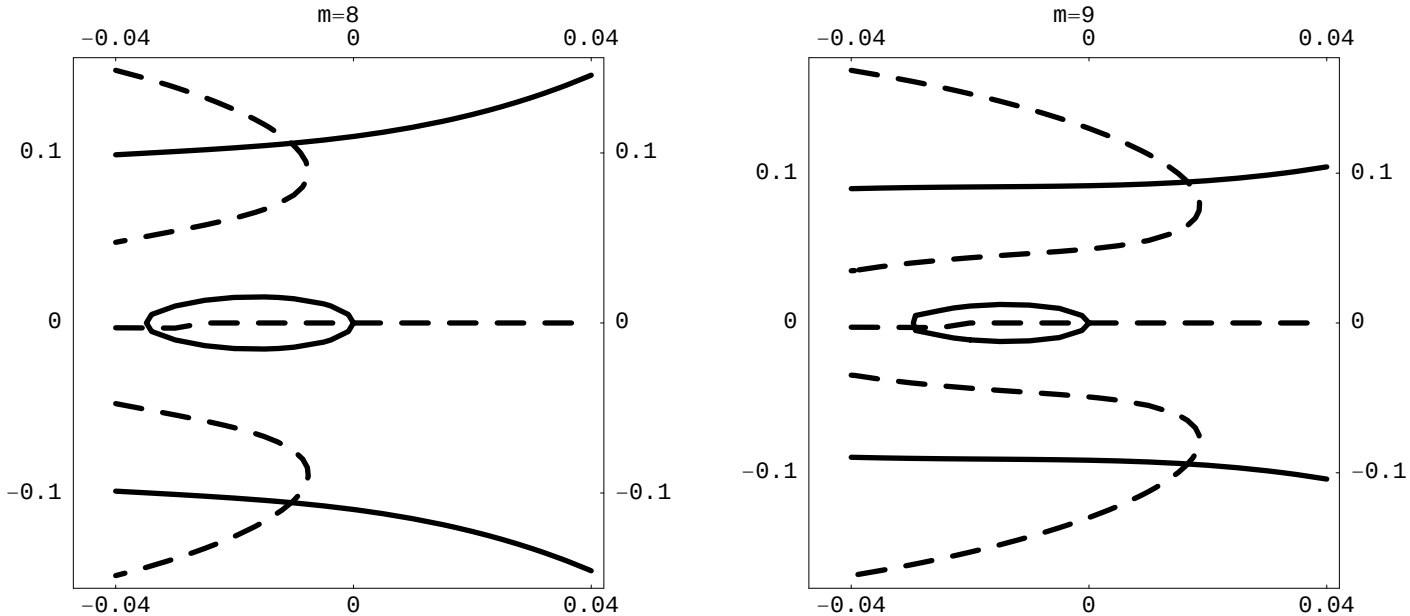
The autocatalytic reaction $U + mV \longrightarrow (m+1)V$ is modeled by

$$u_t = \delta u_{xx} - uv^m\,,$$
$$v_t = v_{xx} + uv^m\,.$$

There is a unique travelling wave solution with $(u, v) \longrightarrow (0, 1)$ as $x \longrightarrow -\infty$ and $(u, v) \longrightarrow (1, 0)$ as $x \longrightarrow +\infty$ for any speed $c \in [c_*, \infty)$.

Evans function evaluated with the Magnus and Gauss–Legendre methods to assess the stability of travelling wave with $c = c_*$.

# Autocatalytic model II



# Stability of rotating Ekman layer

Linearization of the 3d Navier–Stokes equation in a rotating frame about the Ekman layer coupled to a compliant surface leads to

$$\phi'''' - b(x)\phi'' - a(x)\phi + 2\psi' = 0\,,$$
$$\psi'' + \big(\gamma^2 - b(x)\big)\psi - \mathrm{i}\gamma R V'(x)\phi - 2\phi' = 0\,,$$

for $0 \leq x < \infty$ with compliant surface BCs at $x = 0$.

Lifting yields an ODE on $\Lambda^3(\mathbb{C}^6)$, which has dimension 20.
However the dimension of the Grassmannian is only 9.

**(Allen & Bridges '03)**

# Stability of viscous shock waves

A viscous shock wave is a travelling wave solution to the conservation law

$$u_t + \big(f(u)\big)_x = \big(B(u)\, u_x\big)_x \,.$$

As an example consider the cubic model in MHD

$$u_t + \big(|u|^2 u\big)_x = u_{xx} \,.$$

**(Brin '00; Brin & Zumbrun '03)**

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- **(-)** $\dim\left(\Lambda^k(\mathbb{C}^n)\right) = \binom{n}{k}$.
- **(+)** Continuous orthogonalization

**(Humphreys & Zumbrun '05)**