High order integrators for linear stochastic systems

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Introduction

$$\mathrm{d}S = A_0(\tau)S\,\mathrm{d}\tau + A_1(\tau)S\,\mathrm{d}W_1 + A_2(\tau)S\,\mathrm{d}W_2$$

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1. Neumann & Magnus series
2. Global error: Neumann vs Magnus
3. Quadrature & efficiency
4. Numerical experiments
   - Linear SDE
   - Nonlinear Riccati SDE
5. Conclusions

Introduction

Take home message:

> *For a given computational effort, order* 1 *Magnus integrators are the most accurate of all numerical methods.*

Neumann series

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$$S(t) = I + \int_0^t A_0 S\,\mathrm{d}\tau + \int_0^t A_1 S\,\mathrm{d}W_1(\tau) + \int_0^t A_2 S\,\mathrm{d}W_2(\tau)$$

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$$(\mathsf{I} - \mathsf{K}) \circ S = I$$

$$S = (\mathsf{I} - \mathsf{K})^{-1} \circ I$$

$$S = (\mathsf{I} + \mathsf{K} + \mathsf{K}^2 + \mathsf{K}^3 + \mathsf{K}^4 + \cdots) \circ I$$

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- *Peano–Baker series*, *Feynman–Dyson path ordered exponential*, *Chen-Fleiss series* or *Neumann series*

Magnus series

$$S(t) = \exp\left(\sigma(t)\right) \ ,$$

$$\begin{aligned}\sigma(t) &= \ \ln\big(I + \mathsf{K}\circ I + \mathsf{K}^2\circ I + \cdots\big) \\ &= \mathsf{K}\circ I + \mathsf{K}^2\circ I - \tfrac{1}{2}(\mathsf{K}\circ I)^2 + \cdots \ ,\end{aligned}$$

- *Magnus* 1954, *Kunita* 1980, *Ben Arous* 1989, *Burrage* 1999.

Integral operators

$$
\begin{aligned}
\mathsf{K} \circ I &= \mathsf{K}_0 \circ I + \mathsf{K}_1 \circ I + \mathsf{K}_2 \circ I \\
&= \int_0^t A_0(\tau)\,\mathrm{d}\tau + \int_0^t A_1(\tau)\,\mathrm{d}W_1(\tau) + \int_0^t A_2(\tau)\,\mathrm{d}W_2(\tau)
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{K}^2 \circ I &= (\mathsf{K}_0 + \mathsf{K}_1 + \mathsf{K}_2)^2 \circ I \\
&= (\mathsf{K}_0^2 + \mathsf{K}_0\mathsf{K}_1 + \mathsf{K}_1\mathsf{K}_0 + \mathsf{K}_1^2 + \mathsf{K}_1\mathsf{K}_2 + \mathsf{K}_2\mathsf{K}_1 + \cdots) \circ I
\end{aligned}
$$

$$
\mathsf{K}_i\mathsf{K}_j \circ I \equiv \int_0^t A_i(\tau_1) \int_0^{\tau_1} A_j(\tau_2)\,\mathrm{d}W_j(\tau_2)\mathrm{d}W_i(\tau_1)
$$

Constant coefficient, non-commutative case

$$\mathsf{K}_j\mathsf{K}_i \circ I \equiv a_j a_i \underbrace{\int_0^t \int_0^{\tau_1} \mathrm{d}W_i(\tau_2)\mathrm{d}W_j(\tau_1)}_{J_{ij}}$$

$$\mathsf{K}_k\mathsf{K}_j\mathsf{K}_i \circ I \equiv a_k a_j a_i \underbrace{\int_0^t \int_0^{\tau_1} \int_0^{\tau_2} \mathrm{d}W_i(\tau_3)\mathrm{d}W_j(\tau_2)\mathrm{d}W_k(\tau_1)}_{J_{ijk}}$$

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Special non-commutative cases:

$$S(t) = I + a_1 \cdot \int_0^t S(\tau)\,\mathrm{d}W_1(\tau) + \int_0^t S(\tau)\,\mathrm{d}W_2(\tau) \cdot a_2\,,$$

$$S(t) = \exp\big(a_1 W_1(t)\big) \cdot \exp\big(a_2 W_2(t)\big).$$

Neumann expansion (order 3/2)

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S_1 &= a_2 a_1 J_{12} + a_1 a_2 J_{21} , \\
S_{3/2} &= a_0 a_1 J_{10} + a_1 a_0 J_{01} + a_0 a_2 J_{20} + a_2 a_0 J_{02} \\
&\quad + a_1^3 J_{111} + a_2 a_1^2 J_{112} + a_1 a_2 a_1 J_{121} + a_2^2 a_1 J_{122} \\
&\quad + a_1^2 a_2 J_{211} + a_2 a_1 a_2 J_{212} + a_1 a_2^2 J_{221} + a_2^3 J_{222} \\
&\quad + a_0^2 J_{00} + a_1^2 a_0 J_{011} + a_1 a_0 a_1 J_{101} + a_0 a_1^2 J_{110} + a_2^2 a_0 J_{022} \\
&\quad + a_2 a_0 a_2 J_{202} + a_0 a_2^2 J_{220} + a_1^4 J_{1111} + a_2^2 a_1^2 J_{1122} \\
&\quad + a_2 a_1 a_2 a_1 J_{1212} + a_1 a_2 a_1 a_2 J_{2121} + a_1^2 a_2^2 J_{2211} + a_2^4 J_{2222} .
\end{aligned}$$

Relations between Stratonovich integrals

$$
\begin{aligned}
J_{121} &= J_1 J_{12} - 2J_{112}\,, \\
J_{122} &= J_2 J_{12} - \tfrac{1}{2} J_1 J_2^2 + J_{221}\,, \\
J_{210} &= J_0 J_{21} - J_2 J_{01} + J_{012}\,, \\
J_{102} &= J_1 J_{02} - J_{021} - J_{012}\,.
\end{aligned}
$$

- Shuffle algebras: *Gaines* 1995; *Kawski* 2001.

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s_{1/2} &= a_1 J_1 + a_2 J_2 + a_0 J_0\,,\\
s_1 &= \tfrac{1}{2}[a_1, a_2](J_{21} - J_{12})\,,\\
s_{3/2} &= \tfrac{1}{2}[a_0, a_1](J_{10} - J_{01}) + \tfrac{1}{2}[a_0, a_2](J_{20} - J_{02})\\
&\quad + [a_1, [a_1, a_2]]\left(J_{112} - \tfrac{1}{2}J_1 J_{12} + \tfrac{1}{12}J_1^2 J_2\right)\\
&\quad + [a_2, [a_2, a_1]]\left(J_{221} - \tfrac{1}{2}J_2 J_{21} + \tfrac{1}{12}J_2^2 J_1\right)\\
&\quad + [a_1, [a_1, a_0]]\left(J_{110} - \tfrac{1}{2}J_1 J_{10} + \tfrac{1}{12}J_1^2 J_0\right)\\
&\quad + [a_2, [a_2, a_0]]\left(J_{220} - \tfrac{1}{2}J_2 J_{20} + \tfrac{1}{12}J_2^2 J_0\right)\,.
\end{aligned}$$

Numerical SDE schemes

General integrators

- Euler-Maruyama and Milstein methods
- Runge–Kutta type methods (*Kloeden & Platen* 1999)
- Magnus (*Burrage* 1999; *Misawa* 2001; *P-C. Moan* 2004)
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- **Low dimensional computationally favourable basis**

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- **More accurate.**

Global error I

Global interval: $[0, T] = \cup_{n=0}^{N-1} [t_n, t_{n+1}]\,, \quad t_n = nh.$

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Neumann or Magnus:

$$S(t_n, t_{n+1}) = \underbrace{\hat{S}(t_n, t_{n+1})}_{\text{approx}} + R(t_n, t_{n+1})\,.$$

Global error II

$$\mathcal{E}^2 \equiv \sup_{\|Y_0\|=1} \mathbb{E} \left\| \left(\prod_{n=N-1}^{0} S(t_n, t_{n+1}) - \prod_{n=N-1}^{0} \hat{S}(t_n, t_{n+1}) \right) Y_0 \right\|^2$$

$$= \sup_{\|Y_0\|=1} \mathbb{E} \left\| \underbrace{\left(\sum_{n=0}^{N-1} \hat{S}(t_{n+1}, t_N) R(t_n, t_{n+1}) \hat{S}(t_0, t_n) \right)}_{\mathcal{R}} Y_0 \right\|^2 .$$

$$\prod_{n=N-1}^{0} \left(\hat{S}(t_n, t_{n+1}) + R(t_n, t_{n+1}) \right) - \prod_{n=N-1}^{0} \hat{S}(t_n, t_{n+1})$$

Recall: Magnus expansion (order 1)

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&\quad + [a_1, [a_1, a_2]]\left(J_{112} - \tfrac{1}{2}J_1 J_{12} + \tfrac{1}{12}J_1^2 J_2\right) \\
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\end{aligned}$$

Global error III

$$A_\alpha = [a_1, [a_1, a_2]] \qquad \text{and} \qquad J_\alpha = J_{112} - \tfrac{1}{2} J_1 J_{12} + \tfrac{1}{12} J_1^2 J_2 \,.$$

$$\mathcal{R} = \sum_{n=0}^{N-1} \sum_{\alpha} \big(\hat{S}(t_{n+1}, t_N) A_\alpha \hat{S}(t_0, t_n)\big)\, J_\alpha(t_n, t_{n+1}) \,.$$

$$\begin{aligned}
&\mathbb{E}(\mathcal{R}^T \mathcal{R}) \\
&= \sum_{n=0}^{N-1} \sum_{\alpha,\beta} \mathbb{E}\Big(\big(\hat{S}(t_{n+1}, t_N) A_\alpha \hat{S}(t_0, t_n)\big)^T \big(\hat{S}(t_{n+1}, t_N) A_\beta \hat{S}(t_0, t_n)\big) \Big) \cdot \\
&\qquad\qquad\qquad \cdot \mathbb{E}\big(J_\alpha(t_n, t_{n+1})\, J_\beta(t_n, t_{n+1}) \big) \\
&+ \sum_{n \neq m} \sum_{\alpha,\beta} \mathbb{E}\Big(\big(\hat{S}(t_{n+1}, t_N) A_\alpha \hat{S}(t_0, t_n)\big)^T \big(\hat{S}(t_{m+1}, t_N) A_\beta \hat{S}(t_0, t_m)\big) \Big) \cdot \\
&\qquad\qquad\qquad \cdot \mathbb{E}\big(J_\alpha(t_n, t_{n+1}) \big)\, \mathbb{E}\big(J_\beta(t_m, t_{m+1}) \big) \,.
\end{aligned}$$

Magnus more accurate than Neumann?

For order 1/2 schemes:

$$R^{\text{mag}} = \frac{1}{2}[a_1, a_2](J_{21} - J_{12})\,;$$

$$R^{\text{neu}} = R^{\text{mag}} + \underbrace{\frac{1}{2}(a_1 a_2 + a_2 a_1)(J_{21} + J_{12})}_{\hat{R}}\,.$$

Local error:

$$\begin{aligned}\mathbb{E}\big((R^{\text{neu}})^{\mathsf{T}}\, R^{\text{neu}}\big) &= \mathbb{E}\big((R^{\text{mag}} + \hat{R})^{\mathsf{T}}\, (R^{\text{mag}} + \hat{R})\big) \\ &= \mathbb{E}\big((R^{\text{mag}})^{T} R^{\text{mag}}\big) + \mathbb{E}\big(\hat{R}^{T} \hat{R}\big) \\ &\quad + \underbrace{\mathbb{E}\big(\hat{R}^{T} R^{\text{mag}}\big)}_{=0} + \underbrace{\mathbb{E}\big((R^{\text{mag}})^{T} \hat{R}\big)}_{=0}\,.\end{aligned}$$

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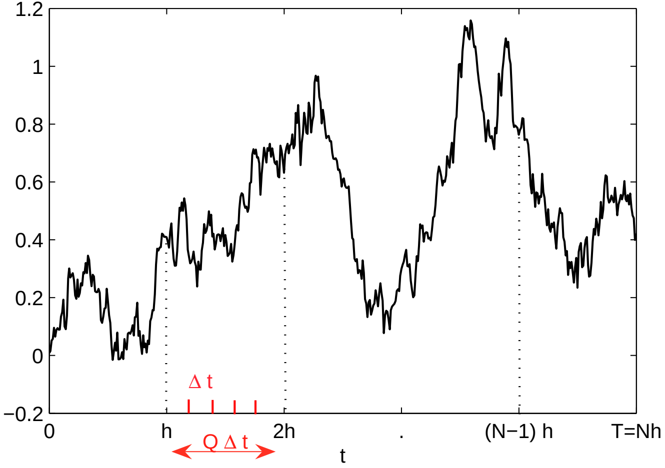
Uniformly accurate Magnus integrator:

$$\begin{aligned}\sigma_1 = {} & a_1 J_1 + a_2 J_2 + a_0 J_0 + \tfrac{1}{2}[a_1, a_2](J_{21} - J_{12}) \\ & + \tfrac{h^2}{12}\big(a_1 a_2^2 a_1 + a_2 a_1^2 a_2\big) \\ & - \tfrac{h^2}{12}\big(a_1 a_0 a_1 + a_2 a_0 a_2\big) \\ & - \tfrac{h^2}{12}\big(a_1 a_2 a_1 a_2 + a_2 a_1 a_2 a_1\big).\end{aligned}$$

Quadrature approximation



Quadrature approximation

Two inherent scales:

- Wiener-discrete-path scale Δt—the *smallest scale* on which the Wiener paths $W_1(t)$ and $W_2(t)$ are generated;
- Time-step scale $h = Q\Delta t$—on which the SDE is stepped forward.

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Practical approach:

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- **Basic idea**: $J \to \mathbb{E}\big(J \big| \mathcal{F}_Q(n)\big)$.

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- **Polygonal volumes.**

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Practical approach:

- $J_{12}(t_n, t_{n+1}) \equiv \int_{t_n}^{t_{n+1}} \big(W_1(\tau) - W_1(t_n)\big)\,\mathrm{d}W_2(\tau)$
- Filtration: $\mathcal{F}_Q = \{W_i(t_n + q\,\Delta t)\colon \text{all } i, n, q\}$.
- Basic idea: $J \to \mathbb{E}\big(J \big| \mathcal{F}_Q(n)\big)$.
- Polygonal volumes.
- **Stump & Hill 2005.**

Quadrature approximation II

$$\hat{J}_{12}(t_n, t_{n+1}) = \frac{1}{2} \sum_{q=0}^{Q-1} \Big(W_1(\tau_{q+1}) + W_1(\tau_q) - 2W_1(\tau_0) \Big) \Delta W_2(t_q)\,.$$

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$$\begin{aligned}
&\mathbb{E}\left(\left| J_{12}(t_n,t_{n+1}) - \hat{J}_{12}(t_n, t_{n+1}) \right|^2 \right) \\
&\qquad = \mathbb{E}\left(\left| \sum_{q=0}^{Q-1} \big(J_{12}(\tau_q, \tau_{q+1}) - \tfrac{1}{2} \Delta W_1(\tau_q) \Delta W_2(\tau_q) \big) \right|^2 \right) \\
&\qquad = \sum_{q=0}^{Q-1} \mathbb{E}\left(\mathrm{Var}\Big[J_{12}(\tau_q, \tau_{q+1}) \Big| \mathcal{F}_Q \Big] \right) \\
&\qquad = \mathcal{O}\big((\Delta t)^2\, Q \big) \\
&\qquad = \mathcal{O}\big(h^2/Q \big)\,.
\end{aligned}$$

Quadrature approximation IV

Quadrature		$\hat{J}_{12}$	$\hat{J}_{112}$	$\hat{J}_{120}$	$\hat{J}_{1112}$
$\mathcal{E}^{\text{loc}}(n)$		$h/Q^{1/2}$	$h^{3/2}/Q^{1/2}$	$h^2/Q^{1/2}$	$h^2/Q^{1/2}$
$\mathcal{U}$	$\mathcal{O}\left(h^{3/2}\right)$	$\mathbf{h^{-1}}$	$\cdots$	$\cdots$	$\cdots$
	$\mathcal{O}\left(h^{2}\right)$	h^{-2}	h^{-1}	$\cdots$	$\cdots$
	$\mathcal{O}\left(h^{5/2}\right)$	$\mathbf{h^{-3}}$	h^{-2}	h^{-1}	h^{-1}

Global error vs computational effort

Theorem

$$\mathcal{U} = \underbrace{\left(k_{\mathcal{U}}\, K_{\mathcal{E}}^{2}\right)\mathcal{E}^{-2}}_{\mathcal{U}^{\text{quad}}} + \underbrace{\left(\left(c_M\, n^2 + c_E\right) K_{\mathcal{E}}^{1/M}\right)\mathcal{E}^{-1/M}}_{\mathcal{U}^{\text{eval}}} .$$

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Proof.

$$\mathcal{E} = K_Q(M)\,\frac{h^{\frac{1}{2}}}{\sqrt{Q}} + K_T(M)\,h^M .$$

$$Q = \frac{1}{h^{2M-1}} \quad \Rightarrow \quad \mathcal{E} = K_{\mathcal{E}}\,h^M .$$

$$\mathcal{U} = \left(k_{\mathcal{U}}\,Q + c_M\,n^2 + c_E\right)N .$$

□

Flops: two Wiener processes

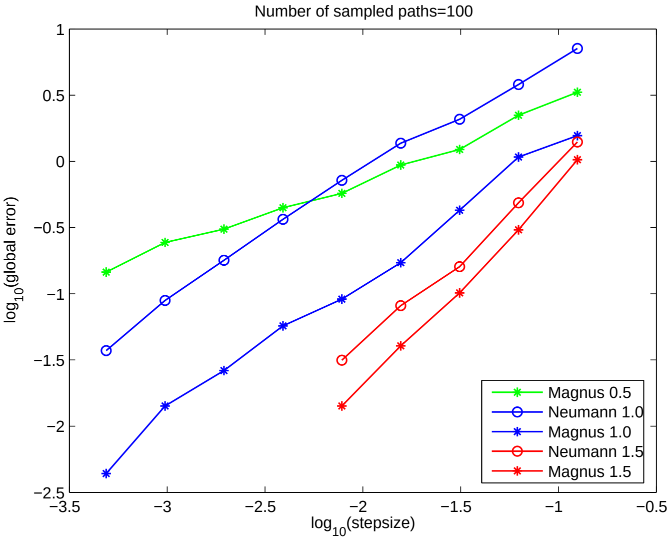
	For each path		
Order	Neumann	Magnus	Runge–Kutta
$\frac{1}{2}$	$9n^2$	$5n^2 + 5n^3$	$\cdots$
1	$13n^2$	$7n^2 + 5n^3$	$\cdots$
$1\frac{1}{2}$	$63n^2$	$19n^2 + 5n^3$	$20n^3 + 37n^2$
2	$95n^2$	$29n^2 + 5n^3$	$\cdots$

Numerical simulations

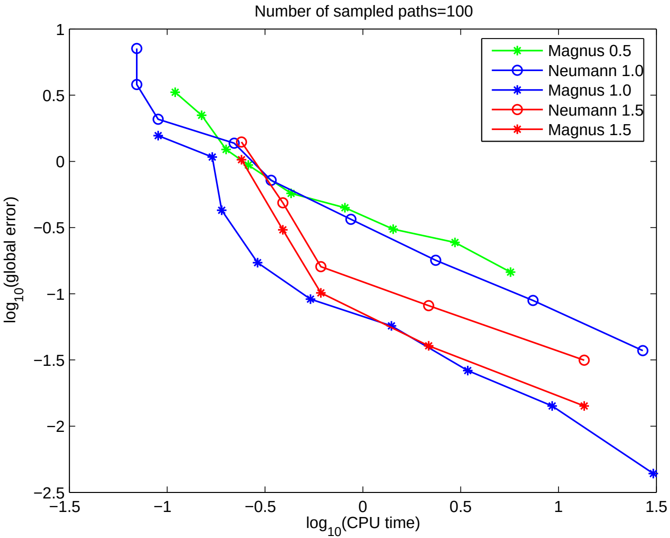
Linear system

$$a_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} , \quad a_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{51}{200} \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} ,$$

Linear system



Linear system



Stochastic Riccati system

$$S(t) = I + \sum_{i=0}^{d} \int_0^t f_i(\tau, S(\tau))\,\mathrm{d}W_i(\tau)\,.$$

$$f_i(t, S) = S(t)A_i(t)S(t) + B_i(t)S(t) + S(t)C_i(t) + D_i(t)\,.$$

- Stochastic linear-quadratic optimal control.
- eg. mean-variance hedging in finance (*Bobrovnytska & Schweizer* 2004; *Kohlmann & Tang* 2003).

Riccati II

If $\mathbb{A}_i(t) \equiv \begin{pmatrix} B_i(t) & D_i(t) \\ -A_i(t) & -C_i(t) \end{pmatrix}$ and $\mathbb{U} = \begin{pmatrix} U \\ V \end{pmatrix}$ satisfies

$$\mathbb{U}(t) = \mathbb{I} + \sum_{i=0}^{d} \int_0^t \mathbb{A}_i(\tau)\mathbb{U}(\tau)\,\mathrm{d}W_i(\tau)\,,$$

then $S = UV^{-1}$ solves the Riccati system.

$$A_0 = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}, \quad C_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad D_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

$D_1 = a_1$ and $D_2 = a_2$.

Riccati III

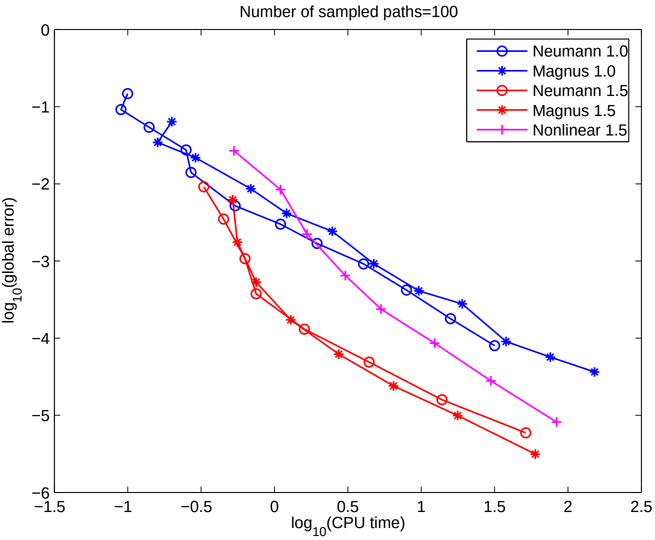
Kloeden & Platen:

$$
\begin{aligned}
S_{n+1} = {} & S_n + f(S_n)h + D_1 J_1 + D_2 J_2 \\
& + \tfrac{h}{4}\big(f(Y_1^+) + f(Y_1^-) + f(Y_2^+) + f(Y_2^-) - 4f(S_n)\big) \\
& + \tfrac{1}{2\sqrt{h}}\Big(\big(f(Y_1^+) - f(Y_1^-)\big)J_{10} + \big(f(Y_2^+) - f(Y_2^-)\big)J_{20}\Big)\ ,
\end{aligned}
$$

$$
Y_j^{\pm} = S_n + \tfrac{h}{2} f(S_n) \pm D_j\sqrt{h}
$$

$$
f(S) = SA_0S + B_0S + SC_0 + D_0\ .
$$

Riccati IV


Number of sampled paths=100
Neumann 1.0
Magnus 1.0
Neumann 1.5
Magnus 1.5
Nonlinear 1.5
log10(global error)
log10(CPU time)
0
−1
−2
−3
−4
−5
−6
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5

Conclusions

Take home message:

For a given computational effort, order 1 *Magnus integrators are the most accurate of all numerical methods.*

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For a given computational effort, order 1 *Magnus integrators are the most accurate of all numerical methods.*

Future directions:

- Variable step scheme (*Gaines & Lyons* 1997).
- Zakai equation: Markov chain filters.
- Lie-group preserving properties (*Castell & Gaines* 1995; *Burrage et al.* 2004; *Misawa* 2001; *Milstein et al.* 2002).
- Backwards SDEs
- Nonlinear SDEs