Grassmannian shooting and the stability of multi-dimensional fronts

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Spectral problems

Parabolic nonlinear systems on $\mathbb{R} \times \mathbb{T}$:

$$\partial_t U = B\,\Delta U + c\,\partial_x U + F(U),$$

Travelling wave U_c. Small perturbations U satisfy:

$$B\,\Delta U + c\,\partial_x U + \mathrm{D}F(U_c)U = \lambda U.$$

Two main solution approaches:

- *Projection.*
- *Shooting.*

Setup

On $\mathbb{R}$: $\quad B\,\Delta U + c\,\partial_x U + \mathrm{D}F(U_c)U = \lambda U$

$$\Leftrightarrow \qquad Y' = A(x;\lambda)\,Y$$

For $\lambda \in \Omega \subseteq \mathbb{C}$: matching condition

$$\begin{aligned}
\mathrm{e}^{\int_0^x \mathrm{Tr}A(\xi;\lambda)\,\mathrm{d}\xi}\,D(\lambda) &:= \det\big(Y_1^- \;\cdots\; Y_k^- \; Y_{k+1}^+ \;\cdots\; Y_n^+\big) \\
&= \det\big(Y^-(x;\lambda)\;\; Y^+(x;\lambda)\big) \\
&= Y_1^- \wedge \cdots \wedge Y_k^- \wedge Y_{k+1}^+ \wedge \cdots \wedge Y_n^+ \\
&= U^-(x;\lambda) \wedge U^+(x;\lambda)
\end{aligned}$$

Numerical issues

- Computational domain.
- Different exponential growth rates.
- Polynomial complexity.
- Flow singularities?!
- Where to match?
- Retaining analyticity.
- How to project transversely.

Stiefel and Grassmann manifolds

- Stiefel manifold:

$$\mathbb{V}(n,k) = \{k\text{-frames centred at the origin}\}.$$

- Grassmann manifold:

$$\mathrm{Gr}(n,k) = \{k\text{-dimensional subspaces of } \mathbb{C}^n\}.$$

- Fibre bundle:

$$\pi\colon \mathbb{V}(n,k) \to \mathrm{Gr}(n,k) \cong \mathbb{V}(n,k)/\mathrm{GL}(k)$$
$$\pi\colon k\text{-frame} \mapsto \text{spanning } k\text{-plane}$$

Representation

$\pi\colon\ Y = y_{\mathfrak{i}^\circ} u \mapsto y_{\mathfrak{i}^\circ}$

Coordinate patches $\mathbb{U}_{\mathfrak{i}}$: multi-index $\mathfrak{i} = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$.

Example: $\mathbb{U}_{\{1,\ldots,k\}}$ uniquely represented by:

$$y_{\mathfrak{i}^\circ} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hat{y}_{k+1,1} & \hat{y}_{k+1,2} & \cdots & \hat{y}_{k+1,k} \\ \hat{y}_{k+2,1} & \hat{y}_{k+2,2} & \cdots & \hat{y}_{k+2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{y}_{n,1} & \hat{y}_{n,2} & \cdots & \hat{y}_{n,k} \end{pmatrix}.$$

Local coordinate chart $\varphi_{\mathfrak{i}}\colon \mathbb{U}_{\mathfrak{i}} \to \mathbb{C}^{(n-k)k}$ given by $\varphi_{\mathfrak{i}}\colon y_{\mathfrak{i}^\circ} \mapsto \hat{y}$.

Grassmannian flows

$$Y' = A(x, Y)\, Y$$

Substitute decomposition $Y = y_{\mathfrak{i}^\circ} u$:

$$y'_{\mathfrak{i}^\circ} u + y_{\mathfrak{i}^\circ} u' = (A_{\mathfrak{i}} + A_{\mathfrak{i}^\circ} \hat{y})\, u$$

Project onto $\mathfrak{i}^\circ$th and $\mathfrak{i}$th rows:

$$\hat{y}' = c + d\,\hat{y} - \hat{y}(a + b\,\hat{y}) \qquad \text{and} \qquad u' = (a + b\,\hat{y})\, u$$

where $a = A_{\mathfrak{i}\times\mathfrak{i}}$, $b = A_{\mathfrak{i}\times\mathfrak{i}^\circ}$, $c = A_{\mathfrak{i}^\circ\times\mathfrak{i}}$ and $d = A_{\mathfrak{i}^\circ\times\mathfrak{i}^\circ}$.

Linear vector field: $A = A(x)$ only $\longrightarrow$ decoupling.

Grassmannian Gaussian elimination method (GGEM)

$$
\begin{tikzcd}
\mathbb{C}^{(n-k)k} \arrow[r, "\varphi_{\mathfrak{i}}^{-1}"] \arrow[d, "Riccati"] & \mathbb{U}_{\mathfrak{i}} \arrow[r, "\mathrm{id}"] \arrow[d, "GGEM"] & \mathbb{V}(n,k) \arrow[d, "RK"] \\
\mathbb{C}^{(n-k)k} & \mathbb{U}_{\mathfrak{i}'} \arrow[l, "\varphi_{\mathfrak{i}'}"'] & \mathbb{V}(n,k) \arrow[l, "\mathrm{QOGE}"']
\end{tikzcd}
$$

Quasi-optimal Gaussian elimination (QOGE)

GE with *free* stepwise max pivot, generates: $Y_{m+1} = y_{\mathfrak{i}^\circ} L$.

$$
\begin{pmatrix}
* & * & * & * & \cdots & * \\
* & * & * & * & \cdots & * \\
* & * & * & * & \cdots & * \\
* & * & * & * & \cdots & * \\
* & * & * & * & \cdots & * \\
* & * & * & * & \cdots & * \\
* & * & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & * & \cdots & * \\
* & * & * & * & \cdots & * \\
* & * & * & * & \cdots & *
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
* & * & * & * & \cdots & * \\
0 & 1 & 0 & 0 & \cdots & 0 \\
* & * & * & * & \cdots & * \\
* & * & * & * & \cdots & * \\
0 & 0 & 1 & 0 & \cdots & 0 \\
* & * & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & * & \cdots & * \\
0 & 0 & 0 & 0 & \cdots & 1 \\
* & * & * & * & \cdots & *
\end{pmatrix}
$$

Applications (planar fronts)

- $D(\lambda) := \mathrm{e}^{-\int_0^x \mathrm{Tr} A(\xi;\lambda)\,\mathrm{d}\xi} \det\big(Y^-(x;\lambda)\ \ Y^+(x;\lambda)\big)$
- $\det\big(Y^-\ Y^+\big) = \det\begin{pmatrix} y_{\mathfrak{i}^\circ_-} & y_{\mathfrak{i}^\circ_+}\end{pmatrix} \cdot \det\, u_{\mathfrak{i}_-} \cdot \det\, u_{\mathfrak{i}_+}$
- $D(\lambda; x_*) := \det\begin{pmatrix} y_{\mathfrak{i}^\circ_-} & y_{\mathfrak{i}^\circ_+}\end{pmatrix} \cdot \det L^- \cdot \det L^+$
- Exponentially rescale $\det L^\pm$

Boussinesq system

PDE: $u_{tt} = (1 - c^2)\, u_{xx} + 2c\, u_{xt} - u_{xxxx} - (u^2)_{xx}$.

Solitary waves: $\bar{u}(x) = \frac{3}{2}(1 - c^2)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{1 - c^2}\, x\right)$.

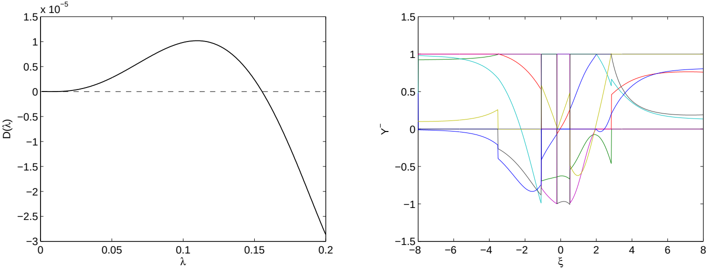


Figure: Evans function for $c = 1/4$ with GGEM-RK and $x_* = 8$ (left panel). Entries of y_i for $\lambda = 0.15543141$ (right panel).

Boussinesq: error vs matching point

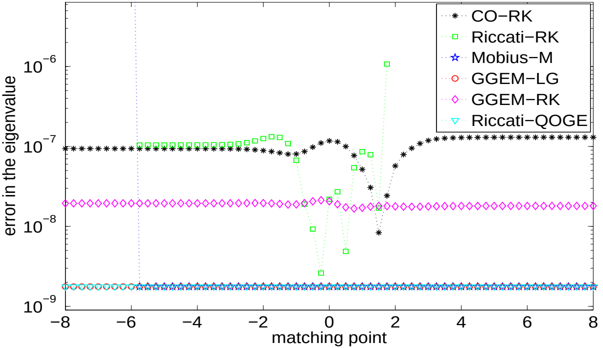


Figure: Error in the eigenvalue for different choices of the matching point: $N = 512$.

Autocatalytic fronts

$$\partial_t u = \delta \Delta u + c\partial_x u - uv^m,$$
$$\partial_t v = \Delta v + c\partial_x v + uv^m.$$

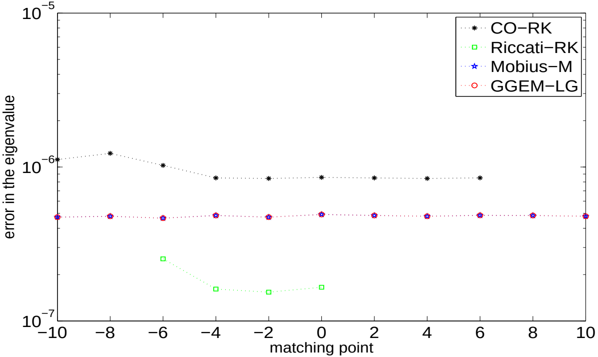


Figure: Error in the eigenvalue when $\delta = 0.1$ and $m = 9$: $N = 256$.

Transverse Fourier basis

On $\mathbb{R} \times \mathbb{T}$ we have:

$$B\Delta U + c\,\partial_x U + \mathrm{D}F(U_c)U = \lambda U.$$

On the Fourier modes $k = -K, -K+1, \ldots, K$:

$$\begin{aligned} \partial_x \hat{U}_k &= \hat{P}_k, \\ \partial_x \hat{P}_k &= \lambda B^{-1} \hat{U}_k + (k/\tilde{L})^2 \hat{U}_k - c\, B^{-1} \hat{P}_k - \sum_{\nu=-K}^{K} B^{-1} \hat{D}_{k-\nu} \hat{U}_\nu, \end{aligned}$$

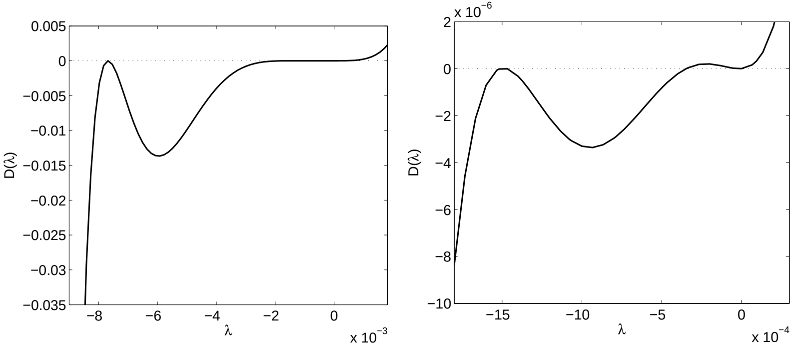
Computing travelling waves: freezing method

Substitute $U(x, y, t) = V(x - \gamma(t), y, t)$ into original PDE:

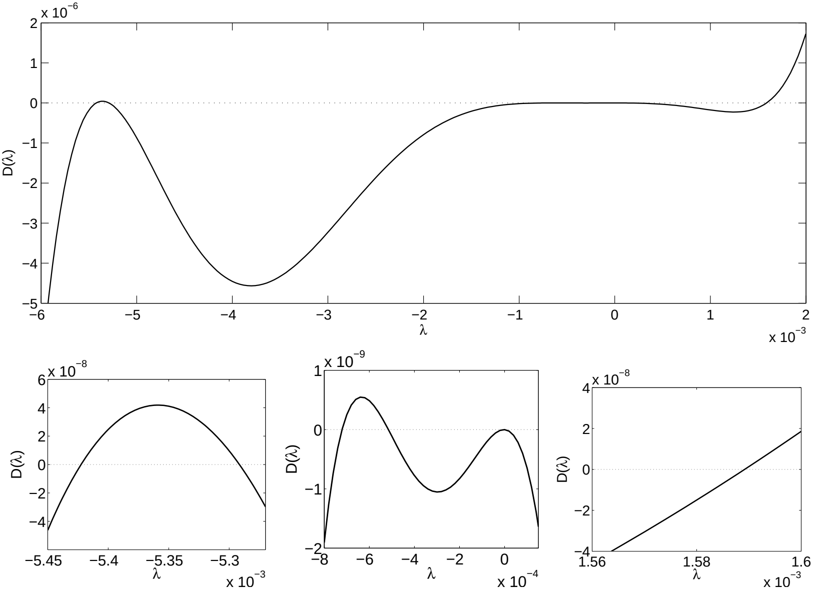
$$\begin{aligned} \partial_t V &= B\,\Delta V + \gamma'(t)\partial_x V + F(V), \\ 0 &= \int_{\mathbb{R}\times\mathbb{T}} \big(\partial_x \hat{V}(x, y, t)\big)^{\mathrm{T}} \big(\hat{V}(x, y, t) - V(x, y, t)\big)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

(Developed by Beyn and Thümmler.)

Wrinkled front: Evans function for $\delta = 2.5$



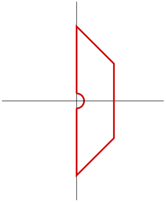
Wrinkled front: Evans function for $\delta = 3$



Wrinkled front: Eigenvalues for $\delta = 3$

K	Eigenvalues (Evans function)				
3	0.001609	-0.000026	-0.000781	-0.001296	-0.000670
4	0.001609	0.000002	-0.000001	-0.000519	-0.000670
5	0.001589	0.000002	-0.000001	-0.000519	-0.000720
6	0.001589	-0.000002	-0.000003	-0.000515	-0.000720
7	0.001589	-0.000002	-0.000003	-0.000515	-0.000721
8	0.001589	-0.000002	-0.000003	-0.000515	-0.000721
9	0.001589	-0.000002	-0.000003	-0.000515	-0.000721
$\vdots$			$\vdots$		
24	0.001589	-0.000002	-0.000003	-0.000515	-0.000721
	Eigenvalues (ARPACK)				
	0.001592	0.000000	0.000000	-0.000514	-0.000719

Wrinkled front: contour integration



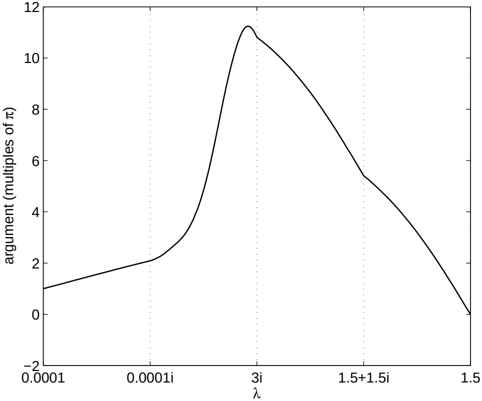


Figure: Left panel: contour. Right panel: $\arg\big(D(\lambda)\big)$ when λ transverses the top half. $\delta = 3$.

Future work

- Multiple sources of error: relative influence?