

# Lie group stochastic integrators

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# Introduction

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- 1 Stochastic Taylor series
- 2 Exponential Lie series
- 3 Castell–Gaines method
- 4 Quadrature & efficiency
- 5 Lie group stochastic integrators
- 6 Conclusions

# Introduction

## Take home messages:

*For numerical SDE methods that are conditioned on two driving Wiener paths, order 1 methods provide the same accuracy as higher order methods for a given computational effort.*

*Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.*

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$$W_t - W_s \sim N(0, \sqrt{t - s})$$

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$$K_i K_j \circ V_j V_i \circ y_0 \equiv \underbrace{\int_0^t \int_0^{\tau_1} dW_{\tau_2}^i dW_{\tau_1}^j}_{J_{ij}} V_j V_i \circ y_0$$

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- ▶ *Magnus 1954, Kunita 1980, Ben Arous 1989, Castell 1993, Burrage 1999.*

# Numerical SDE schemes

## General integrators

- ▶ Euler-Maruyama and Milstein methods
- ▶ Runge–Kutta type methods (*Kloeden & Platen* 1999)
- ▶ Exponential Lie series methods (*Burrage* 1999; *Misawa* 2001; *P-C. Moan* 2004)
- ▶ Castell–Gaines method

# Castell–Gaines method

**Truncated exponential Lie series across  $[t_n, t_{n+1}]$ :**

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**Castell–Gaines:**

$$u'(\tau) = \hat{\psi}_{t_n, t_{n+1}} \circ u(\tau)$$

**across**  $\tau \in [0, 1]$  **with**  $u(0) = y_{t_n}$  **gives**  $u(1) \approx y_{t_{n+1}}$ .

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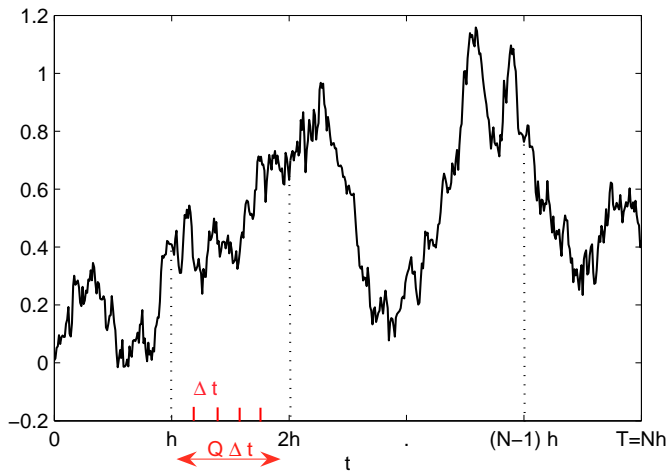
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**Need approximations for iterated integrals.**

## Uniformly accurate exponential Lie integrator:

$$\begin{aligned}\hat{\psi}^* &= J_1 V_1 + J_2 V_2 + J_0 V_0 + \frac{1}{2}(J_{21} - J_{12})[V_1, V_2] \\ &\quad + \frac{h^2}{12} ([V_1, [V_1, V_0]] + [V_2, [V_2, V_0]]) \\ &\quad + \frac{h^2}{12} (V_1[V_2, [V_2, V_1]] + V_2[V_1, [V_1, V_2]]) .\end{aligned}$$

# Quadrature approximation





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## Two inherent scales:

- ▶ Quadrature scale  $\Delta t$ —the *smallest scale* on which the Wiener paths  $W^1$  and  $W^2$  are generated;
- ▶ Time-step scale  $h = Q\Delta t$ —on which the SDE is stepped forward.

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- ▶ Basic idea:  $J \rightarrow \mathbb{E}(J | \mathcal{F}_Q(n))$ .
- ▶ Polygonal volumes.
- ▶ **Wiktorsson 2001; Stump & Hill 2005.**

## Quadrature approximation II

$$\begin{aligned}\mathbb{E}\left(\left|J_{12}(t_n, t_{n+1}) - \hat{J}_{12}(t_n, t_{n+1})\right|^2\right) \\ &= \sum_{q=0}^{Q-1} \mathbb{E}\left(\text{Var}[J_{12}(\tau_q, \tau_{q+1}) | \mathcal{F}_Q]\right) \\ &= \mathcal{O}((\Delta t)^2 Q) \\ &= \mathcal{O}(h^2/Q).\end{aligned}$$



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**Clark & Cameron 1980**

# Global error vs computational effort

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Proof.

$$\mathcal{E} = K_Q(M) \frac{h^{\frac{1}{2}}}{\sqrt{Q}}.$$

$$Q = \frac{1}{h^{2M-1}} \Rightarrow \mathcal{E} = K_{\mathcal{E}} h^M.$$

$$\mathcal{U} = (k_{\mathcal{U}} Q + \text{eval}) N = k_{\mathcal{U}} \frac{1}{h^{2M-1}} \frac{T}{h} = \mathcal{O}(\mathcal{E}^{-2}).$$

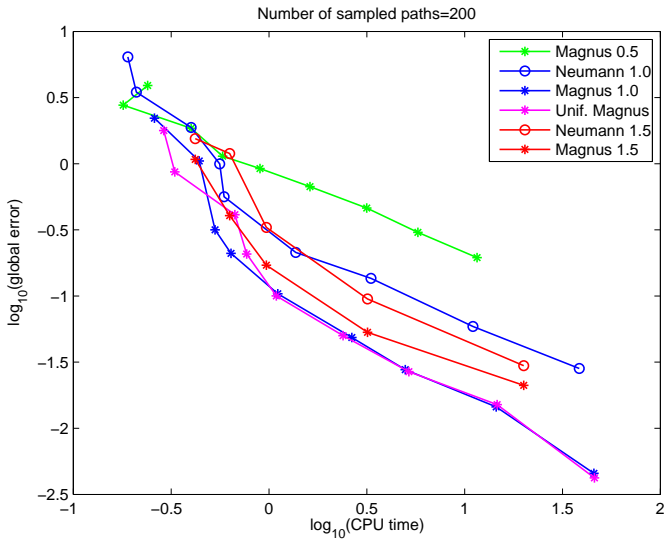
□

## Linear system

$$V_i(y) = a_i y.$$

$$a_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{51}{200} \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix},$$

# Linear system



# Lie group action

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- ▶  $\Lambda(\text{Id}, y) = y$  and  $\Lambda(R, \Lambda(S, y)) = \Lambda(RS, y)$ .
- ▶  $R(\tau) \in \mathcal{G}$ :  $R(0) = S$  and  $R'(0) = a$ . If  $y = \Lambda(S, y_0) \Leftrightarrow y_0 = \Lambda(S^{-1}, y)$ :

$$\begin{aligned}(\partial_1 \Lambda)(S, y_0) \circ a &\equiv \partial_\tau \Lambda(R(\tau), y_0) \Big|_{\tau=0} \\ &= \partial_\tau \Lambda(R(\tau), \Lambda(S^{-1}, y)) \Big|_{\tau=0} \\ &= \partial_\tau \Lambda(R(\tau)S^{-1}, y) \Big|_{\tau=0} \\ &= (\partial_1 \Lambda)(\text{Id}, y) \circ (aS^{-1}).\end{aligned}$$



## Pullback to the Lie group I

$$y_t = y_0 + \sum_{i=0}^d \int_0^t V_i(\tau, y_\tau) dW_\tau^i$$

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## Examples

$$\mathcal{G} = \text{Diff}(\mathcal{M}), \quad \Lambda(S, y) = S \circ y$$
$$\implies y_t = S_t \circ y_0 \quad \text{with} \quad S_t \equiv \varphi_t.$$



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$$V_i(t, y) = a_i(t, y) y \quad \text{with} \quad a_i \in \mathfrak{so}(n), \quad \mathcal{M} = \mathcal{G} = \text{SO}(n) :$$

$$\Lambda(S, y) = Sy \quad \implies \quad (\partial_1 \Lambda)(\text{Id}, y) \circ a = ay \quad \implies \quad y_t = S_t y_0.$$

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# Stochastic Taylor Munthe-Kaas method

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**Algorithm:**

$$\hat{\sigma}_{t_n, t_{n+1}} = (J_0 v_0 + J_1 v_1 + J_2 v_2 + \frac{1}{2} J_1^2 v_1^2 + J_{12} v_2 v_1 + J_{21} v_1 v_2 + \frac{1}{2} J_2^2 v_2^2) \circ O.$$

# Stochastic Taylor Munthe-Kaas method

$$\sigma_t = \sum_{i=0}^d \int_0^t \text{dexp}_{\sigma_\tau}^{-1} \circ a_i(\tau, \Lambda(\exp \sigma_\tau, y_0)) \, dW_\tau^i,$$

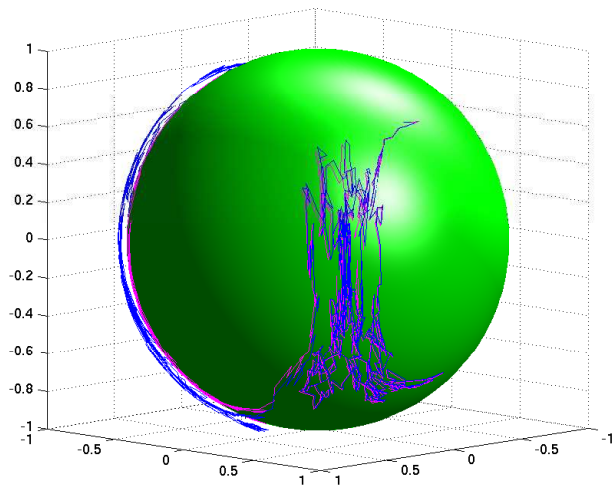
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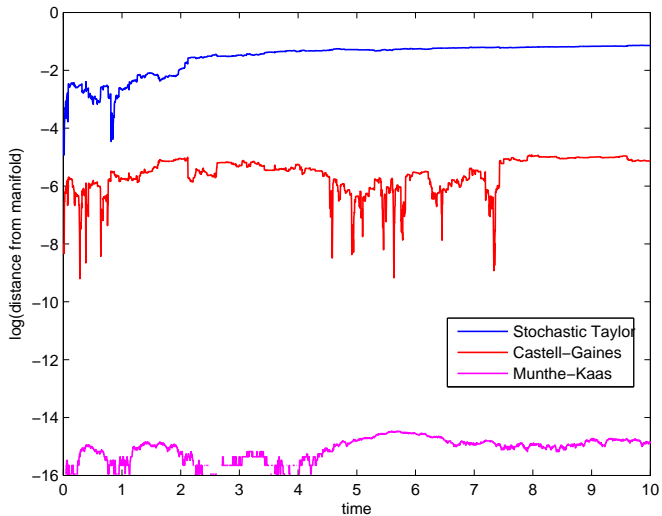
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$$y_{t_{n+1}} \approx \Lambda(\exp \hat{\sigma}_{t_n, t_{n+1}}, y_{t_n})$$

# Rigid body



# Rigid body



# Conclusions

## Take home messages:

*For numerical SDE methods that are conditioned on two driving Wiener paths, order 1 methods provide the same accuracy as higher order methods for a given computational effort.*

*Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.*

# Conclusions

## Take home messages:

*For numerical SDE methods that are conditioned on two driving Wiener paths, order 1 methods provide the same accuracy as higher order methods for a given computational effort.*

*Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.*

## Future directions:

- ▶ Variable step scheme (*Gaines & Lyons 1997*).
- ▶ Zakai equation: Markov chain filters.
- ▶ Backwards SDEs

# Accuracy I

For order 1/2 schemes:

$$R^{\text{els}} = \frac{1}{2}(J_{21} - J_{12})[V_1, V_2] \circ y_0;$$

$$R^{\text{st}} = R^{\text{els}} + \underbrace{\frac{1}{2}(J_{21} + J_{12})}_{\hat{R}}(V_1 V_2 + V_2 V_1) \circ y_0.$$

Local error:

$$\begin{aligned}\mathbb{E}((R^{\text{st}})^T R^{\text{st}}) &= \mathbb{E}((R^{\text{els}} + \hat{R})^T (R^{\text{els}} + \hat{R})) \\ &= \mathbb{E}((R^{\text{els}})^T R^{\text{els}}) + \mathbb{E}(\hat{R}^T \hat{R}) \\ &\quad + \underbrace{\mathbb{E}(\hat{R}^T R^{\text{els}})}_{=0} + \underbrace{\mathbb{E}((R^{\text{els}})^T \hat{R})}_{=0}.\end{aligned}$$



# Quadrature approximation

Quadrature		$\hat{J}_{12}$	$\hat{J}_{112}$	$\hat{J}_{120}$	$\hat{J}_{1112}$
$\mathcal{E}^{\text{loc}}(n)$		$h/Q^{1/2}$	$h^{3/2}/Q^{1/2}$	$h^2/Q^{1/2}$	$h^2/Q^{1/2}$
$\mathcal{U}$	$\mathcal{O}(h^{3/2})$	$h^{-1}$	...	...	...
	$\mathcal{O}(h^2)$	$h^{-2}$	$h^{-1}$	...	...
	$\mathcal{O}(h^{5/2})$	$h^{-3}$	$h^{-2}$	$h^{-1}$	$h^{-1}$

# Global error I

**Global interval:**  $[0, T] = \cup_{n=0}^{N-1} [t_n, t_{n+1}]$ ,  $t_n = nh$ .

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$$R = \exp(\psi) - \exp(\hat{\psi}) = \exp(\hat{\psi} + \rho) - \exp(\hat{\psi}) = \rho + \mathcal{O}(\psi\rho).$$

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**Stochastic Taylor or exponential Lie series:**

$$y_{t_n, t_{n+1}} = \hat{y}_{t_n, t_{n+1}} + R_{t_n, t_{n+1}}.$$

## Global error II

$$\begin{aligned}\mathcal{E}^2 &\equiv \mathbb{E} \left\| \left( \prod_{n=N-1}^0 \varphi_{t_n, t_{n+1}} - \prod_{n=N-1}^0 \hat{\varphi}_{t_n, t_{n+1}} \right) \circ y_0 \right\|^2 \\ &= \mathbb{E} \left\| \underbrace{\left( \sum_{n=0}^{N-1} \varphi_{t_{n+1}, t_N} R_{t_n, t_{n+1}} \hat{\varphi}_{t_0, t_n} \right)}_{\mathcal{R}} \circ y_0 \right\|^2.\end{aligned}$$

## Recall: exponential Lie series (order 1)

$$y_t = \exp(\psi_t) \circ y_0, \quad \psi_t = \psi_{1/2} + \psi_1 + \psi_{3/2} + \cdots$$

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$$\psi_1 = \frac{1}{2}(J_{21} - J_{12})[V_1, V_2],$$

$$\begin{aligned} \psi_{3/2} = & \frac{1}{2}(J_{10} - J_{01})[V_0, V_1] + \frac{1}{2}(J_{20} - J_{02})[V_0, V_2] \\ & + (J_{112} - \frac{1}{2}J_1 J_{12} + \frac{1}{12}J_1^2 J_2) [V_1, [V_1, V_2]] \\ & + (J_{221} - \frac{1}{2}J_2 J_{21} + \frac{1}{12}J_2^2 J_1) [V_2, [V_2, V_1]] \\ & + (J_{110} - \frac{1}{2}J_1 J_{10} + \frac{1}{12}J_1^2 J_0) [V_1, [V_1, V_0]] \\ & + (J_{220} - \frac{1}{2}J_2 J_{20} + \frac{1}{12}J_2^2 J_0) [V_2, [V_2, V_0]]. \end{aligned}$$

## Global error III

$$\mathcal{R} = \sum_{n=0}^{N-1} \sum_{\alpha} (\varphi_{t_{n+1}, t_N} V_{\alpha} \varphi_{t_0, t_n} \circ y_0) J_{\alpha}(t_n, t_{n+1}).$$

$$\mathbb{E}(\mathcal{R}^T \mathcal{R})$$

$$\begin{aligned} &= \sum_{n=0}^{N-1} \sum_{\alpha, \beta} \mathbb{E} \left( (\varphi_{t_{n+1}, t_N} V_{\alpha} \varphi_{t_0, t_n} \circ y_0)^T (\varphi_{t_{n+1}, t_N} V_{\beta} \varphi_{t_0, t_n} \circ y_0) \right) \\ &\quad \cdot \mathbb{E}(J_{\alpha}(t_n, t_{n+1}) J_{\beta}(t_n, t_{n+1})) \\ &+ \sum_{n \neq m} \sum_{\alpha, \beta} \mathbb{E} \left( (\varphi_{t_{n+1}, t_N} V_{\alpha} \varphi_{t_0, t_n} \circ y_0)^T (\varphi_{t_{m+1}, t_N} V_{\beta} \varphi_{t_0, t_m} \circ y_0) \right) \\ &\quad \cdot \mathbb{E}(J_{\alpha}(t_n, t_{n+1})) \mathbb{E}(J_{\beta}(t_m, t_{m+1})). \end{aligned}$$

# Stochastic Riccati system

$$S(t) = I + \sum_{i=0}^d \int_0^t f_i(\tau, S(\tau)) dW_i(\tau).$$

$$f_i(t, S) = S(t)A_i(t)S(t) + B_i(t)S(t) + S(t)C_i(t) + D_i(t).$$

- ▶ Stochastic linear-quadratic optimal control.
- ▶ eg. mean-variance hedging in finance (*Bobrovnytska & Schweizer 2004; Kohlmann & Tang 2003*).

## Riccati II

If  $\mathbb{A}_i(t) \equiv \begin{pmatrix} B_i(t) & D_i(t) \\ -A_i(t) & -C_i(t) \end{pmatrix}$  and  $\mathbb{U} = \begin{pmatrix} U \\ V \end{pmatrix}$  satisfies

$$\mathbb{U}(t) = \mathbb{I} + \sum_{i=0}^d \int_0^t \mathbb{A}_i(\tau) \mathbb{U}(\tau) dW_i(\tau),$$

then  $S = UV^{-1}$  solves the Riccati system.

$$A_0 = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}, \quad C_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad D_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

$D_1 = a_1$  and  $D_2 = a_2$ .

## Riccati III

*Kloeden & Platen:*

$$\begin{aligned} S_{n+1} &= S_n + f(S_n)h + D_1 J_1 + D_2 J_2 \\ &+ \frac{h}{4} (f(Y_1^+) + f(Y_1^-) + f(Y_2^+) + f(Y_2^-) - 4f(S_n)) \\ &+ \frac{1}{2\sqrt{h}} ((f(Y_1^+) - f(Y_1^-))J_{10} + (f(Y_2^+) - f(Y_2^-))J_{20}) , \end{aligned}$$

$$Y_j^\pm = S_n + \frac{h}{2} f(S_n) \pm D_j \sqrt{h}$$

$$f(S) = SA_0 S + B_0 S + SC_0 + D_0 .$$

# Riccati IV

