Stochastic expansions and Hopf algebras

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Outline

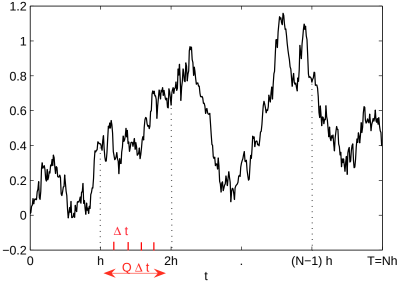
Stochastic differential equations

$$\mathrm{d}y_t = V_0(y_t)\,\mathrm{d}t + V_1(y_t)\,\mathrm{d}W_t^1 + \cdots + V_d(y_t)\,\mathrm{d}W_t^d$$

$$y_t = y_0 + \sum_{i=0}^{d} \int_0^t V_i(y_\tau)\,\mathrm{d}W_\tau^i$$

- Solution process: $y\colon \mathbb{R}_+ \to \mathbb{R}^N$
- Non-commuting vector fields: $V_i = \sum_{j=1}^n V_i^j(y)\partial_y$
- d-dimensional driving signal: $(W^1, \ldots, W^d)$
- Convention: $W_t^0 \equiv t$

Wiener process



- $W_t - W_s \sim \mathrm{N}(0, \sqrt{t-s})$
- Independent increments
- Continuous, nowhere differentiable (w.p. 1)

Stochastic chain rule

$$y_t = y_0 + \sum_i \int_0^t V_i \circ y_\tau \, \mathrm{d}W_\tau^i$$

Itô lemma (stochastic chain rule) $\Rightarrow$

$$f \circ y_t = f \circ y_0 + \sum_j \int_0^t V_j \circ f \circ y_\tau \, \mathrm{d}W_\tau^j$$

E.g. choose $f = V_i \Rightarrow$

$$V_i \circ y_t = V_i \circ y_0 + \sum_j \int_0^t V_j \circ V_i \circ y_\tau \, \mathrm{d}W_\tau^j$$

$$y_t = y_0 + \sum_i \int_0^t \left(V_i \circ y_0 + \sum_j \int_0^{\tau_1} V_j \circ V_i \circ y_{\tau_2} \, \mathrm{d}W_{\tau_2}^j \right) \, \mathrm{d}W_{\tau_1}^i$$

Stochastic Taylor series

$$y_t = y_0 + \sum_i \int_0^t \mathrm{d}W^i_{\tau_1} V_i \circ y_0 + \sum_{i,j} \int_0^t \int_0^{\tau_1} V_j \circ V_i \circ y_{\tau_2} \, \mathrm{d}W^j_{\tau_2} \, \mathrm{d}W^i_{\tau_1}$$

Now choose $f = V_j \circ V_i \Rightarrow$

$$y_t = y_0 + \sum_i \underbrace{\int_0^t \mathrm{d}W^i_{\tau_1}}_{J_i(t)} V_i \circ y_0 + \sum_{i,j} \underbrace{\int_0^t \int_0^{\tau_1} \mathrm{d}W^j_{\tau_2} \, \mathrm{d}W^i_{\tau_1}}_{J_{ji}(t)} V_j \circ V_i \circ y_0 + \cdots$$

$$J_i(t) \equiv \int_0^t \mathrm{d}W^i_{\tau_1} = W^i_t - \underbrace{W^i_0}_{0}$$

Stochastic Taylor series II

$$J_{ii}(t) \equiv \int_0^t \int_0^{\tau_1} \mathrm{d}W^i_{\tau_2}\,\mathrm{d}W^i_{\tau_1} \equiv \tfrac{1}{2}(W^i_t)^2$$
$$J_{ij}(t) \equiv \int_0^t \int_0^{\tau_1} \mathrm{d}W^j_{\tau_2}\,\mathrm{d}W^i_{\tau_1} \equiv \int_0^t W^j_{\tau_1}\,\mathrm{d}W^i_{\tau_1}$$

$J_{ij}(t) \longrightarrow \hat{J}_{ij}$

- Left Riemann sum
- Karhunen–Loeve
- K–L + tail distribution approx. (Wiktorsson)

Stochastic Taylor series III

$$y_t = y_0 + \sum_i \underbrace{\int_0^t \mathrm{d}W^i_{\tau_1}}_{J_i(t)} V_i \circ y_0 + \sum_{i,j} \underbrace{\int_0^t \int_0^{\tau_1} \mathrm{d}W^j_{\tau_2}\, \mathrm{d}W^i_{\tau_1}}_{J_{ji}(t)} V_j \circ V_i \circ y_0 + \cdots$$

Strong approximation:

- Euler-Maruyama: inc. $\hat{J}_i(t)$
- Milstein method: inc. $\hat{J}_i(t)$ & $\hat{J}_{ij}$

Deterministic case: V_0 only $\Rightarrow$

$$\begin{aligned} y_t &= y_0 + tV_0 \circ y_0 + \tfrac{1}{2} t^2 V_0^2 + \tfrac{1}{6} t^3 V_0^3 + \cdots \\ &= \exp(tV_0) \circ y_0. \end{aligned}$$

Flow map and exponential Lie series

$$y_t = \varphi_t \circ y_0$$

$$\varphi_t = \mathrm{id} + \sum_i J_i(t) V_i + \sum_{i,j} J_{ij}(t) V_i \circ V_j + \sum_{i,j,k} J_{ijk}(t) V_i \circ V_j \circ V_k + \cdots$$

Exponential Lie series:

$$\begin{aligned}
\psi_t &= \ln \varphi_t \\
&= (\varphi_t - \mathrm{id}) - \tfrac{1}{2}(\varphi_t - \mathrm{id})^2 + \tfrac{1}{3}(\varphi_t - \mathrm{id})^3 + \cdots \\
&= \sum_i J_i V_i + \sum_{i,j} J_{ij} V_i V_j + \cdots - \tfrac{1}{2}\Big(\sum_i J_i V_i + \cdots\Big)^2 + \cdots \\
&= \sum_i J_i V_i + \sum_{i>j} \tfrac{1}{2}(J_{ij} - J_{ji})[V_i, V_j] + \cdots
\end{aligned}$$

Castell–Gaines (ODE) method

Truncated exponential Lie series across $[t_n, t_{n+1}]$:

$$\hat{\psi}_{t_n,t_{n+1}} = \sum_i \hat{J}_i V_i + \sum_{i>j} + \tfrac{1}{2}(\hat{J}_{ij} - \hat{J}_{ji})[V_i, V_j] \,.$$

Approximate solution:

$$y_{t_{n+1}} \approx \exp(\hat{\psi}_{t_n,t_{n+1}}) \circ y_{t_n} \,.$$

Castell–Gaines: solve ODE

$$u'(\tau) = \hat{\psi}_{t_n,t_{n+1}} \circ u(\tau)$$

across $\tau \in [0,1]$ with $u(0) = y_{t_n}$ gives $u(1) \approx y_{t_{n+1}}$.

Basic idea

- Construct the new series $\psi_t = F(\varphi_t)$
- Truncate and use $\hat{J}_i$, $\hat{J}_{ij}$ to produce $\hat{\psi}_t$
- Approximate flow-map is $\hat{\varphi}_t = F^{-1}(\hat{\psi}_t)$
- "Flow error" is the flow remainder $R_t = \varphi_t - \hat{\varphi}_t$
- Approximate solution is $\hat{y}_t = \varphi_t \circ y_0$
- Error/remainder is $R_t \circ y_0$
- Mean-square error measure is

$$\|R_t \circ y_0\|_{L^2}^2 \equiv \mathbb{E}(R_t \circ y_0)^{\mathrm{T}}(R_t \circ y_0)$$

Basic idea

$$\varphi = \mathrm{id} + \sum_{w \in \mathbb{A}^+} J_w \, V_w$$

Here $\mathbb{A}^+ = \{\text{non-empty words over } \mathbb{A} = \{0, 1, \ldots, d\}\}$

$$\psi = F(\varphi) = \sum_{k=1}^{\infty} C_k \, (\varphi - \mathrm{id})^k = \sum_{w \in \mathbb{A}^+} K_w \, V_w$$

Goal: choose C_k so that

$$\|R \circ y_0\|_{L^2}^2 \leq \|R^{\mathrm{st}} \circ y_0\|_{L^2}^2$$

Sinh-log series?

- Diffusion vector fields commute: exponential Lie series has smaller mean-square error (orders 1/2, 1, 3/2)
- Do not commute: counter-example for order 1
- Now *assume no drift*: $\mathbb{A} = \{1, \ldots, d\}$

Set

$$\psi = \sinh \log \varphi \equiv \sum_{k=1}^{\infty} C_k (\varphi - id)^k$$

$$\hat{\varphi} \equiv \hat{\psi} + \sqrt{\mathrm{id} + \hat{\psi}^2}$$

Hopf algebra of words

Shuffle relations:

$$\begin{aligned} J_{a_1} J_{a_2} &= J_{a_1 a_2} + J_{a_2 a_1} \\ J_{a_1} J_{a_2 a_3} &= J_{a_1 a_2 a_3} + J_{a_2 a_1 a_3} + J_{a_2 a_3 a_1} \\ J_{a_1 a_2} J_{a_3 a_4} &= J_{a_1 a_2 a_3 a_4} + J_{a_1 a_3 a_2 a_4} + J_{a_1 a_3 a_4 a_2} \\ &\quad + J_{a_3 a_1 a_4 a_2} + J_{a_3 a_1 a_2 a_4} + J_{a_3 a_4 a_1 a_2} \end{aligned}$$

Shuffle product: $J_u J_v \longrightarrow u \sqcup\!\sqcup\, v$

Two *Hopf algebra* structures, set:

$$\mathcal{H} = \mathbb{R}\langle \mathbb{A} \rangle \otimes \mathbb{R}\langle \mathbb{A} \rangle$$

$$(u \otimes x)(v \otimes y) = (u \sqcup\!\sqcup\, v) \otimes (xy)\,,$$

Pullback to Hopf algebra

$$\varphi = 1 \otimes 1 + \sum_{w \in \mathbb{A}^+} w \otimes w$$

$$\begin{aligned}
\Rightarrow \quad \psi &= \sum_{k \geq 1} C_k \big(\varphi - 1 \otimes 1\big)^k \\
&= \sum_{k \geq 1} C_k \left(\sum_{w \in \mathbb{A}^+} w \otimes w \right)^k \\
&= \sum_{k \geq 1} C_k \sum_{u_1, \ldots, u_k \in \mathbb{A}^+} (u_1 ⧢ \ldots ⧢ u_k) \otimes (u_1 \ldots u_k) \\
&= \sum_{w \in \mathbb{A}^*} \left(\sum_{k=1}^{|w|} C_k \sum_{\substack{u_1, \ldots, u_k \in \mathbb{A}^+ \\ w = u_1 \ldots u_k}} u_1 ⧢ \ldots ⧢ u_k \right) \otimes w
\end{aligned}$$

Concatenation-shuffle operator algebra

$$K \circ w \equiv \sum_{k=1}^{|w|} C_k \sum_{\substack{u_1,\ldots,u_k \in \mathbb{A}^+ \\ w = u_1 \ldots u_k}} u_1 ⧢ \ldots ⧢ u_k$$

$$u_1 ⧢ u_2 ⧢ \ldots ⧢ u_k = c^{|u_1|-1} s c^{|u_2|-1} s \ldots s c^{|u_k|-1} \otimes w$$

Warning:

$$(b_1 \otimes u_1)(b_2 \otimes u_2) = (b_1 s b_2) \otimes (u_1 u_2)\,,$$

Coefficients

$|w| = n + 1$:

$$\begin{aligned} K \circ w &= \sum_{k=1}^{|w|} C_k \sum_{\substack{u_1,\ldots,u_k \in \mathbb{A}^+ \\ w = u_1 \ldots u_k}} u_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup u_k \\ &\to \sum_{k=1}^{n+1} C_k \left(c^{n-(k-1)} \sqcup\!\sqcup s^{k-1}\right) \otimes w \\ &= \sum_{k=0}^{n} C_{k+1} \left(c^{n-k} \sqcup\!\sqcup s^k\right) \otimes w \\ &= \left(\sum_{k=0}^{n} C_{k+1} (c^{n-k} \sqcup\!\sqcup s^k)\right) \otimes w \,. \end{aligned}$$

Sinh-log coefficients

Lemma

With

$$C_k = \begin{cases} 1\,, & k = 1\,, \\ \frac{1}{2}(-1)^{k+1}\,, & k \geq 2\,, \\ \frac{1}{2}(-1)^{n+1} + \epsilon\,, & k = n\,, \end{cases}$$

then

$$K = \frac{1}{2}\left(c^n - \alpha^n\right) + \epsilon\, s^n$$

where

$$\alpha^n \circ (a_1 \ldots a_{n+1}) \equiv \alpha \circ (a_1 \ldots a_{n+1}) \equiv (-1)^n a_{n+1} \ldots a_1 .$$

Proof: step 1

Partial integration formula:

$$a_1 \dots a_{n+1} = (a_1 \dots a_n) \sqcup\!\sqcup\, a_{n+1} - (a_1 \dots a_{n-1}) \sqcup\!\sqcup\, (a_{n+1} a_n) + \dots + (-1)^n a_{n+1} \dots a_1$$

$$c^n = -c^{n-1}s - c^{n-2}s\alpha - c^{n-3}s\alpha^2 - \dots - \alpha^n$$

$$\alpha^n = -c^n - \sum_{k=0}^{n-1} c^k s \alpha^{n-k-1}$$

Proof: step 2

Antipode polynomial:

$$\alpha^n \equiv -(c-s)^n\,.$$

True for $n = 1, 2$, assume true for $k = 1, 2, \ldots, n-1$. Direct expansion $\Rightarrow$

$$(c-s)^n = c^n - \sum_{k=0}^{n-1} c^k s (c-s)^{n-k-1}$$

Use induction.

Proof: step 3

Recall:

$$\begin{aligned} K &= \tfrac{1}{2}c^n + \tfrac{1}{2}\sum_{k=0}^{n}(-1)^k(c^{n-k} \sqcup\!\sqcup\, s^k) + \epsilon\, s^n \\ &= \tfrac{1}{2}c^n + \tfrac{1}{2}(c-s)^n + \epsilon\, s^n \\ &= \tfrac{1}{2}c^n - \tfrac{1}{2}\alpha^n + \epsilon\, s^n \end{aligned}$$

Hence

$$K_w = \tfrac{1}{2}\big(J_w - J_{\alpha\circ w}\big) + \epsilon \prod_{i=1}^{n+1} J_{w_i}$$

Sinh-log remainder is smaller I

$$\|R^{st} \circ y_0\|_{L^2}^2 = \|R^{sl} \circ y_0\|_{L^2}^2 + E$$

where with $R^{\mathrm{st}} = R^{\mathrm{sl}} + \bar{R}$:

$$E \equiv \mathbb{E}\left(\bar{R}\circ y_0\right)^{\mathrm{T}}\left(R^{\mathrm{sl}}\circ y_0\right)+\mathbb{E}\left(R^{\mathrm{sl}}\circ y_0\right)^{\mathrm{T}}\left(\bar{R}\circ y_0\right)+\mathbb{E}\left(\bar{R}\circ y_0\right)^{\mathrm{T}}\left(\bar{R}\circ y_0\right)$$

$$R^{\mathrm{sl}} = \sum_{\substack{w\in\mathbb{A}^+\\ |w|\geq n+1}} K_w\, V_w + \cdots$$

$$\bar{R} = \sum_{\substack{w\in\mathbb{A}^+\\ |w|= n+1}} \underbrace{J_w - K_w}_{\bar{J}_w}\, V_w$$

Sinh-log remainder is smaller II

$$E = \sum_{\substack{u,v \in \mathbb{A}^+ \\ |u|=|v|=n+1}} \mathbb{E}\left(\bar{J}_u K_v + K_u \bar{J}_v + \bar{J}_u \bar{J}_v\right) \left(V_u \circ y_0\right)^T \left(V_v \circ y_0\right)$$

Recall $K_u = \frac{1}{2}(J_u - J_{\alpha \circ u}) + \epsilon \prod_{i=1}^{n+1} J_{u_i} \Rightarrow$

$$\begin{aligned}
\bar{J}_u K_v + K_u \bar{J}_v + \bar{J}_u \bar{J}_v \\
&= \epsilon^0 \left(\tfrac{1}{2}(J_u J_v - J_{\alpha \circ u} J_{\alpha \circ v}) + \tfrac{1}{4}(J_u + J_{\alpha \circ u})(J_v + J_{\alpha \circ v}) \right) \\
&\quad - \epsilon^1 \left(\tfrac{1}{2}(J_u - J_{\alpha \circ u}) \prod_{i=1}^{n+1} J_{v_i} + \tfrac{1}{2}(J_v - J_{\alpha \circ v}) \prod_{i=1}^{n+1} J_{u_i} \right) \\
&\quad - \epsilon^2 \left(\prod_{i,j=1}^{n+1} J_{u_i} J_{v_j} \right)
\end{aligned}$$

Concluding remarks

- Drift?
- Measure?
- Lévy processes
- Fractional Brownian motion
- Combinatorics $\leftrightarrow$ analysis