Computing the Evans function using Grassmannians

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Acknowledgments

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Outline

Take home message:

Shooting methods are about to take off!

Stability of travelling waves

$$U_t = BU_{\xi\xi} + cU_\xi + F(U)$$

- Travelling wave in moving frame: $U(\xi, t) = U_c(\xi)$
- Perturbation ansatz:

$$U(\xi, t) = U_c(\xi) + \hat{\mathrm{U}}(\xi)\,\mathrm{e}^{\lambda t}$$

- Small perturbations satisfy:

$$\lambda \hat{\mathrm{U}} = \left[B\partial_{\xi\xi} + c\, I\partial_\xi + DF(U_c(\xi))\right] \hat{\mathrm{U}}$$

with $\hat{\mathrm{U}}(\xi) \to 0$ as $\xi \to \pm\infty$.

Strategies

- **Projection:** Project the spectral equations onto a finite dimensional basis, which by construction satisfies the boundary conditions. Solve the matrix eigenvalue problem.
  - $(+)$ Applies to structures of any dimension.
  - $(-)$ Increasing accuracy or adaptation is costly and complicated; need to re-project onto the finer or adapted basis.
- **Shooting:** And matching.
  - $(+)$ Much easier to fine-tune accuracy and adaptivity.
  - $(-)$ Essentially one-dimensional method.

Shooting reformulation

$$\lambda \hat{\mathrm{U}} = \left[B\partial_{\xi\xi} + c\, I\partial_{\xi} + DF(U_c(\xi)) \right] \hat{\mathrm{U}}$$

Write $Y = \left(\hat{\mathrm{U}}, \hat{\mathrm{U}}_{\xi}\right)$ then

$$Y' = A(\xi;\lambda)\, Y$$
$$Y \to 0\,, \quad \xi \to \pm\infty$$

where

$$A(\xi;\lambda) = \begin{pmatrix} O & I \\ B^{-1}\left(\lambda - DF(U_c(\xi))\right) & -c\, B^{-1} \end{pmatrix}$$

The Evans function

Limiting systems

$$A_\pm(\lambda) = \lim_{\xi \to \pm\infty} A(\xi, \lambda)$$

- Assume A_- has a k-dimensional unstable manifold
- A_+ has an $(2n-k)$-dimensional stable manifold
- Look for intersection under the “evolution” of the BVP

Wronskian:

$$\begin{aligned} D(\lambda) &= \mathrm{e}^{-\int_0^\xi \mathrm{Tr}\, A(x,\lambda)\mathrm{d}x} \cdot \det\left(Y_1^- \cdots Y_k^- \quad Y_{k+1}^+ \cdots Y_{2n-k}^+\right) \\ &= \mathrm{e}^{-\int_0^\xi \mathrm{Tr}\, A(x,\lambda)\mathrm{d}x} \cdot \underbrace{\left(Y_1^- \wedge \cdots \wedge Y_k^-\right)}_{U_-(\xi;\lambda)} \wedge \underbrace{\left(Y_{k+1}^+ \wedge \cdots \wedge Y_{2n-k}^+\right)}_{U_+(\xi;\lambda)} \end{aligned}$$

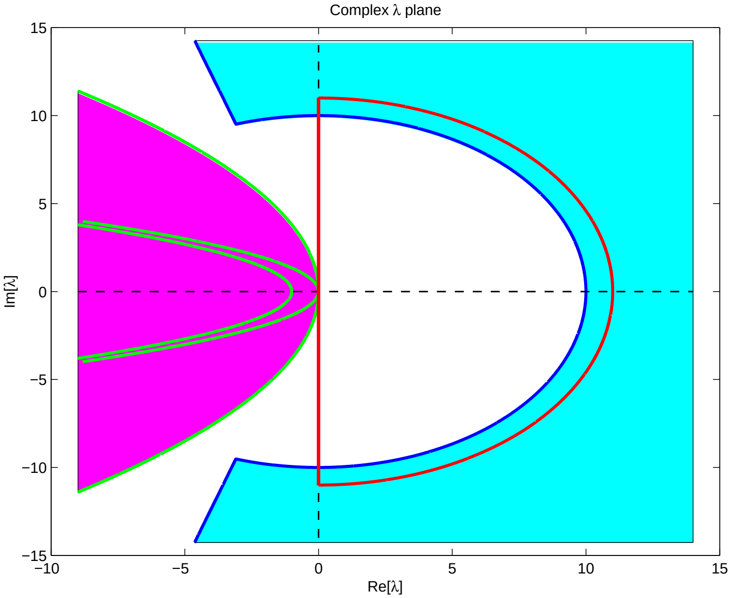
(Prefactor ensures ξ-independence.)

Properties of the Evans function

(Evans, 1975; Alexander, Gardner & Jones, 1990)

- Zeros correspond to eigenvalues
- Analytic to the right of the essential spectrum
- Can use argument principle to determine number of zeros in the right half plane

Spectrum of the linearized travelling wave operator



Numerical issues

$$Y' = A(\xi;\lambda)\,Y$$

- Rescale by expected exponential growth, i.e. $Y = \mathrm{e}^{\mu x} Z$:

$$Z' = \big(A(\xi;\lambda) - \mu I\big)\,Z$$

- Computing second unstable basis function numerically unstable
- Resolution: integrate exterior products $U \in \bigwedge^k \mathbb{C}^n$

$$D(\lambda) = \mathrm{e}^{-\int_0^\xi \mathrm{Tr}\,A(x,\lambda)\mathrm{d}x} \cdot \underbrace{\left(Y_1^- \wedge \cdots \wedge Y_k^-\right)}_{U_-(\xi;\lambda)} \wedge \underbrace{\left(Y_{k+1}^+ \wedge \cdots \wedge Y_{2n-k}^+\right)}_{U_+(\xi;\lambda)}$$

Multi-dimensional stability

- $\dim\left(\bigwedge^k \mathbb{C}^n\right) = \frac{n!}{k!(n-k)!}$
- AIM meeting May 2005: Stability criteria for multi-dimensional waves and patterns (C.K.R.T. Jones, Y. Latushkin, B. Sandstede)
- Humpherys & Zumbrun (2006): Continuous orthogonalization
- Drury–Oja flow (Drury, Davey, Bridges–Reich)

Idea came from

- Schiff and Shnider:

$$\text{Möbius projection}: \mathrm{GL}(n) \to \text{Grassmannian}$$

- Greenberg and Marletta: minimal information for Evans function

Principle fibre bundle

$$\pi : \mathbb{V}(n, k) \rightarrow \mathrm{Gr}(n, k)$$

- base space $\mathrm{Gr}(n, k)$
- $\forall\ y \in \mathrm{Gr}(n, k)$: $\pi^{-1}(y)$ homeomorphic to fibre space $\mathrm{GL}(k)$
- projection map π: natural quotient map sending each k-frame centered at the origin to the k-plane it spans

Stiefel and Grassmannian manifolds

$$\begin{pmatrix} Y_{11} & \cdots & Y_{1k} \\ \vdots & & \vdots \\ Y_{n1} & \cdots & Y_{nk} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ y_{k+1,1} & y_{k+1,2} & \cdots & y_{k+1,k} \\ y_{k+2,1} & y_{k+2,2} & \cdots & y_{k+2,k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1} & y_{n,2} & \cdots & y_{n,k} \end{pmatrix}$$

Grassmannian manifold $\mathrm{Gr}(n, k)$

- $Y = [Y_1 Y_2 \cdots Y_k]$
- Natural decomposition $\mathbb{V}(n, k) \cong \mathrm{Gr}(n, k) \times \mathrm{GL}(k)$:

$$Y = \begin{pmatrix} U \\ L \end{pmatrix} = \begin{pmatrix} I_k \\ y \end{pmatrix} U$$

  where $y = LU^{-1} \in \mathbb{C}^{(n-k)\times k}$ and $U \in \mathrm{GL}(k)$.
- Coordinatization of $\mathrm{Gr}(n, k)$ implicitly chosen.

Tangent space decomposition

$\mathbb{V}(n,k) = \mathrm{Gr}(n,k) \times \mathrm{GL}(k) \Rightarrow$ induced decomposition

$$\mathrm{T}_Y \mathbb{V}(n,k) = \mathbb{H}_Y \oplus \mathbb{V}_Y$$

$$V = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathrm{T}_Y \mathbb{V}(n,k)$$

$$\mathbb{H}_Y = \left\{ V_{\mathrm{h}} = \begin{pmatrix} \mathrm{O}_k \\ \eta \end{pmatrix} : \eta \in \mathbb{C}^{(n-k)\times k} \right\}$$
$$\mathbb{V}_Y = \left\{ V_{\perp} = \begin{pmatrix} \xi \\ \mathrm{O} \end{pmatrix} : \xi \in \mathfrak{gl}(k) \right\}$$

Riccati flow

$$V(x,Y) = \begin{pmatrix} a(x,Y) & b(x,Y) \\ c(x,Y) & d(x,Y) \end{pmatrix} Y\,, \qquad Y \in \mathbb{V}(n,k)$$

$$V_{\mathrm{h}} = \begin{pmatrix} \mathrm{O}_k & \mathrm{O}_{k\times(n-k)} \\ -y & I_{n-k} \end{pmatrix} V = \begin{pmatrix} \mathrm{O}_k \\ c + d\,y - y(a + b\,y) \end{pmatrix} U$$

$$V_{\perp} = \begin{pmatrix} I_k & \mathrm{O}_{k\times(n-k)} \\ y & \mathrm{O}_{n-k} \end{pmatrix} V = \begin{pmatrix} I_k \\ y \end{pmatrix} (a + b\,y) U$$

$\Rightarrow$ coupled flow on $\mathrm{Gr}(n,k) \times \mathrm{GL}(k)$

$$\begin{aligned} y' &= c + d\,y - y(a + b\,y) \\ U' &= (a + b\,y)\,U \end{aligned}$$

Continuous orthogonalization

- $Y = [Y_1 Y_2 \cdots Y_k]$
- Polar decomposition: $Y = \Omega\,\alpha$
- Stiefel manifold $\mathbb{V}(n,k)$: $\Omega^*\Omega = I$
- Substitute ansatz into $Y' = AY$:

$$\Omega' = (I - \Omega\Omega^*)A\Omega$$

- With slaved linear ODE for $\gamma \equiv \det\alpha$.

Push forward the linear vector field to $\mathrm{Gr}(n, k)$

$$Y' = A(x)Y$$

$$\begin{pmatrix} U \\ L \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} U \\ L \end{pmatrix}$$

$$\Rightarrow \qquad \begin{aligned} y' &= \left(LU^{-1}\right)' \\ &= L'U^{-1} - LU^{-1}U'U^{-1} \\ &= (cU + dL)U^{-1} - y(aU + bL)U^{-1} \\ &= c + dy - y(a + by) \end{aligned}$$

Evans function

$$
\begin{aligned}
D(\lambda) &\equiv \det\big(Y^- \; Y^+\big) \\
&= \det \begin{pmatrix} I_k & y^+ \\ y^- & I_{n-k} \end{pmatrix} \cdot \det U^- \cdot \det L^+
\end{aligned}
$$

Boussinesq system I

$$u_{tt} = u_{xx} - u_{xxxx} - (u^2)_{xx},$$

$$\bar{u}(\xi) = \tfrac{3}{2}(1 - c^2)\mathrm{sech}^2\left(\tfrac{1}{2}\sqrt{1 - c^2}\,\xi\right)$$

waves stable when $1/2 < |c| < 1$ and unstable when $|c| < 1/2$.

Boussinesq system II

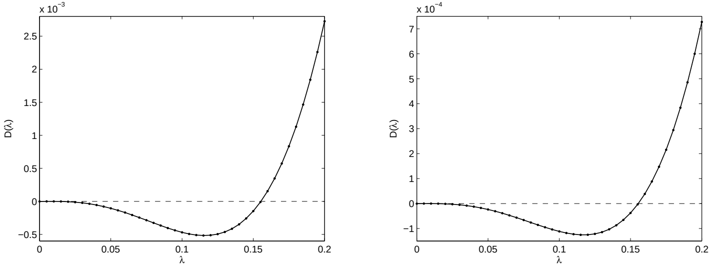


Figure: Riccati (left) and continuous orthogonalization (right).

Autocatalytic fronts

$$u_t = \delta u_{xx} + c u_x - u v^m$$
$$v_t = v_{xx} + c v_x + u v^m$$

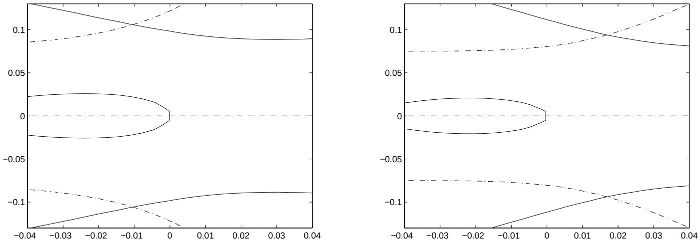


Figure: $\delta = 0.1$: $m = 8$ (left) and $m = 9$ (right).

Ekman boundary layer I

$$u_t + uu_x + vu_y + wu_z + \frac{1}{\mathrm{R_o}} p_x - \frac{2}{\mathrm{R_o}} v = \frac{1}{\mathrm{R_e}} (u_{xx} + u_{yy}) + \frac{1}{\mathrm{R_o}} u_{zz}$$
$$v_t + uv_x + vv_y + wv_z + \frac{1}{\mathrm{R_o}} p_y + \frac{2}{\mathrm{R_o}} u = \frac{1}{\mathrm{R_e}} (v_{xx} + v_{yy}) + \frac{1}{\mathrm{R_o}} v_{zz}$$
$$w_t + uw_x + vw_y + ww_z + \frac{1}{\mathrm{R_o E_k}} p_z = \frac{1}{\mathrm{R_e}} (w_{xx} + w_{yy}) + \frac{1}{\mathrm{R_o}} w_{zz}$$

Set $\mathrm{R_e} = \mathrm{R_o}$, $\mathrm{E_k} = 1$.

Base state:

$$V(z) = \cos(\epsilon)\big(1 - \exp(-z)\cos(z)\big) + \sin(\epsilon)\exp(-z)\sin(z)$$
$$U(z) = -\sin(\epsilon)\big(1 - \exp(-z)\cos(z)\big) + \cos(\epsilon)\exp(-z)\sin(z)$$

Ekman boundary layer II

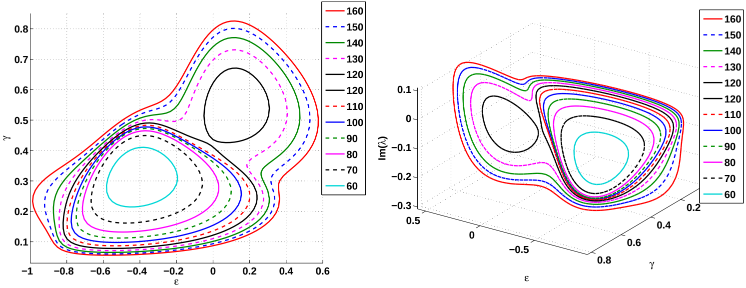


Figure: Neutral curves for rigid wall, $\mathrm{R_e}$ fixed (values indicated).

Future work: multi-d I

Cylindrical domain $\mathbb{R} \times \mathbb{T}$:

$$\partial_t U = B\,\Delta U + c\,\partial_x U + F(U)$$

$$B\Delta U + c\,\partial_x U + \mathrm{D}F(U_c(x,y))U = \lambda U$$

Multi-d II

$$U(x,y) = \sum_{k=-\infty}^{\infty} \hat{U}_k(x)\, \mathrm{e}^{iky/\tilde{L}}$$

$$\mathrm{D}F(U_c(x,y)) = \sum_{k=-\infty}^{\infty} \hat{D}_k(x)\, \mathrm{e}^{iky/\tilde{L}}$$

Multi-d II

$$\partial_x \hat{U}_k = \hat{P}_k$$

$$\partial_x \hat{P}_k = \lambda B^{-1} \hat{U}_k + (k/\tilde{L})^2 \hat{U}_k - c\, B^{-1} \hat{P}_k - \sum_{\nu=-K}^{K} B^{-1} \hat{D}_{k-\nu} \hat{U}_\nu$$

$$\partial_x \begin{pmatrix} \hat{U} \\ \hat{P} \end{pmatrix} = \begin{pmatrix} O_{2(2K+1)} & I_{2(2K+1)} \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} \hat{U} \\ \hat{P} \end{pmatrix}$$

Multi-d III

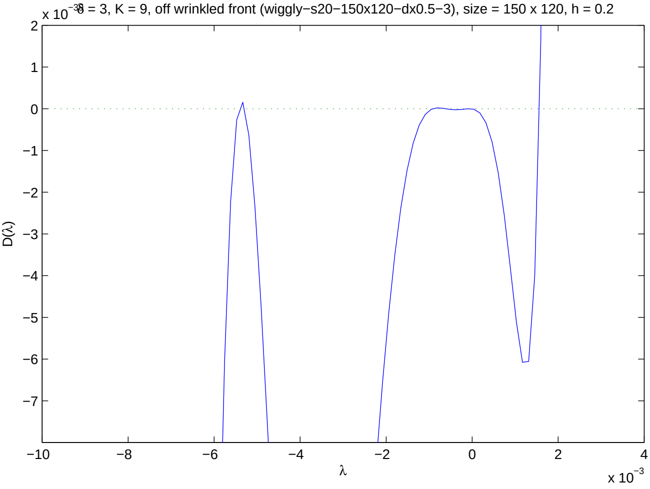


Figure: Evans function along real λ-axis