High order stochastic integrators for linear systems

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Outline

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- Neumann and Magnus SDE integrators
- Global errors:
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  - Quadrature approximation
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Introduction: linear Stratonovich system

$$S(t) = I + \int_0^t A_0(\tau)S(\tau)\,\mathrm{d}W_0(\tau) + \int_0^t A_1(\tau)S(\tau)\,\mathrm{d}W_1(\tau) + \int_0^t A_2(\tau)S(\tau)\,\mathrm{d}W_2(\tau)$$

$$S = I + \mathsf{K}_0 \circ S + \mathsf{K}_1 \circ S + \mathsf{K}_2 \circ S$$

$$S = I + \mathsf{K} \circ S$$

$$\mathsf{K} \equiv \mathsf{K}_0 + \mathsf{K}_1 + \mathsf{K}_2 \quad \text{and} \quad W_0(t) \equiv t$$

Neumann series

$$S = I + \mathsf{K} \circ S$$

$$(\mathsf{I} - \mathsf{K}) \circ S = I$$

$$S = (\mathsf{I} - \mathsf{K})^{-1} \circ I$$

$$S = (\mathsf{I} + \mathsf{K} + \mathsf{K}^2 + \mathsf{K}^3 + \mathsf{K}^4 + \cdots) \circ I$$

$$S(t) = I + \mathsf{K} \circ I + \mathsf{K}^2 \circ I + \mathsf{K}^3 \circ I + \mathsf{K}^4 \circ I + \cdots$$

(*Peano–Baker series*, *Feynman–Dyson path ordered exponential* or *Chen-Fleiss series*)

Magnus series

$$S(t) = \exp\left(\sigma(t)\right) \,,$$

$$\begin{aligned}\sigma(t) = {} & \ln\big(I + \mathsf{K}\circ I + \mathsf{K}^2\circ I + \cdots\big) \\ & = \mathsf{K}\circ I + \mathsf{K}^2\circ I - \tfrac{1}{2}(\mathsf{K}\circ I)^2 + \cdots \,,\end{aligned}$$

(Magnus 1954, Kunita 1980, Ben Arous 1989, Burrage 1999)

Integral operators

$$\mathsf{K} \circ I = \mathsf{K}_0 \circ I + \mathsf{K}_1 \circ I + \mathsf{K}_2 \circ I$$

$$= \int_0^t A_0(\tau)\,\mathrm{d}W_0(\tau) + \int_0^t A_1(\tau)\,\mathrm{d}W_1(\tau) + \int_0^t A_2(\tau)\,\mathrm{d}W_2(\tau)$$

$$\mathsf{K}^2 \circ I = (\mathsf{K}_0 + \mathsf{K}_1 + \mathsf{K}_2)^2 \circ I$$

$$= (\mathsf{K}_0^2 + \mathsf{K}_0\mathsf{K}_1 + \mathsf{K}_1\mathsf{K}_0 + \mathsf{K}_1^2 + \mathsf{K}_1\mathsf{K}_2 + \mathsf{K}_2\mathsf{K}_1 + \mathsf{K}_2^2 + \mathsf{K}_2\mathsf{K}_0 + \mathsf{K}_0\mathsf{K}_2) \circ I$$

$$\mathsf{K}_i\mathsf{K}_j \circ I \equiv \int_0^t A_i(\tau_1) \int_0^{\tau_1} A_j(\tau_2)\,\mathrm{d}W_j(\tau_2)\mathrm{d}W_i(\tau_1)$$

Constant coefficient, non-commutative case

$$\mathsf{K}_i\mathsf{K}_j \circ I \equiv a_i a_j \underbrace{\int_0^t \int_0^{\tau_1} \mathrm{d}W_j(\tau_2)\mathrm{d}W_i(\tau_1)}_{J_{ji}}$$

$$S^{\text{neu}}(t) = \sum_{n=0}^{\infty} \sum_{p \in \mathbb{P}_n} a_{p_n} \cdots a_{p_1} J_{p_1 \cdots p_n}(t)\,,$$

Special non-commutative cases:

$$S(t) = I + a_1 \cdot \int_0^t S(\tau)\,\mathrm{d}W_1(\tau) + \int_0^t S(\tau)\,\mathrm{d}W_2(\tau) \cdot a_2\,,$$

$$S(t) = \exp\bigl(a_1 W_1(t)\bigr) \cdot \exp\bigl(a_2 W_2(t)\bigr).$$

Neumann expansion (order 2)

$$S^{\rm neu}(t) \approx I + S_{1/2} + S_1 + S_{3/2} + S_2\,,$$

$$S_{1/2} = a_1 J_1 + a_2 J_2 + a_0 J_0 + a_1^2 J_{11} + a_2^2 J_{22}\,,$$

$$S_1 = a_2 a_1 J_{12} + a_1 a_2 J_{21}\,,$$

$$\begin{aligned} S_{3/2} = {} & a_0^2 J_{00} + a_0 a_1 J_{10} + a_1 a_0 J_{01} + a_0 a_2 J_{20} + a_2 a_0 J_{02} \\ & + a_1^3 J_{111} + a_2 a_1^2 J_{112} + a_1 a_2 a_1 J_{121} + a_2^2 a_1 J_{122} \\ & + a_1^2 a_2 J_{211} + a_2 a_1 a_2 J_{212} + a_1 a_2^2 J_{221} + a_2^3 J_{222} \\ & + a_1^2 a_0 J_{011} + a_1 a_0 a_1 J_{101} + a_0 a_1^2 J_{110} + a_2^2 a_0 J_{022} \\ & + a_2 a_0 a_2 J_{202} + a_0 a_2^2 J_{220} + a_1^4 J_{1111} + a_2^2 a_1^2 J_{1122} \\ & + a_2 a_1 a_2 a_1 J_{1212} + a_1 a_2 a_1 a_2 J_{2121} + a_1^2 a_2^2 J_{2211} + a_2^4 J_{2222}\,, \end{aligned}$$

$$\begin{aligned} S_2 = {} & a_0 a_1 a_2 J_{210} + a_0 a_2 a_1 J_{120} + a_1 a_0 a_2 J_{201} + a_1 a_2 a_0 J_{021} \\ & + a_2 a_0 a_1 J_{102} + a_2 a_1 a_0 J_{012} + a_2 a_1^3 J_{1112} + a_1^2 a_2 a_1 J_{1121} \\ & + a_1^2 a_2 a_1 J_{1211} + a_1 a_2^2 a_1 J_{1221} + a_2^3 a_1 J_{1222} + a_2 a_1^3 J_{2111} \\ & + a_2 a_1^2 a_2 J_{2112} + a_2^2 a_1 a_2 J_{2122} + a_2 a_1 a_2^2 J_{2212} + a_1 a_2^3 J_{2221}\,. \end{aligned}$$

Relations between Stratonovich integrals

$$
\begin{aligned}
J_{121} &= J_1 J_{12} - 2J_{112}\,, \\
J_{122} &= J_2 J_{12} - \tfrac{1}{2} J_1 J_2^2 + J_{221}\,, \\
J_{210} &= J_0 J_{21} - J_2 J_{01} + J_{012}\,, \\
J_{102} &= J_1 J_{02} - J_{021} - J_{012}\,,
\end{aligned}
$$

$$
\begin{aligned}
J_{1121} &= J_{112} J_1 - 3J_{1112}\,, \\
J_{1211} &= \tfrac{1}{2} J_1^2 J_{12} - 2J_{112} J_1 + 3J_{1112}\,, \\
J_{1212} &= \tfrac{1}{2} J_{12}^2 - 2J_{1122}\,, \\
J_{1221} &= (J_2 J_{12} - \tfrac{1}{2} J_1 J_2^2 + J_{221}) J_1 - \tfrac{1}{2} J_{12}^2\,, \\
J_{1222} &= J_{122} J_2 - \tfrac{1}{2} J_{12} J_2^2 + \tfrac{1}{6} J_1 J_2^3 - J_{2221}\,.
\end{aligned}
$$

(Kloeden & Platen 1999; Gaines 1995; Kawski 2001)

Magnus expansion (order 2)

$$S^{\mathrm{mag}}(t) = \exp\big(\sigma(t)\big)\,, \qquad \sigma(t) = s_{1/2} + s_1 + s_{3/2} + s_2 + \cdots$$

$$s_{1/2} = a_1 J_1 + a_2 J_2 + a_0 J_0\,,$$

$$s_1 = \tfrac{1}{2}[a_1, a_2](J_{21} - J_{12})\,,$$

$$\begin{aligned} s_{3/2} = {} & \tfrac{1}{2}[a_0, a_1](J_{10} - J_{01}) + \tfrac{1}{2}[a_0, a_2](J_{20} - J_{02}) \\ & + [a_1, [a_1, a_2]]\left(J_{112} - \tfrac{1}{2}J_1 J_{12} + \tfrac{1}{12}J_1^2 J_2\right) \\ & + [a_2, [a_2, a_1]]\left(J_{221} - \tfrac{1}{2}J_2 J_{21} + \tfrac{1}{12}J_2^2 J_1\right) \\ & + [a_1, [a_1, a_0]]\left(J_{110} - \tfrac{1}{2}J_1 J_{10} + \tfrac{1}{12}J_1^2 J_0\right) \\ & + [a_2, [a_2, a_0]]\left(J_{220} - \tfrac{1}{2}J_2 J_{20} + \tfrac{1}{12}J_2^2 J_0\right)\,, \end{aligned}$$

$$\begin{aligned} s_2 = {} & + [a_2, [a_1, a_0]]\left(J_{120} + \tfrac{1}{2}J_1 J_{02} + \tfrac{1}{2}J_0 J_{21} - \tfrac{2}{3}J_0 J_1 J_2\right) \\ & + [a_1, [a_2, a_0]]\left(J_{210} + \tfrac{1}{2}J_2 J_{01} + \tfrac{1}{2}J_0 J_{12} - \tfrac{2}{3}J_0 J_1 J_2\right) \\ & - [a_1, [a_1, [a_1, a_2]]]\left(J_{1112} - \tfrac{1}{2}J_1 J_{112} + \tfrac{1}{12}J_1^2 J_{12}\right) \\ & - [a_2, [a_2, [a_2, a_1]]]\left(J_{2221} - \tfrac{1}{2}J_2 J_{221} + \tfrac{1}{12}J_2^2 J_{21}\right) \\ & + [a_1, [a_2, [a_1, a_2]]]\left(\tfrac{1}{24}J_1^2 J_2^2 - \tfrac{1}{2}J_2 J_{112} + \tfrac{1}{6}J_1 J_2 J_{21} - \tfrac{1}{2}J_1 J_{221} + J_{1122}\right)\,. \end{aligned}$$

Numerical SDE schemes

General integrators

- Euler-Maruyama and Milstein methods
- Runge–Kutta type methods (Kloeden & Platen 1999)
- Magnus (Burrage 1999; Misawa 2001; P-C. Moan 2004)

Linear systems: Neumann $\equiv$ stochastic Taylor $\equiv$ Runge–Kutta

Magnus integrators

- Not overly complex (Burrage 1999)
- Low dimensional computationally favourable basis

$$\{J_0, J_1, J_2, J_{12}, J_{01}, J_{02}, J_{112}, J_{221}, J_{110}, J_{220}, J_{120}, J_{210}, J_{1112}, J_{2221}, J_{1122}\} \,.$$

- More accurate for the same computational expense for smaller systems

Implementation: Global errors I

Global interval:

$$[0,T] = \cup_{n=0}^{N-1} [t_n, t_{n+1}]\,, \quad t_n = nh$$

Global error:

$$\mathcal{E} = \mathbb{E} \left\| \prod_{n=N-1}^{0} S(t_n, t_{n+1}) - \prod_{n=N-1}^{0} S^{\text{approx}}(t_n, t_{n+1}) \right\|$$

$$= \mathcal{E}^{\text{trunc}} + \mathcal{E}^{\text{quad}}\,,$$

Local error:

$$\begin{aligned} \mathrm{e}^{\sigma(t_n,t_{n+1})} - \mathrm{e}^{\sigma_M(t_n,t_{n+1})} &= \mathrm{e}^{\sigma_M(t_n,t_{n+1}) + R_M(t_n,t_{n+1})} - \mathrm{e}^{\sigma_M(t_n,t_{n+1})} \\ &= R_M(t_n, t_{n+1}) + \mathcal{O}(\sigma_M R_M)\,. \end{aligned}$$

Global errors II

$$\mathcal{E} = \mathbb{E}\left\| \prod_{n=N-1}^{0} \mathrm{e}^{\sigma(t_n,t_{n+1})} - \prod_{n=N-1}^{0} \mathrm{e}^{\sigma_M(t_n,t_{n+1})} \right\|$$

$$= \mathbb{E}\left\| \sum_{n=0}^{N-1} R_M(t_n,t_{n+1}) \right\|$$

$$\leq \sum_{\alpha} \|a_\alpha\| \, \mathbb{E}\left| \sum_{n=0}^{N-1} J_\alpha(t_n,t_{n+1}) \right| .$$

Global errors III

$$\begin{aligned}
\mathbb{E}\big(J_{120}\big) = \mathbb{E}\big(J_1 J_{02}\big) = \mathbb{E}\big(J_0 J_{21}\big) = \mathbb{E}\big(J_0 J_1 J_2\big) &= 0\,,\\
\mathbb{E}\big(J_{1112}\big) = \mathbb{E}\big(J_1 J_{112}\big) = \mathbb{E}\big(J_1^2 J_{12}\big) &= 0\,,\\
\mathbb{E}\big(\tfrac{1}{24}J_1^2 J_2^2 + \tfrac{1}{6}J_1 J_2 J_{21} + J_{1122}\big) = \mathbb{E}\big(J_2 J_{112}\big) = \mathbb{E}\big(J_1 J_{221}\big) &= 0\,,\\
\mathbb{E}\big(J_{110} - \tfrac{1}{2}J_1 J_{10} + \tfrac{1}{12}J_1^2 J_0\big) &= \tfrac{1}{12}h^2\,.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left|\sum_{n=0}^{N-1} J_\alpha(t_n, t_{n+1})\right| &\leq \left(\mathbb{E}\sum_{n=0}^{N-1}\big(J_\alpha(t_n, t_{n+1})\big)^2\right)^{\frac{1}{2}}\\
&= \left(\sum_{n=0}^{N-1}\mathrm{Var}\big(J_\alpha(t_n, t_{n+1})\big)\right)^{\frac{1}{2}}\\
&= \sqrt{N}h^2\\
&= \sqrt{T}h^{3/2}\,.
\end{aligned}$$

Magnus more accurate than Neumann?

For 1/2 order schemes:

$$R^{\mathrm{mag}}(t_n, t_{n+1}) = \tfrac{1}{2}[a_1, a_2](J_{21} - J_{12})\,;$$

$$R^{\mathrm{neu}}(t_n, t_{n+1}) = R^{\mathrm{mag}}(t_n, t_{n+1}) + \underbrace{\tfrac{1}{2}(a_1 a_2 + a_2 a_1)(J_{21} + J_{12})}_{\hat{R}}\,.$$

Using Variance as error measure, for all initial data x_0:

$$\begin{aligned}
\mathbb{E}\big(\langle R^{\mathrm{neu}}\, x_0\,,\, R^{\mathrm{neu}}\, x_0\rangle\big) &= \mathbb{E}\big(x_0^{\mathrm{T}}\,(R^{\mathrm{neu}})^{\mathrm{T}}\, R^{\mathrm{neu}}\, x_0\big) \\
&= \mathbb{E}\big(x_0^{\mathrm{T}}\,(R^{\mathrm{mag}} + \hat{R})^{\mathrm{T}}\,(R^{\mathrm{mag}} + \hat{R})\, x_0\big) \\
&= x_0^{\mathrm{T}}\,\Big(\mathbb{E}\big((R^{\mathrm{mag}})^2\big) + \mathbb{E}\big((\hat{R})^2\big) \\
&\qquad + \underbrace{\mathbb{E}\big(\hat{R}\cdot R^{\mathrm{mag}}\big)}_{=0} + \underbrace{\mathbb{E}\big(R^{\mathrm{mag}}\cdot \hat{R}\big)}_{=0}\Big)\, x_0\,.
\end{aligned}$$

Quadrature approximation

Two inherent scales:

- Wiener-discrete-path scale Δt—the *smallest scale* on which the Wiener paths $W_1(t)$ and $W_2(t)$ are generated;
- Time-step scale $h = Q\Delta t$—on which the SDE is stepped forward.

- $J_{12}(t_n, t_{n+1}) \equiv \int_{t_n}^{t_{n+1}} \big(W_1(\tau) - W_1(t_n)\big)\,\mathrm{d}W_2(\tau)$
- Filtration: $\mathcal{F}_Q(n) = \{W_i(t_n + q\,\Delta t) \colon i = 1, 2\,; q = 0, \ldots, Q\}$
- Basic idea: $J \;\rightarrow\; \mathbb{E}\big(J \big| \mathcal{F}_Q(n)\big)$
- Polygonal volumes

Quadrature approximation II

$$\hat{J}_{12}(t_n,t_{n+1}) = \tfrac{1}{2}\sum_{q=0}^{Q-1}\Big(\big(W_1(\tau_{q+1})-W_1(\tau_0)\big)+\big(W_1(\tau_q)-W_(\tau_0)\big)\Big)\Delta W_2(t_q)\,;$$

$$\begin{aligned}\hat{J}_{112}(t_n,t_{n+1}) = \tfrac{1}{6}\sum_{q=0}^{Q-1}\bigg(\Big(&\big(W_1(\tau_{q+1})-W_1(\tau_0)\big)^2\\ &+\big(W_1(\tau_{q+1})-W_1(\tau_0)\big)\big(W_1(\tau_q)-W_1(\tau_0)\big)\\ &+\big(W_1(\tau_q)-W_1(\tau_0)\big)^2\Big)\Delta W_2(\tau_q)\bigg)\\ &+\tfrac{1}{12}\Delta t\big(W_2(t_{n+1})-W_2(t_n)\big)\,;\end{aligned}$$

$$\begin{aligned}\hat{J}_{120}(t_n,t_{n+1}) = \tfrac{1}{2}\Delta t\sum_{q=0}^{Q-1}(Q-q-1)\Big(&\Delta W_1(\tau_q)+2\big(W_1(\tau_q)-W_1(\tau_0)\big)\Big)\Delta W_2(\tau_q)\\ &+\tfrac{1}{6}\Delta t\sum_{q=0}^{Q-1}\Big(\Delta W_1(\tau_q)+3\big(W_1(\tau_q)-W_1(\tau_0)\big)\Big)\Delta W_2(\tau_q)\,.\end{aligned}$$

Quadrature approximation III

$$
\begin{aligned}
\mathcal{E}_{12}(n) &\equiv \mathbb{E}\Big(\big|J_{12}(t_n,t_{n+1})-\hat{J}_{12}(t_n,t_{n+1})\big|\Big)\\
&= \mathbb{E}\left(\left|\sum_{q=0}^{Q-1}\big(J_{12}(\tau_q,\tau_{q+1})-\tfrac{1}{2}\Delta W_1(\tau_q)\Delta W_2(\tau_q)\big)\right|\right)\\
&\leq \left(\sum_{q=0}^{Q-1}\mathbb{E}\big|J_{12}(\tau_q,\tau_{q+1})-\tfrac{1}{2}\Delta W_1(\tau_q)\Delta W_2(\tau_q)\big|^2\right)^{1/2}\\
&= \left(\sum_{q=0}^{Q-1}\mathbb{E}\Big(\mathbb{E}\Big(\big|J_{12}(\tau_q,\tau_{q+1})-\tfrac{1}{2}\Delta W_1(\tau_q)\Delta W_2(\tau_q)\big|^2\Big|\mathcal{F}_Q(n)\Big)\Big)\right)^{1/2}\\
&= \left(\sum_{q=0}^{Q-1}\mathbb{E}\Big(\mathrm{Var}\big(J_{12}(\tau_q,\tau_{q+1})\big|\mathcal{F}_Q(n)\big)\Big)\right)^{1/2}\\
&= \left(\sum_{q=0}^{Q-1}\mathbb{E}\Big(\tfrac{1}{6}\Delta t\big((\Delta W_1(\tau_q))^2+(\Delta W_2(\tau_q))^2\big)\Big)\right)^{1/2}\\
&= \mathcal{O}\big(\Delta t\,\sqrt{Q}\big)\\
&= \mathcal{O}\big(h/\sqrt{Q}\big)\,,
\end{aligned}
$$

Quadrature approximation IV

Quadrature		$\hat{J}_{12}$	$\hat{J}_{112}$	$\hat{J}_{120}$	$\hat{J}_{1112}$	$\hat{J}_{1122}$
$\mathcal{E}^{\text{loc}}(n)$		$h/Q^{1/2}$	$h^{3/2}/Q^{1/2}$	$h^2/Q^{1/2}$	$h^2/Q^{1/2}$	$h^2/Q^{1/2}$
	$\mathcal{O}\big(h^{3/2}\big)$	h^{-1}	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\mathcal{U}$	$\mathcal{O}\big(h^{2}\big)$	h^{-2}	h^{-1}	$\cdots$	$\cdots$	$\cdots$
	$\mathcal{O}\big(h^{5/2}\big)$	h^{-3}	h^{-2}	h^{-1}	h^{-1}	h^{-1}

Error vs effort

Locally

$$\begin{aligned} \mathcal{E}^{\mathrm{loc}} \sim h^{3/2} \quad &\text{and} \quad \mathcal{U} \sim h^{-1} \quad &\Longrightarrow \log \mathcal{E}^{\mathrm{loc}} \sim -\tfrac{3}{2} \log \mathcal{U}\,, \\ \mathcal{E}^{\mathrm{loc}} \sim h^{2} \quad &\text{and} \quad \mathcal{U} \sim h^{-2} \quad &\Longrightarrow \log \mathcal{E}^{\mathrm{loc}} \sim - \log \mathcal{U}\,, \\ \mathcal{E}^{\mathrm{loc}} \sim h^{5/2} \quad &\text{and} \quad \mathcal{U} \sim h^{-3} \quad &\Longrightarrow \log \mathcal{E}^{\mathrm{loc}} \sim -\tfrac{5}{6} \log \mathcal{U}\,. \end{aligned}$$

Globally

$$\mathcal{E}^{\mathrm{quad}} = \mathcal{O}\left(\frac{h^{\alpha}}{\sqrt{Q}}\right)\sqrt{N} = \mathcal{O}\left(\frac{h^{\alpha-\frac{1}{2}}}{\sqrt{Q}}\sqrt{T}\right).$$

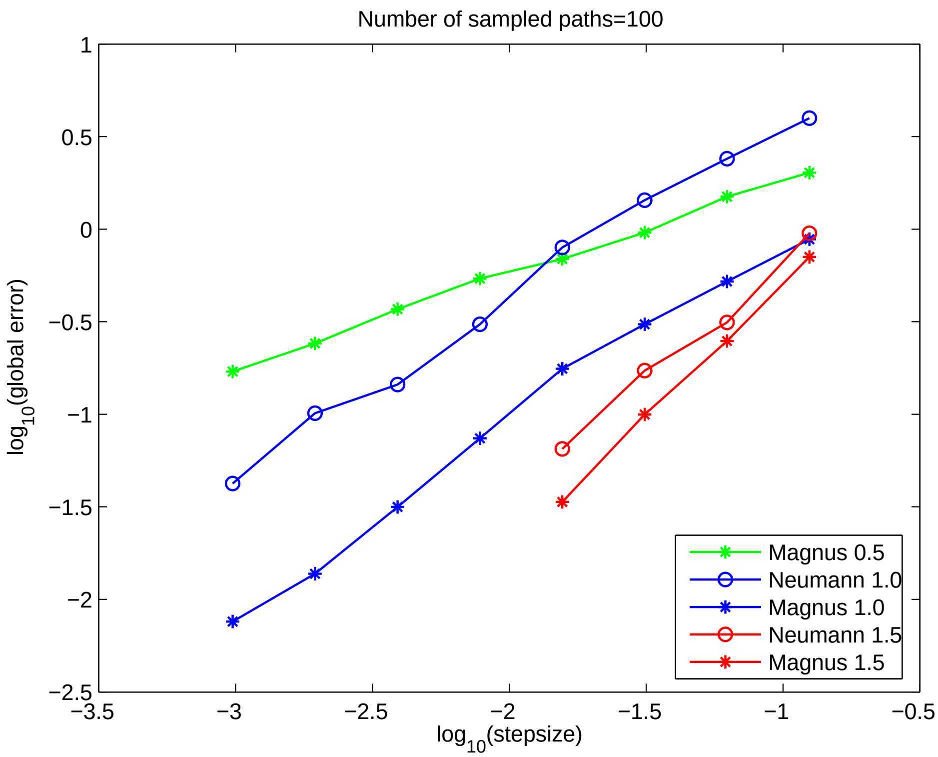
$$\mathcal{U} \sim QN\,.$$

Numerical simulations

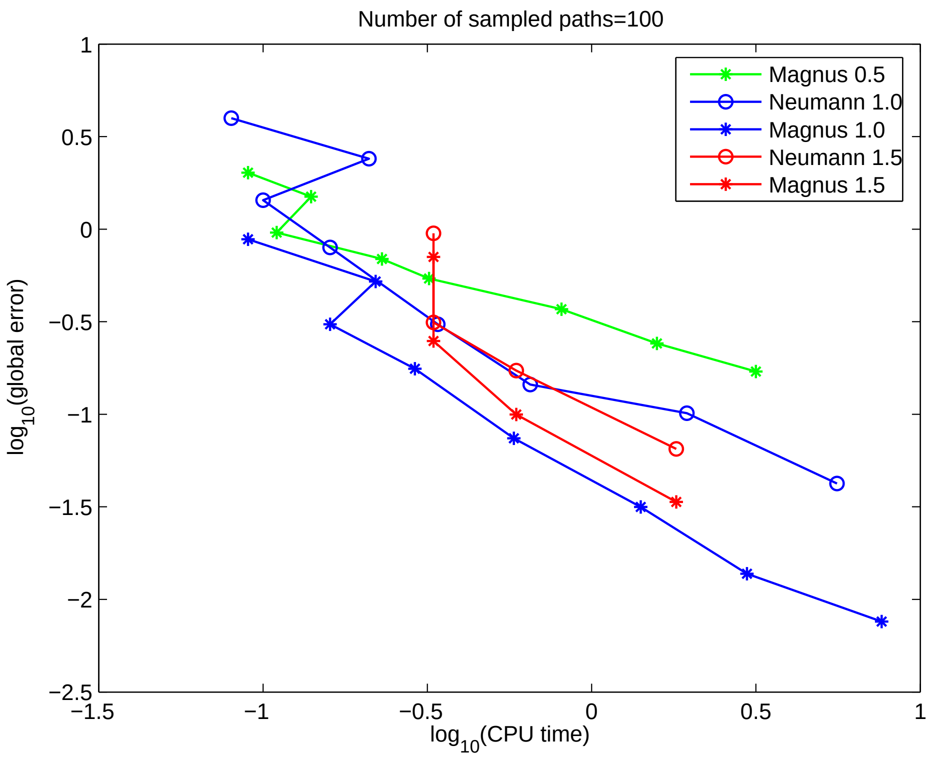
Linear system

$$a_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \qquad a_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{51}{200} \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix},$$

Linear system



Linear system



Flops: two Wiener processes

	For each path			Independent of the path	
Order	Neumann	Magnus	Runge–Kutta	Neumann	Magnus
$\frac{1}{2}$	$9n^2$	$5n^2 + 5n^3$	$\cdots$	n/a	n/a
1	$13n^2$	$7n^2 + 5n^3$	$\cdots$	$4n^3$	$2n^3$
$1\frac{1}{2}$	$63n^2$	$19n^2 + 5n^3$	$20n^3 + 37n^2$	$29n^3$	$14n^3$
2	$95n^2$	$29n^2 + 5n^3$	$\cdots$	$45n^3$	$26n^3$

Stochastic Riccati system

$$S(t) = I + \sum_{i=0}^{d} \int_0^t \big(S(\tau)A_i(\tau)S(\tau) + B_i(\tau)S(\tau) + S(\tau)C_i(\tau) + D_i(\tau)\big)\, \mathrm{d}W_i(\tau)\,.$$

- Stochastic linear-quadratic optimal control.
- eg. mean-variance hedging in finance.
- Bobrovnytska & Schweizer 2004; Kohlmann & Tang 2003.

Riccati II

If $\mathbb{A}_i(t) \equiv \begin{pmatrix} B_i(t) & D_i(t) \\ -A_i(t) & -C_i(t) \end{pmatrix}$ and $\mathbb{U} = \begin{pmatrix} U \\ V \end{pmatrix}$ satisfies

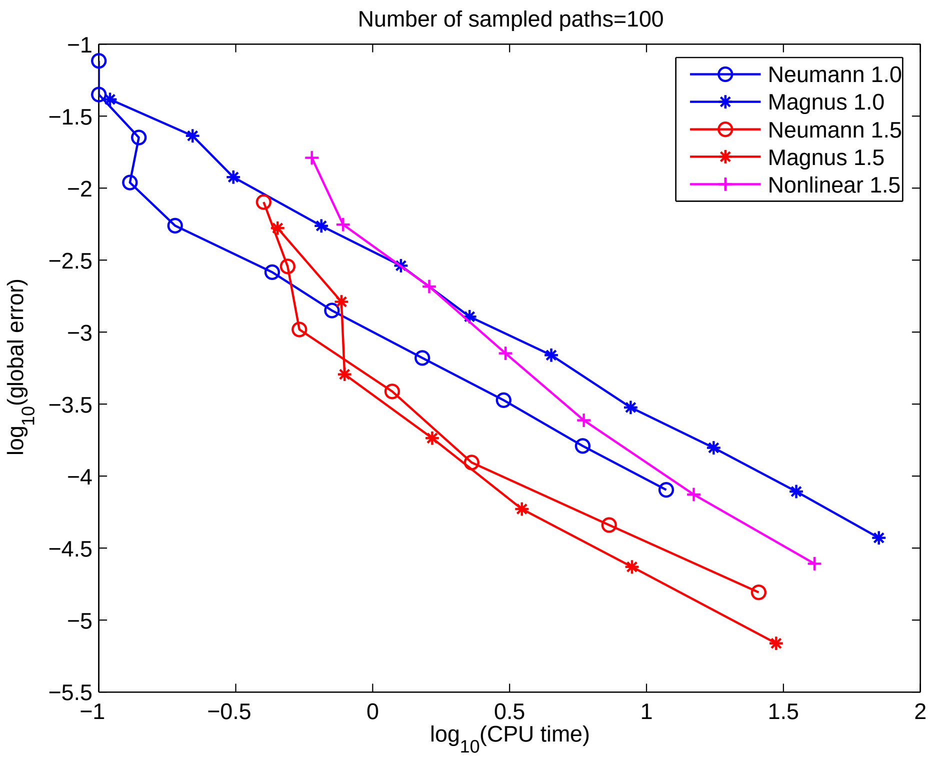
$$\mathbb{U}(t) = \mathbb{I} + \sum_{i=0}^{d} \int_0^t \mathbb{A}_i(\tau)\mathbb{U}(\tau)\,\mathrm{d}W_i(\tau)\,,$$

then $S = UV^{-1}$ solves the Riccati system.

$$A_0 = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}, \qquad C_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & -1 \end{pmatrix} \qquad \text{and} \qquad D_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

$D_1 = a_1$ and $D_2 = a_2$.

Riccati III


Number of sampled paths=100
log10(global error)
log10(CPU time)
Neumann 1.0
Magnus 1.0
Neumann 1.5
Magnus 1.5
Nonlinear 1.5
−1
−1.5
−2
−2.5
−3
−3.5
−4
−4.5
−5
−5.5
−1
−0.5
0
0.5
1
1.5
2

Riccati IV

Kloeden & Platen:

$$\begin{aligned} S_{n+1} = S_n &+ f(S_n)h + D_1 J_1 + D_2 J_2 \\ &+ \tfrac{h}{4}\big(f(Y_1^+) + f(Y_1^-) + f(Y_2^+) + f(Y_2^-) - 4f(S_n)\big) \\ &+ \tfrac{1}{2\sqrt{h}}\left(\big(f(Y_1^+) - f(Y_1^-)\big)J_{10} + \big(f(Y_2^+) - f(Y_2^-)\big)J_{20}\right) , \end{aligned}$$

$$Y_j^{\pm} = S_n + \tfrac{h}{2} f(S_n) \pm D_j \sqrt{h}$$

$$f(S) = S A_0 S + B_0 S + S C_0 + D_0 \,.$$

Conclusions

- Error vs time? Parallelization.
- Variable step scheme (Gaines & Lyons 1997).
- Zakai equation: Markov chain filters.
- Lie-group preserving properties (Castell & Gaines 1995; Burrage *et al.* 2004; Misawa 2001; Milstein *et al.* 2002).
- Driven by Lèvy processes
- Backwards SDEs
- Nonlinear SDEs