Computing the Maslov index for large systems

Margaret Beck and Simon J.A. Malham

Stirling: December 13th and 14th

Spectral problems

Elliptic operator on $\mathbb{R} \times \mathbb{T}$:

$$H := B\,(\partial_x^2 + \partial_y^2) + d\,\partial_x + V(x,y).$$

Eigenvalue problem:

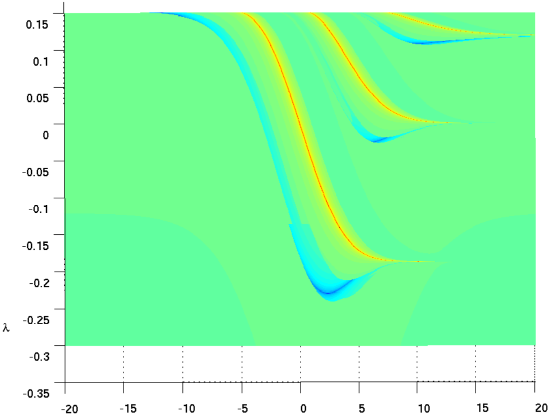
$$(H - \lambda)q = O \qquad \Leftrightarrow \qquad (\partial_x - A)\begin{pmatrix} q \\ p \end{pmatrix} = O$$

where

$$\partial - A \colon H^1(\mathbb{R};\mathbb{H}) \to L^2(\mathbb{R};\mathbb{H}).$$

First case: *H selfadjoint* and $A \colon \mathbb{R} \to \mathfrak{sp}(\mathbb{R}^{2n})$.

Oscillation index



Total frame matrix

Path of symplectic frames generated by:

$$\partial \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

Measure intersection of solution subspaces via

$$\begin{pmatrix} q & q_0 \\ p & p_0 \end{pmatrix},$$

Definition (Total frame matrix)

$$\begin{pmatrix} q' & O \\ p' & \mathrm{id} \end{pmatrix} = \begin{pmatrix} \mathrm{id} & q_0 p_0^{-1} \\ O & p_0^{-1} \end{pmatrix} \begin{pmatrix} q & q_0 \\ p & p_0 \end{pmatrix},$$

Lagrangian Grassmannian

Fibering: $\mathrm{GL}(\mathbb{R}^n) \to \mathrm{V}(\mathbb{R}^{2n}) \to \Lambda(\mathbb{R}^{2n})$.

Chart $s\colon q \mapsto p$ is real symmetric:

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{id} \\ s \end{pmatrix},$$

Cayley transform $\mathrm{Cay}\colon \mathrm{D}(\mathbb{R}^n) \to \mathrm{U}_{\mathrm{sym}}(\mathbb{C}^n)$:

$$\mathrm{Cay}\colon s \mapsto \frac{\mathrm{id} - \mathrm{i}\, s}{\mathrm{id} + \mathrm{i}\, s}$$

Arnol'd $\implies$ $\mathrm{Det}^2 := \det \circ \mathrm{Cay}$.

Riccati flow in the chart

Evolutionary problem for $\partial - A$ using $p = s\,q$:

$$\begin{aligned}\partial q &= (a + bs)\,q,\\ \partial p &= (c + ds)\,q.\end{aligned}$$

Then

$$\begin{aligned}(\partial s)\,q &= \partial(sq) - s\,\partial q\\ &= (c + ds)\,q - s\,(a + bs)\,q\end{aligned}$$

$$\Leftrightarrow \qquad \partial s = c + ds - s(a + bs).$$

Equivalent Maslov index contributions

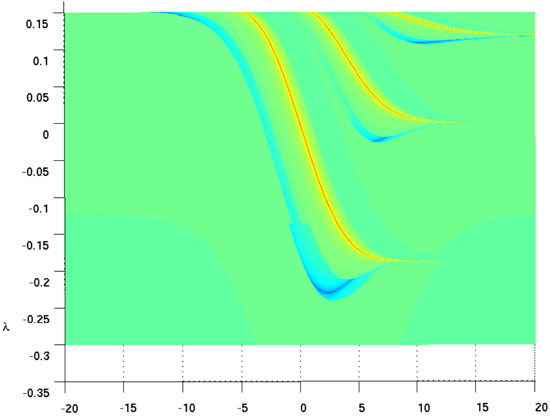
Theorem (Beck & M. 2013)

When the Lagrangian plane intersects the standard reference plane, the following quantites are equal (ignoring direction of crossing):

1. *Unsigned integer jump value in the Maslov index;*
2. *Rank loss in the matrix q;*
3. *Rank loss in the total frame matrix;*
4. *Dimension of intersection of Lagrangian and standard reference planes;*
5. *Number of singular eigenvalues of s.*

Singularities

First observed by Veerle Ledoux (Ledoux & M. 2009).



Cayley reprise

$$\mathrm{Cay}\colon \mathrm{D}(\mathbb{R}^n) \to \mathrm{U}_{\mathrm{sym}}(\mathbb{C}^n)$$
$$\implies \det \mathrm{Cay}\colon \mathrm{D}(\mathbb{R}^n) \to \mathbb{S}^1.$$

$\implies$ in coordinates:

$$\mathrm{e}^{\mathrm{i}\theta} = \det \mathrm{Cay}\, s$$

Then

$$\det \mathrm{Cay} = \exp \operatorname{tr} \log \mathrm{Cay},$$

and

$$\implies \log \mathrm{Cay}\, s = \log(\mathrm{id} - \mathrm{i}\, s) - \log(\mathrm{id} + \mathrm{i}\, s)$$
$$= 2\mathrm{i} \arctan s,$$

Trace formula

$\Longrightarrow$

Lemma (Beck & M. 2013)

The following maps $\mathrm{D}(\mathbb{R}^n) \to \mathrm{U}(\mathbb{C}^1)$ *are equivalent:*

$$\log \det \mathrm{Cay} = 2\mathrm{i}\,\mathrm{tr}\,\mathrm{arctan}.$$

The angle between the evolving Lagrangian and standard reference plane is

$$\theta = -2\,\mathrm{tr}\,\mathrm{arctan}\, s.$$

Symmetric unitary flow

Set

$$u := \frac{\mathrm{id} - \mathrm{i}\, s}{\mathrm{id} + \mathrm{i}\, s}.$$

Lemma (Beck & M. 2013)

Map $u\colon \mathbb{R} \to \mathrm{U}_{\mathrm{sym}}(\mathbb{C}^n)$ *satisfies*

$$\partial u = C + Du - u(D^* + C^* u),$$

where

$$C := \tfrac{1}{2}\big(a - d - \mathrm{i}(b + c)\big) \qquad \text{and} \qquad D := \tfrac{1}{2}\big(a + d + \mathrm{i}(b - c)\big).$$

Pullback to the unitary Lie algebra

Natural Lie group action $L\colon (g, u) \mapsto gug^{\mathsf{T}}$ (not isospectral).

Corres. Lie algebra action

$$\ell\colon (\xi, u) \mapsto \xi u - u\xi^*.$$

Following Munthe–Kaas:

Theorem (Beck & M. 2013)

Flow pullback sequence:

1. *Manifold:* $\partial u = \xi u - u\xi^*$ *where* $\xi := D - \frac{1}{2}(uC^* - Cu^\dagger)$.
2. *Lie group:* $\partial g = \xi\, g$.
3. *Lie algebra:* $\partial \sigma = \mathrm{dexp}_\sigma^{-1} \circ \xi$.

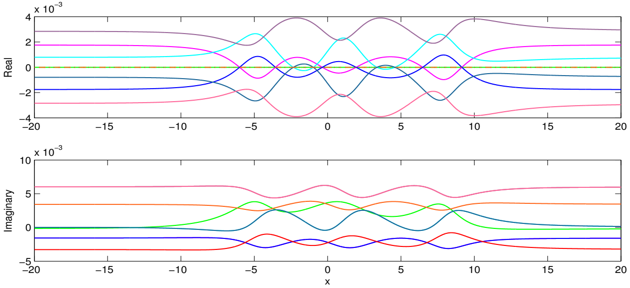
Here $\mathrm{dexp}_\sigma^{-1} := \mathrm{ad}_\sigma / (\exp \mathrm{ad}_\sigma - id)$.

Trace of Lie algebra flow

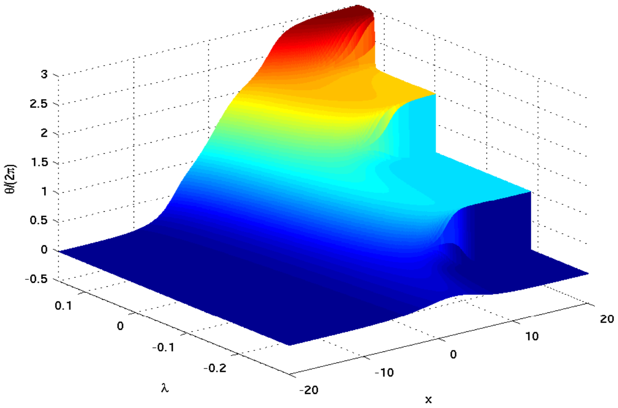
Corollary (Beck & M. 2013)

Angle between evolving Lagrangian and standard reference plane is

$$\theta = -2\mathrm{i}\,\mathrm{tr}\sigma + \theta_0.$$



Trace computation



Infinite dimensional case

Second case: $\mathbb{H} = L^2(\mathbb{T};\mathbb{C}^{2n}) \cong (\ell^2)^{2n}$;

$$q(x,y) \rightsquigarrow \hat{q}(x,k)$$

Evolutionary problem:

$$(\partial - A)\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} = O,$$

where

$$A\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} = \begin{pmatrix} O & \mathrm{id} \\ B^{-1}(\lambda + K^2 - \hat{V}\star) & -B^{-1}d\end{pmatrix}\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}.$$

Perturb map $s\colon q \to p$ to use $s = s_0 + \hat{s}$.

Superpotentials

Maps $s\colon q \to p$ and $r\colon p \to q$ satisfy

$$\partial s = c + ds - s(a + bs) \qquad \text{and} \qquad \partial r = b + ar - r(d + cr).$$

Define

$$U := \begin{pmatrix} \mathrm{id} & r \\ s & \mathrm{id} \end{pmatrix}.$$

Then

$$U^{-1}(\partial - A)\,U = \begin{pmatrix} \partial - (a + bs) & O \\ O & \partial - (d + cr). \end{pmatrix}$$

(Beck & M. 2013)

References

1. N.D. Aparicio, S.J.A. Malham and M. Oliver, *Numerical evaluation of the Evans function by Magnus integration*, BIT 45 (2005), pp. 219–258.
2. N.J. Balmforth, R.V. Craster and S.J.A. Malham, *Unsteady fronts in an autocatalytic system*, Proc. R. Soc. Lond A. 455 (1999), pp. 1401–1433.
3. M. Beck and S.J.A. Malham, *Computing the Maslov index for large systems*, submitted to Proc. AMS 2013.
4. V. Ledoux, S.J.A. Malham and V. Thümmler, *Grassmannian spectral shooting*, Mathematics of Computation 79 (2010), pp. 1585–1619.
5. V. Ledoux, S.J.A. Malham, J. Niesen and V. Thümmler, *Computing stability of multi-dimensional travelling waves*, SIADS 8(1) (2009), pp. 480–507.