Grassmannian shooting and the stability of multi-dimensional fronts

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Spectral problems

Parabolic nonlinear systems on $\mathbb{R} \times \mathbb{T}$:

$$\partial_t U = B\,\Delta U + c\,\partial_x U + F(U),$$

Travelling wave U_c. Small perturbations U satisfy:

$$B\,\Delta U + c\,\partial_x U + \mathrm{D}F(U_c)U = \lambda U.$$

Main solution approaches:

- *Projection.*
- *Shooting.*
- *Iteration.*

Setup

On $\mathbb{R}$: $\quad B\,\Delta U + c\,\partial_x U + \mathrm{D}F(U_c)U = \lambda U$

$$\Leftrightarrow \qquad Y' = A(x;\lambda)\,Y$$

For $\lambda \in \Omega \subseteq \mathbb{C}$: matching condition

$$\begin{aligned}
\mathrm{e}^{\int_0^x \mathrm{Tr}A(\xi;\lambda)\,\mathrm{d}\xi}\,D(\lambda) &:= \det\Big(Y_1^- \;\cdots\; Y_k^- \; Y_{k+1}^+ \;\cdots\; Y_n^+\Big) \\
&= \det\Big(Y^-(x;\lambda)\;\; Y^+(x;\lambda)\Big) \\
&= Y_1^- \wedge \cdots \wedge Y_k^- \wedge Y_{k+1}^+ \wedge \cdots \wedge Y_n^+ \\
&= U^-(x;\lambda) \wedge U^+(x;\lambda)
\end{aligned}$$

Setup (exterior)

Note $U^+ \in (\mathbb{C}^n)^{\wedge(n-k)}$, while $\star U^+ \in (\mathbb{C}^n)^{\wedge k}$

$$(\star U^+)' = -\Big(A^{(k)}(x;\lambda)\Big)^{\dagger}(\star U^+)$$

Then modulo analytic non-zero factors:

$$D(\lambda) := \Big\langle U^-, \star U^+ \Big\rangle_{(\mathbb{C}^n)^{\wedge k}}$$

$$= \det \begin{pmatrix} \langle Y_1^-, \star U_1^+ \rangle_{\mathbb{C}^n} & \cdots & \langle Y_1^-, \star U_k^+ \rangle_{\mathbb{C}^n} \\ \vdots & \ddots & \vdots \\ \langle Y_k^-, \star U_1^+ \rangle_{\mathbb{C}^n} & \cdots & \langle Y_k^-, \star U_k^+ \rangle_{\mathbb{C}^n} \end{pmatrix}$$

Synonyms

Carrying the same information are:

- Evans determinant function;
- Miss-distance function;
- Fredholm determinant;
- Titchmarsh–Weyl matrix-function;
- Dirichlet-to-Neumann map.

Numerical issues

- Computational domain.
- Different exponential growth rates.
- Polynomial complexity.
- Flow singularities?!
- Where to match?
- Retaining analyticity.
- How to project transversely.

Stiefel and Grassmann manifolds

- Stiefel manifold:

$$\mathbb{V}(n,k) = \{k\text{-frames centred at the origin}\}.$$

- Grassmann manifold:

$$\mathrm{Gr}(n,k) = \{k\text{-dimensional subspaces of } \mathbb{C}^n\}.$$

- Fibre bundle:

$$\pi\colon \mathbb{V}(n,k) \to \mathrm{Gr}(n,k) \cong \mathbb{V}(n,k)/\mathrm{GL}(k)$$

$$\pi\colon k\text{-frame} \mapsto \text{spanning } k\text{-plane}$$

Representation

$\pi : Y = y_{\mathfrak{i}^\circ} u \mapsto y_{\mathfrak{i}^\circ}$

Coordinate patches $\mathbb{U}_{\mathfrak{i}}$: multi-index $\mathfrak{i} = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$.

Example: $\mathbb{U}_{\{1,\ldots,k\}}$ uniquely represented by:

$$y_{\mathfrak{i}^\circ} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hat{y}_{k+1,1} & \hat{y}_{k+1,2} & \cdots & \hat{y}_{k+1,k} \\ \hat{y}_{k+2,1} & \hat{y}_{k+2,2} & \cdots & \hat{y}_{k+2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{y}_{n,1} & \hat{y}_{n,2} & \cdots & \hat{y}_{n,k} \end{pmatrix}.$$

Local coordinate chart $\varphi_{\mathfrak{i}} : \mathbb{U}_{\mathfrak{i}} \to \mathbb{C}^{(n-k)k}$ given by $\varphi_{\mathfrak{i}} : y_{\mathfrak{i}^\circ} \mapsto \hat{y}$.

Grassmannian flows

$$Y' = A(x, Y)\, Y$$

Substitute decomposition $Y = y_{\mathfrak{i}^\circ} u$:

$$y'_{\mathfrak{i}^\circ} u + y_{\mathfrak{i}^\circ} u' = (A_{\mathfrak{i}} + A_{\mathfrak{i}^\circ} \hat{y})\, u$$

Project onto $\mathfrak{i}^\circ$th and $\mathfrak{i}$th rows:

$$\hat{y}' = c + d\,\hat{y} - \hat{y}(a + b\,\hat{y}) \qquad \text{and} \qquad u' = (a + b\,\hat{y})\, u$$

where $a = A_{\mathfrak{i}\times\mathfrak{i}}$, $b = A_{\mathfrak{i}\times\mathfrak{i}^\circ}$, $c = A_{\mathfrak{i}^\circ\times\mathfrak{i}}$ and $d = A_{\mathfrak{i}^\circ\times\mathfrak{i}^\circ}$.

Linear vector field: $A = A(x)$ only $\longrightarrow$ decoupling.

Grassmannian Gaussian elimination method (GGEM)

$$
\begin{array}{ccccc}
\mathbb{C}^{(n-k)k} & \xrightarrow{\ \varphi_{\mathfrak{i}}^{-1}\ } & \mathbb{U}_{\mathfrak{i}} & \xrightarrow{\ \mathrm{id}\ } & \mathbb{V}(n,k) \\
\Big\downarrow \textit{Riccati} & & \Big\downarrow \textit{GGEM} & & \Big\downarrow \textit{RK} \\
\mathbb{C}^{(n-k)k} & \xleftarrow[\ \varphi_{\mathfrak{i}'}\ ]{} & \mathbb{U}_{\mathfrak{i}'} & \xleftarrow[\ \mathrm{QOGE}\ ]{} & \mathbb{V}(n,k)
\end{array}
$$

Quasi-optimal Gaussian elimination (QOGE)

GE with *free* stepwise max pivot, generates: $Y_{m+1} = y_{\mathfrak{i}^\circ} L$.

$$\begin{pmatrix} * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & * & \cdots & * \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ * & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ * & * & * & * & \cdots & * \end{pmatrix}$$

Applications (planar fronts)

- $D(\lambda) := \mathrm{e}^{-\int_0^x \mathrm{Tr} A(\xi;\lambda)\,\mathrm{d}\xi} \det\left(Y^-(x;\lambda)\ \ Y^+(x;\lambda)\right)$
- $\det\left(Y^-\ Y^+\right) = \det\left(y_{\mathfrak{i}^\circ_-}\quad y_{\mathfrak{i}^\circ_+}\right) \cdot \det u_{\mathfrak{i}_-} \cdot \det u_{\mathfrak{i}_+}$
- $D(\lambda; x_*) := \det\left(y_{\mathfrak{i}^\circ_-}\quad y_{\mathfrak{i}^\circ_+}\right) \cdot \det L^- \cdot \det L^+$
- Exponentially rescale $\det L^\pm$

Boussinesq system

PDE: $u_{tt} = (1 - c^2)\, u_{xx} + 2c\, u_{xt} - u_{xxxx} - (u^2)_{xx}$.

Solitary waves: $\bar{u}(x) = \frac{3}{2}(1 - c^2)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{1 - c^2}\, x\right)$.

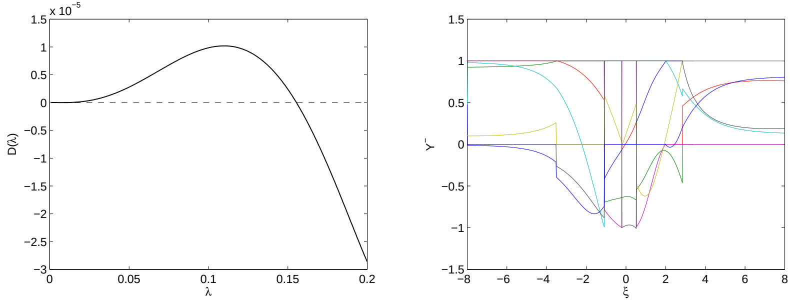


Figure: Evans function for $c = 1/4$ with GGEM-RK and $x_* = 8$ (left panel). Entries of y_i for $\lambda = 0.15543141$ (right panel).

Boussinesq: error vs matching point

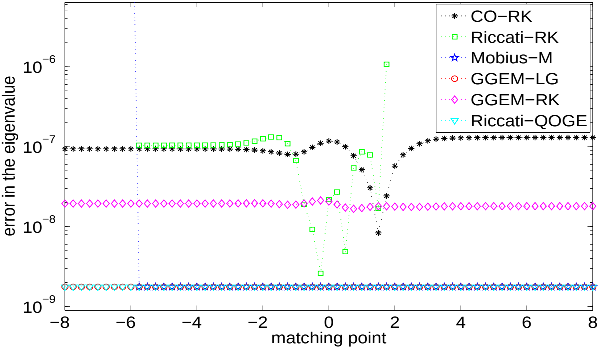


Figure: Error in the eigenvalue for different choices of the matching point: $N = 512$.

Autocatalytic fronts

$$\partial_t u = \delta\Delta u + c\partial_x u - uv^m,$$
$$\partial_t v = \Delta v + c\partial_x v + uv^m.$$

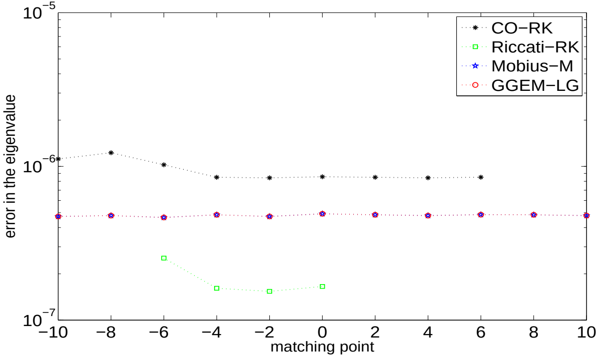


Figure: Error in the eigenvalue when $\delta = 0.1$ and $m = 9$: $N = 256$.

Transverse Fourier basis

On $\mathbb{R} \times \mathbb{T}$ we have:

$$B\Delta U + c\,\partial_x U + \mathrm{D}F(U_c)U = \lambda U.$$

On the Fourier modes $k = -K, -K+1, \ldots, K$:

$$\partial_x \hat{U}_k = \hat{P}_k,$$

$$\partial_x \hat{P}_k = \lambda B^{-1}\hat{U}_k + (k/\tilde{L})^2 \hat{U}_k - c\,B^{-1}\hat{P}_k - \sum_{\nu=-K}^{K} B^{-1}\hat{D}_{k-\nu}\hat{U}_\nu,$$

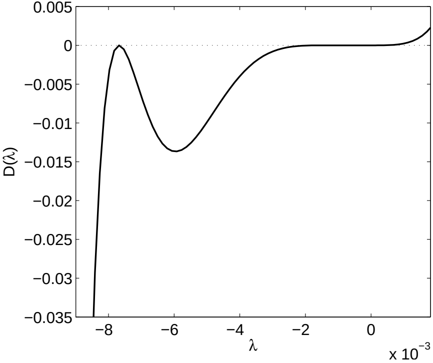
Computing travelling waves: freezing method

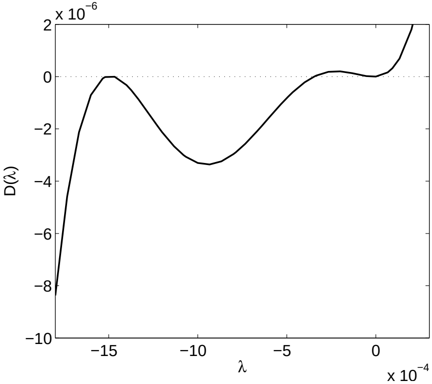
Substitute $U(x,y,t) = V(x-\gamma(t),y,t)$ into original PDE:

$$\partial_t V = B\,\Delta V + \gamma'(t)\partial_x V + F(V),$$
$$0 = \int_{\mathbb{R}\times\mathbb{T}} \Big(\partial_x \hat{V}(x,y,t)\Big)^{\mathrm{T}} \Big(\hat{V}(x,y,t) - V(x,y,t)\Big)\,\mathrm{d}x\,\mathrm{d}y.$$

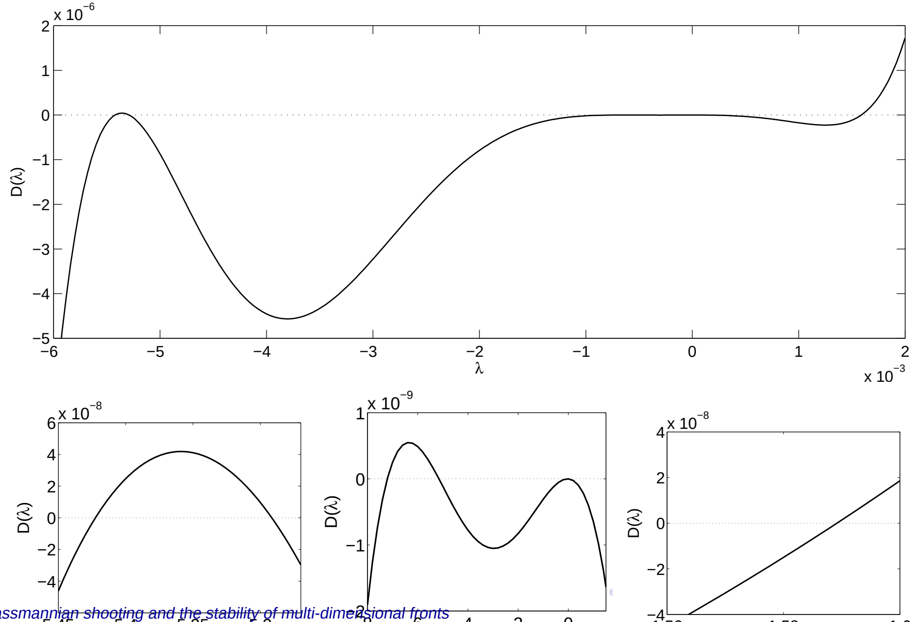
(Developed by Beyn and Thümmler.)

Wrinkled front: Evans function for $\delta = 2.5$





Wrinkled front: Evans function for $\delta = 3$



Wrinkled front: Eigenvalues for $\delta = 3$

K	Eigenvalues (Evans function)				
3	0.001609	−0.000026	−0.000781	−0.001296	−0.000670
4	0.001609	0.000002	−0.000001	−0.000519	−0.000670
5	0.001589	0.000002	−0.000001	−0.000519	−0.000720
6	0.001589	−0.000002	−0.000003	−0.000515	−0.000720
7	0.001589	−0.000002	−0.000003	−0.000515	−0.000721
8	0.001589	−0.000002	−0.000003	−0.000515	−0.000721
9	0.001589	−0.000002	−0.000003	−0.000515	−0.000721
⋮			⋮		
24	0.001589	−0.000002	−0.000003	−0.000515	−0.000721
	Eigenvalues (ARPACK)				
	0.001592	0.000000	0.000000	−0.000514	−0.000719

Wrinkled front: contour integration

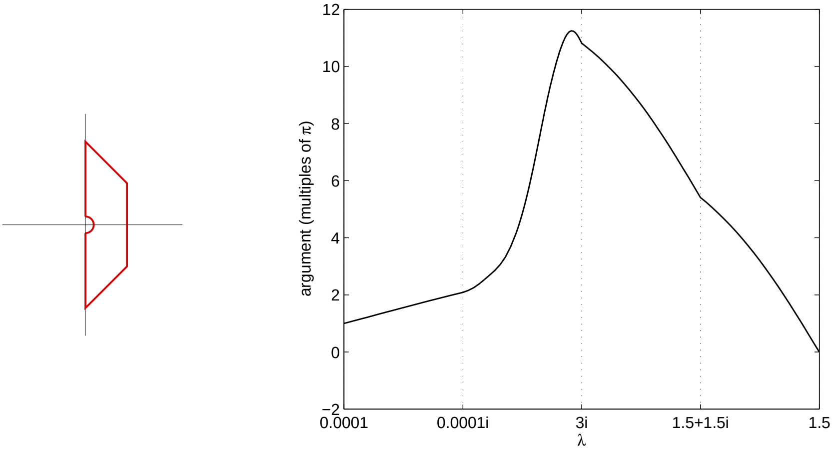


Figure: Left panel: contour. Right panel: $\arg\big(D(\lambda)\big)$ when λ transverses the top half. $\delta = 3$.

Concluding topics

- Schubert varieties and cohomological ring of cycles;
- Fredholm determinant, Bornemann;

Schubert cycles and singularties

Matching condition:

$$\begin{aligned} D(\lambda; x) &:= \det \begin{pmatrix} I_k & I_{n-k} \\ y(x;\lambda) & B(\lambda) \end{pmatrix} \\ &= -\det\big(y - B(\lambda)\big) \\ &= -\det \hat{y} \end{aligned}$$

Here $\hat{y}$ is the Dirichlet to Neumann map:

$$\hat{y}' = \hat{c} + \hat{d}\hat{y} - \hat{y}\hat{a} - \hat{y}b\hat{y}$$

Fredholm determinant

$$\begin{aligned} & Y' = \big(A_0(x) + A_1(x;\lambda)\big)\, Y \\ \Leftrightarrow \quad & D_{A_1} Y = A_0 Y \\ \Leftrightarrow \quad & \big(\mathrm{id} - (K_{A_1} \circ A_0)\big)\, Y = O \end{aligned}$$

Compute the Fredholm determinant for $K = -K_{A_1} \circ A_0$:

$$D(\lambda) := \det(\mathrm{id} + K)$$

For Hilbert space $\mathbb{H}$:

$$\mathrm{tr} K := \sum_{i \geqslant 1} \langle \varphi_i, K\varphi_i \rangle_{\mathbb{H}}$$

Fredholm expansion

$$\det(\mathrm{id} + K) = \sum_{m \geqslant 0} \mathrm{tr}(K^{\wedge m})$$

where

$$\begin{aligned}
\mathrm{tr} K^{\wedge m} &= \sum_{i_1 < \cdots < i_m} \left\langle \varphi_{i_1} \wedge \ldots \wedge \varphi_{i_m}, K\varphi_{i_1} \wedge \ldots \wedge K\varphi_{i_m} \right\rangle_{\mathbb{H}^{\wedge m}} \\
&= \sum_{i_1 < \cdots < i_m} \det\left[\langle \varphi_{i_p}, K\varphi_{i_q} \rangle_{\mathbb{H}} \right]_{p,q \in \{1,\ldots,m\}} \\
&= \frac{1}{m!} \int_{\mathbb{R}^m} \det\left[\hat{K}(x_i, x_j) \right] \mathrm{d}x_1 \ldots \mathrm{d}x_m
\end{aligned}$$

Bornemann, apply quadrature to:

$$(\mathrm{id} + K)\, Y = f$$