# Fredholm determinants and computing the stability of travelling waves


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# Spectral problems

Parabolic nonlinear systems on $\mathbb{R} \times \mathbb{T}$:

$$\partial_t U = B\,\Delta U + c\,\partial_x U + F(U).$$

Travelling wave $U_c$. Small perturbations $U$ satisfy:

$$B\,\Delta U + c\,\partial_x U + \mathrm{D}F(U_c)U = \lambda U.$$

Main solution approaches:

- *Projection and iteration.*
- *Shooting and matching.*
- *Operator determinants.*

# Setup

On $\mathbb{R}$: $\quad B\,\partial_{xx}U + c\,\partial_x U + \mathrm{D}F(U_c)U = \lambda U$

$$\Leftrightarrow \qquad \begin{aligned} \partial_x U &= P, \\ \partial_x P &= B^{-1}\big(\lambda - \mathrm{D}F(U_c)\big)U - cB^{-1}P. \end{aligned}$$

$$\Leftrightarrow \qquad Y' = A(x;\lambda)\,Y.$$

For $\lambda \in \mathbb{C}$: matching condition

$$\begin{aligned} E(\lambda) &:= \mathrm{e}^{\int_0^x \mathrm{Tr}A(\xi;\lambda)\,\mathrm{d}\xi}\,\mathrm{det}\big(Y_1^- \cdots Y_k^-\, Y_{k+1}^+ \cdots Y_n^+\big) \\ &= \mathrm{e}^{\int_0^x \mathrm{Tr}A(\xi;\lambda)\,\mathrm{d}\xi}\,\mathrm{det}\big(Y^-\ \ Y^+\big) \end{aligned}$$

# Synonyms

Carrying the same information are:

- In one dimension $d = 1$:
  - Evans matrix;
  - Matrix of transmission coefficients;
  - Fredholm operators;
  - Titchmarsh–Weyl matrix-function;
  - Grassmannian Riccati flow.
- In multi-dimensions $d > 1$:
  - Fredholm operators;
  - Dirichlet-to-Neumann map;
  - Fredholm Grassmannian flow.

# Numerical issues

- Computational domain.
- Different exponential growth rates.
- Polynomial complexity.
- Flow singularities?!
- Where to match?
- Retaining analyticity?
- How to project transversely.
- How to approximate the Fredholm Grassmannian flow.

# Fredholm determinants

Eval problem: $(L_0 + v(x) - \lambda\,\mathrm{id})u = 0$

Inverse $K_0(\lambda) := (L_0 - \lambda\,\mathrm{id})^{-1}$ is compact

$\lambda$ eval $\Leftrightarrow \det_{\mathrm{F}}\big(\mathrm{id} + K_0(\lambda) \circ v\big) = 0$

Operator $\partial_x - \big(A_0(\lambda) + R(x)\big)$ is Fredholm

Inverse $K_0(\lambda) := (\partial_x - A_0(\lambda))^{-1}$ is compact

$\lambda$ eval $\Leftrightarrow \det_{\mathrm{F}}\big(\mathrm{id} + K_0(\lambda) \circ R\big) = 0$

# Trace class and Hilbert–Schmidt operators

For Hilbert space $\mathbb{H}$:

$$\operatorname{tr} K := \sum_{i \geqslant 1} \langle \varphi_i, K\varphi_i \rangle_{\mathbb{H}} = \int_{\mathbb{R}} \operatorname{tr} G(x; x)\, \mathrm{d}x$$

Trace class operators: $\|K\|_{\mathbb{J}_1(\mathbb{H})} := \operatorname{tr} |K|$

Hilbert–Schmidt operators: $\|K\|_{\mathbb{J}_2(\mathbb{H})} := \operatorname{tr} K^* K$

Birman–Schwinger operator is trace class:

$$K(\lambda) := |R|^{1/2} K_0(\lambda) \circ R|R|^{-1/2}$$

# Evans function and operator determinants

## Definition

Set $d(\lambda) := \det_{\mathrm{F}}\big(\mathrm{id} + K(\lambda)\big)$ and $E(\lambda) = \det_{\mathbb{C}^n}\big(Y^- \ \ Y^+\big)$.

## Theorem (Issa Karambal (2012a))

1. *If $D(\lambda)$ is matrix of transmission coefficients:*

$$d(\lambda) \equiv \det_{\mathbb{C}^n} D(\lambda)$$

2. *If $c(\lambda) := \det_{\mathbb{C}^n}\big(Y_0^- \ \ Y_0^+\big)$ then*

$$d(\lambda) = \frac{E(\lambda)}{c(\lambda)}$$

i.e. $\det_{\mathrm{F}}(L_0 + v(x) - \lambda \mathrm{id}) = \det_{\mathrm{F}}(L_0 - \lambda\, \mathrm{id}) \det_1\big(\mathrm{id} - K(\lambda) \circ v\big)$

# Fredholm expansion

$$\det(\mathrm{id} + K) = \sum_{m \geqslant 0} \operatorname{tr} K^{\wedge m}$$

where

$$\begin{aligned}
\operatorname{tr} K^{\wedge m} &= \sum_{i_1 < \cdots < i_m} \left\langle \varphi_{i_1} \wedge \ldots \wedge \varphi_{i_m}, K\varphi_{i_1} \wedge \ldots \wedge K\varphi_{i_m} \right\rangle_{\mathbb{H}^{\wedge m}} \\
&= \sum_{i_1 < \cdots < i_m} \det\Big[ \langle \varphi_{i_p}, K\varphi_{i_q} \rangle_{\mathbb{H}} \Big]_{p,q \in \{1,\ldots,m\}} \\
&= \frac{1}{m!} \int_{\mathbb{R}^m} \det\Big[ G(x_i, x_j) \Big] \, \mathrm{d}x_1 \ldots \mathrm{d}x_m
\end{aligned}$$

Bornemann 2010, Karambal 2012b

# Stiefel and Grassmann manifolds

- Stiefel manifold:

$$\mathbb{V}(n,k) = \{k\text{-frames centred at the origin}\}.$$

- Grassmann manifold:

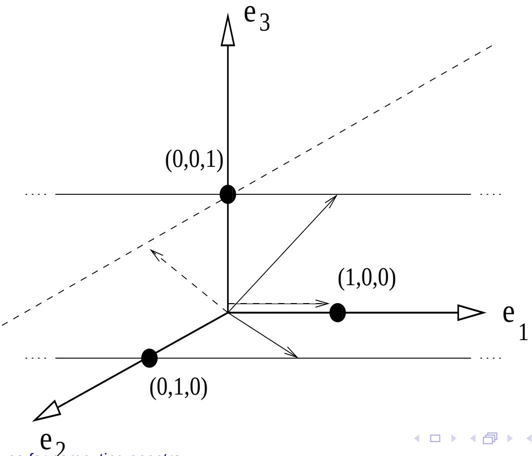
$$\mathrm{Gr}(n,k) = \{k\text{-dimensional subspaces of } \mathbb{C}^n\}.$$

- Fibre bundle:

$$\pi : \mathbb{V}(n,k) \to \mathrm{Gr}(n,k) \cong \mathbb{V}(n,k)/\mathrm{GL}(k)$$
$$\pi : k\text{-frame} \mapsto \text{spanning } k\text{-plane}$$

Example: $\mathrm{Gr}(3,2)$

$$\begin{pmatrix} * & 1 & 0 \\ * & 0 & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$



# Representation II

$\pi\colon Y = y_{\mathfrak{i}^\circ} u \mapsto y_{\mathfrak{i}^\circ}$

Example coordinate patch:

$$y_{\mathfrak{i}^\circ} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hat{y}_{k+1,1} & \hat{y}_{k+1,2} & \cdots & \hat{y}_{k+1,k} \\ \hat{y}_{k+2,1} & \hat{y}_{k+2,2} & \cdots & \hat{y}_{k+2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{y}_{n,1} & \hat{y}_{n,2} & \cdots & \hat{y}_{n,k} \end{pmatrix}.$$

Local chart $\mathbb{U}_{\mathfrak{i}} \to \mathbb{C}^{(n-k)k}$ given by $y_{\mathfrak{i}^\circ} \mapsto \hat{y}$.

# Grassmannian flows

$$Y' = A(x, Y)\, Y$$

Substitute decomposition $Y = y_{\mathfrak{i}^\circ} u$:

$$y'_{\mathfrak{i}^\circ} u + y_{\mathfrak{i}^\circ} u' = (A_{\mathfrak{i}} + A_{\mathfrak{i}^\circ} \hat{y})\, u$$

Project onto $\mathfrak{i}^\circ$th and $\mathfrak{i}$th rows:

$$\hat{y}' = c + d\,\hat{y} - \hat{y}(a + b\,\hat{y}) \qquad \text{and} \qquad u' = (a + b\,\hat{y})\, u$$

where $a = A_{\mathfrak{i}\times\mathfrak{i}}$, $b = A_{\mathfrak{i}\times\mathfrak{i}^\circ}$, $c = A_{\mathfrak{i}^\circ\times\mathfrak{i}}$ and $d = A_{\mathfrak{i}^\circ\times\mathfrak{i}^\circ}$.

# Boussinesq system

PDE: $u_{tt} = (1 - c^2)\, u_{xx} + 2c\, u_{xt} - u_{xxxx} - (u^2)_{xx}$.

Solitary waves with $\mathrm{sech}^2$ profile.

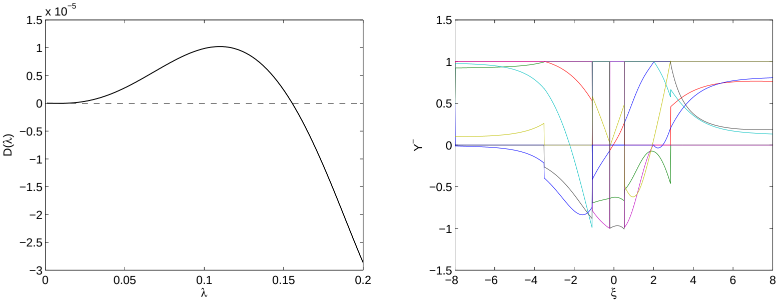



Figure: Evans function for $c = 1/4$ with GGEM-RK and $x_* = 8$ (left panel). Entries of $y_{\mathfrak{i}}$ for $\lambda = 0.15543141$ (right panel).

# Boussinesq: error vs matching point

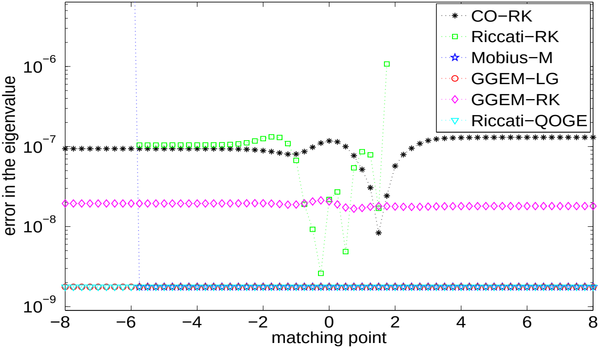



Figure: Error in the eigenvalue for different choices of the matching point: $N = 512$.

# Autocatalytic fronts

$$\partial_t u = \delta\Delta u + c\partial_x u - uv^m,$$
$$\partial_t v = \Delta v + c\partial_x v + uv^m.$$

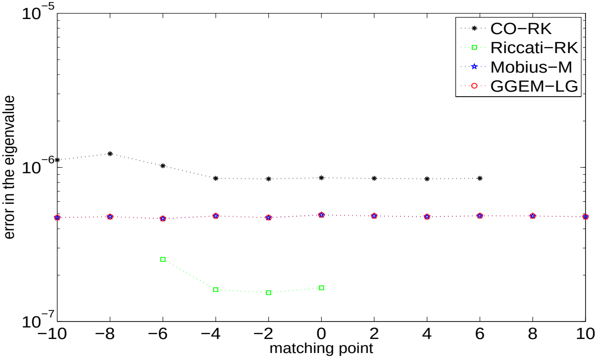



Figure: Error in the eigenvalue when $\delta = 0.1$ and $m = 9$: $N = 256$.

# Transverse Fourier basis

On $\mathbb{R} \times \mathbb{T}$ we have:

$$B\Delta U + c\,\partial_x U + \mathrm{D}F(U_c)U = \lambda U.$$

On the Fourier modes $k = -K, -K+1, \dots, K$:

$$\partial_x \hat{U}_k = \hat{P}_k,$$

$$\partial_x \hat{P}_k = \lambda B^{-1}\hat{U}_k + (k/\tilde{L})^2 \hat{U}_k - \sum_{\nu=-K}^{K} B^{-1}\hat{D}_{k-\nu}\hat{U}_\nu - c\,B^{-1}\hat{P}_k.$$

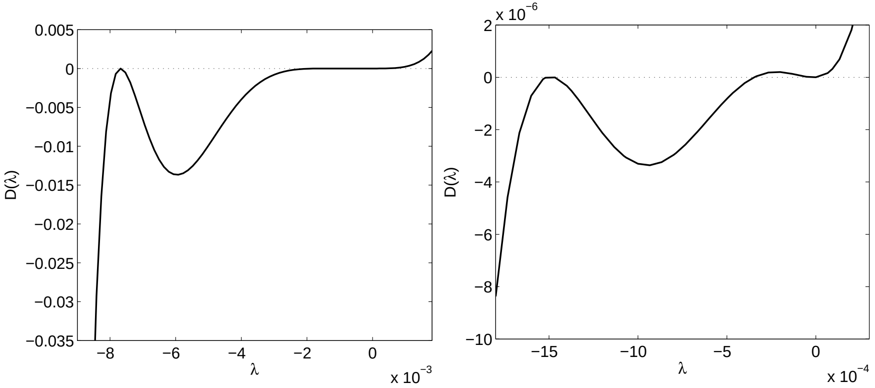
# Computing travelling waves: freezing method

Substitute $U(x, y, t) = V(x - \gamma(t), y, t)$ into original PDE:

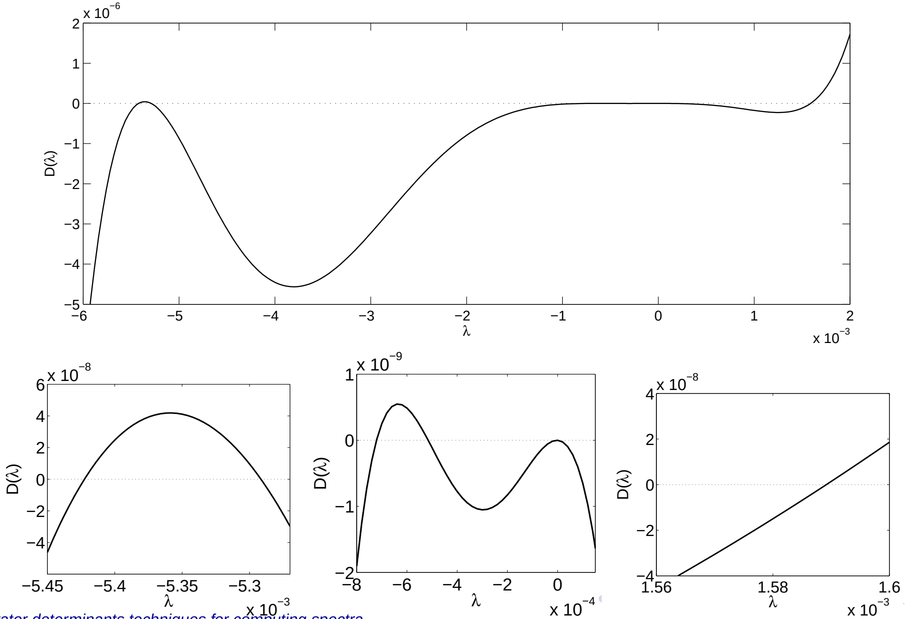
$$\partial_t V = B\,\Delta V + \gamma'(t)\partial_x V + F(V),$$
$$0 = \int_{\mathbb{R}\times\mathbb{T}} \big(\partial_x \hat{V}(x, y, t)\big)^{\mathrm{T}} \big(\hat{V}(x, y, t) - V(x, y, t)\big)\,\mathrm{d}x\,\mathrm{d}y.$$

(Developed by Beyn and Thümmler.)

# Wrinkled front: Evans function for $\delta = 2.5$



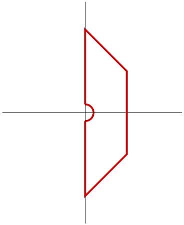
# Wrinkled front: Evans function for $\delta = 3$



# Wrinkled front: Eigenvalues for $\delta = 3$

| $K$ | Eigenvalues (Evans function) | | | | |
|---|---|---|---|---|---|
| 3 | 0.001609 | −0.000026 | −0.000781 | −0.001296 | −0.000670 |
| 4 | 0.001609 | 0.000002 | −0.000001 | −0.000519 | −0.000670 |
| 5 | 0.001589 | 0.000002 | −0.000001 | −0.000519 | −0.000720 |
| 6 | 0.001589 | −0.000002 | −0.000003 | −0.000515 | −0.000720 |
| 7 | 0.001589 | −0.000002 | −0.000003 | −0.000515 | −0.000721 |
| 8 | 0.001589 | −0.000002 | −0.000003 | −0.000515 | −0.000721 |
| 9 | 0.001589 | −0.000002 | −0.000003 | −0.000515 | −0.000721 |
| ⋮ | | | ⋮ | | |
| 24 | 0.001589 | −0.000002 | −0.000003 | −0.000515 | −0.000721 |
| | Eigenvalues (ARPACK) | | | | |
| | 0.001592 | 0.000000 | 0.000000 | −0.000514 | −0.000719 |

# Wrinkled front: contour integration



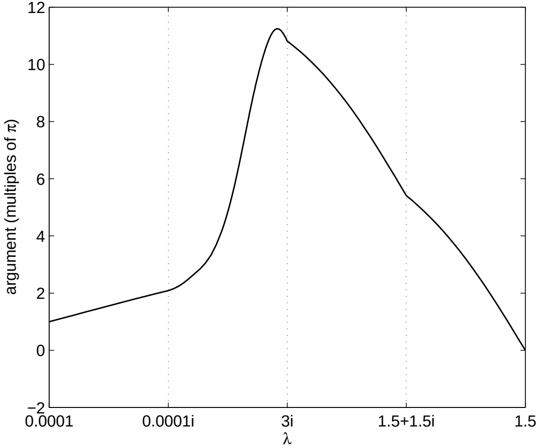
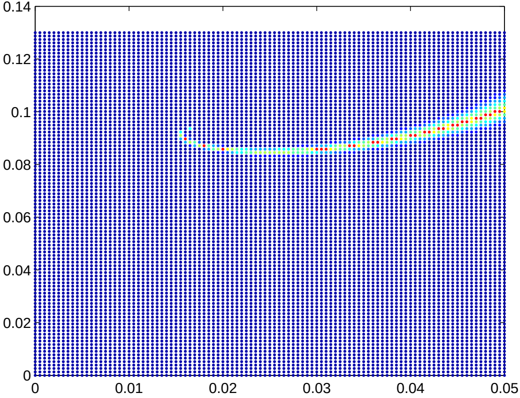



Figure: Left panel: contour. Right panel: $\arg\big(D(\lambda)\big)$ when $\lambda$ transverses the top half. $\delta = 3$.

# Singularities and Schubert cycles




Ledoux & M. 2009, Aljasser 2012 and Beck & M. 2012

# Review: Grassmannian

Sufficient to compute the subspace spanned by $Y^- := [Y_1^- \cdots Y_k^-]$

With respect to $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{n-k}$ graph of $y \in \mathrm{Lin}(\mathbb{C}^k; \mathbb{C}^{n-k})$, i.e.

$$\left\{ (\mathrm{id}_{\mathbb{C}^k} \oplus y)(z) \colon z \in \mathbb{C}^k \right\}$$

realizes such a $k$-plane.

$\mathrm{GL}(\mathbb{C}^n)$ acts on $\mathrm{Gr}(n,k)$ via the Mobius map

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} \mathrm{id} \\ y \end{pmatrix} = \begin{pmatrix} \mathrm{id} \\ (\gamma + \delta y)(\alpha + \beta y)^{-1} \end{pmatrix}$$

where $\alpha \in \mathrm{GL}(\mathbb{C}^k)$, $\delta \in \mathrm{GL}(\mathbb{C}^{n-k})$, $\beta \in \mathbb{C}^{k \times (n-k)}$ and $\gamma \in \mathbb{C}^{(n-k) \times k}$

# Multi-dimensional scenario

Suppose $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$

eg. $\mathbb{H}_1 = H^{s+\frac{1}{2}}(\partial\Omega)$ and $\mathbb{H}_2 = H^{s-\frac{1}{2}}(\partial\Omega)$

$\mathrm{Gr}(\mathbb{H})$ is a Hilbert manifold modelled on $\mathbb{J}_2(\mathbb{H}_1; \mathbb{H}_2)$

For $y \in \mathbb{J}_2(\mathbb{H}_1; \mathbb{H}_2)$: $\left\{(\mathrm{id}_{\mathbb{H}_1} \oplus y)(z) : z \in \mathbb{H}_1\right\}$ carves a subspace of $\mathbb{H}$

$\mathrm{GL}_{\mathrm{res}}(\mathbb{H})$ acts on $\mathrm{Gr}(\mathbb{H})$ via the Mobius map

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} \mathrm{id} \\ y \end{pmatrix} = \begin{pmatrix} \mathrm{id} \\ (\gamma + \delta y)(\alpha + \beta y)^{-1} \end{pmatrix}$$

where $\alpha \in \mathrm{Fred}(\mathbb{H}_1)$, $\delta \in \mathrm{Fred}(\mathbb{H}_2)$, $\beta \in \mathbb{J}_2(\mathbb{H}_2; \mathbb{H}_1)$ and $\gamma \in \mathbb{J}_2(\mathbb{H}_1; \mathbb{H}_2)$.

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