Evans function review

Part I: History, construction, properties and applications

Strathclyde, April 18th 2005

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Acknowledgments

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Goal:

Construct the discrete spectrum of general linear differential operators with associated boundary conditions.

Outline

- Introduction: Sturm–Louiville problems
- Miss-distance function
- Discretization vs shooting
- Example application: reaction-diffusion systems
- Evans function: definition and properties
- Key landmarks: advances and applications
- Further refinements and numerical construction

1 Introduction

Sturm–Liouville problems I

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\,\frac{\mathrm{d}u}{\mathrm{d}x}\right) + q(x)\,u = \lambda\,w(x)\,u\,,$$

on $a \le x \le b$, plus (regular) boundary conditions

$$\begin{aligned} a_1 u(a) &= a_2 p(a) u'(a)\,, \\ b_1 u(b) &= b_2 p(b) u'(b)\,. \end{aligned}$$

Liouville normal form (Schrödinger equation):

$$-u'' + q(x)\,u = \lambda\,u\,,$$

plus boundary conditions.

Sturm–Liouville problems II

Equivalent first order system:

$$U' = \begin{pmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{pmatrix} U \,,$$

plus boundary conditions.

Values of λ for which there is a non-trivial solution subject to the boundary conditions is an *eigenvalue* and corresponding solutions *eigenfunctions*.

Schrödinger equation: energy levels, wave functions, bound states and resonances.

Sturm–Liouville problems III

The linear operator

$$L \equiv \frac{1}{w(x)}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)\right) + q(x)$$

on $a < x < b$ is formally self-adjoint wrt $w(x)$.

- eigenvalues are simple
- $\lambda_0 < \lambda_1 < \lambda_2 < \dots$
- eigenfunctions form orthogonal set

For two solutions

$$L\,u = \lambda\,u \qquad \text{and} \qquad L\,v = \lambda\,v\,,$$

for same value of λ, the **Wronskian**

$$W(u,v) \equiv pv'u - pu'v \;=\; \text{constant}\,.$$

Shooting: basic idea

Sturm-Liouville eigenvalue problem solved over $[a, b]$ for a succession of values of λ which are adjusted until the boundary conditions at both ends are satisfied $\Rightarrow$ we've found an eigenvalue.

Simplest version

- Choose values of $u(a)$ and $p(a)u'(a)$ satisfying the left-hand boundary conditions

  $$p(a)u'_{\mathrm{L}}(a) = a_1 \qquad u_{\mathrm{L}}(a) = a_2\,,$$

  and solve this initial value problem $\Rightarrow u_{\mathrm{L}}(x, \lambda)$.

- At $x = b$ define the **miss-distance function**

  $$D(\lambda) \equiv b_1 u_{\mathrm{L}}(b, \lambda) - b_2 (p u'_{\mathrm{L}})(b, \lambda)\,.$$

Alternative

- Can shoot from both ends towards a middle *matching point* $x = c \in [a, b]$—with the right-hand solution satisfying
$$p(b)u_{\mathrm{R}}'(b) = b_1 \qquad u_{\mathrm{R}}(b) = b_2\,.$$
- Natural choice for **miss-distance function** is the **Wronskian determinant**
$$D(\lambda) \equiv \det \begin{pmatrix} u_{\mathrm{L}}(c,\lambda) & u_{\mathrm{R}}(c,\lambda) \\ pu_{\mathrm{L}}'(c,\lambda) & pu_{\mathrm{R}}'(c,\lambda) \end{pmatrix}\,.$$
- This is zero when multiplying u_{R} by a suitable scalar factor makes it a continuation of u_{L} for $x \geq c \Rightarrow$ we have an eigenfunction, λ an eigenvalue.
- i.e. u_{R} and u_{L} are linearly dependent.
- $D(\lambda)$ is independent of c by the constancy of the Wronskian; however choice of c does have numerical accuracy implications.

Prüfer methods, Pruess methods (1975).

Non-selfadjoint Sturm-Liouville problems

Greenberg & Marletta (2001):

$$p_{2m}(x)u^{(2m)} + \cdots + p_0(x)u = \lambda\, w(x)\, u\,,$$

plus $2m$ separated boundary conditions

$$\sum_{j=0}^{2m-1} a_{ij} y^{(j)}(a) = 0\,, \qquad \sum_{j=0}^{2m-1} b_{ij} y^{(j)}(b) = 0\,,$$

$i = 1, \ldots, m$.

- Reformulate as a first order system

$$\begin{pmatrix} U \\ V \end{pmatrix}' = A(x,\lambda)\begin{pmatrix} U \\ V \end{pmatrix}$$

  with boundary conditions (in matrix form)

$$a_1 U(a) + a_2 V(a) = O \qquad b_1 U(b) + b_2 V(b) = O\,.$$

- Natural choice for **miss-distance function** is the **Wronskian determinant**

$$D(\lambda) \equiv \det\begin{pmatrix} U_{\mathrm{L}}(c,\lambda) & U_{\mathrm{R}}(c,\lambda) \\ V_{\mathrm{L}}(c,\lambda) & V_{\mathrm{R}}(c,\lambda) \end{pmatrix}\,.$$

2 Discretization vs shooting

Discretization of L using finite differences or finite elements $\Rightarrow$ matrix eigenvalue problem.

Advantages

- Simple to set up; especially on a finite interval and a uniform mesh.
- Many applications potential is well behaved and methods competitive.
- Extrapolation and sophisticated correction techniques $\Rightarrow$ even higher eigenvalues can be computed efficiently, error $\mathcal{O}(\lambda^4 h^2) \to \mathcal{O}(\lambda^2 h^4)$.

Disadvantages

- Replace an infinite dimensional problem by a finite dimensional one (dimension=# of mesh points).
- Spurious eigenvalues (can be excised).
- Ill-suited to singular problems.
- Mesh reduction very expensive (unless adaptive variable mesh used).

Shooting methods

- Higher approximations with uniform error bounds.
- Higher order methods.
- More versatility.

3 Example application: reaction-diffusion systems

$$U_t = BU_{\xi\xi} + cU_\xi + F(U)$$

Example: Autocatalytic two-component system

$$\begin{aligned} u_t &= \delta u_{\xi\xi} + cu_\xi - uv^m \\ v_t &= v_{\xi\xi} + cv_\xi + uv^m \end{aligned}$$

Front-type boundary conditions

$$\begin{aligned} (u,v) &\to (1,0) \quad \text{as } x \to -\infty \\ (u,v) &\to (0,1) \quad \text{as } x \to +\infty \end{aligned}$$

Travelling wave in moving frame

$$U(\xi,t) = U_c(\xi)$$

Stability of travelling waves

Perturbation ansatz:

$$U(\xi,t) = U_c(\xi) + \hat{U}(\xi)\,\mathrm{e}^{\lambda t}$$

Plugging this into the reaction-diffusion system

$$U_t = BU_{\xi\xi} + cU_\xi + F(U)$$

and ignoring quadratic and higher powers in $\hat{U}$ yields

$$\lambda\hat{U} = \big[\underbrace{B\partial_{\xi\xi} + c\,I\partial_\xi + DF(U_c(\xi))}_{L}\big]\,\hat{U}$$

with $\hat{U}(\xi) \to 0$ as $\xi \to \pm\infty$.

Reformulation

$$Y' = A(\xi,\lambda)\,Y\,,$$

where $Y = (\hat{U}, \hat{U}_\xi)$, and

$$A(\xi,\lambda) = \begin{pmatrix} O & I \\ B^{-1}\big(\lambda - DF(U_c(\xi))\big) & -c\,B^{-1} \end{pmatrix}$$

Spectrum of the linear operator I

For a general non-selfadjoint linear differential operator L:

- Resolvent operator:

$$R_\lambda \equiv (L - \lambda I)^{-1}\,.$$

- Resolvent set:

$$r(L) \equiv \{\lambda \in \mathbb{C}\colon \|R_\lambda\| < \infty\}\ .$$

- Spectrum:

$$\sigma \equiv \mathbb{C} \backslash r(L)\,.$$

- Discrete spectrum (eigenvalues):

$$\sigma_{\mathrm{discrete}} \equiv \{\lambda \in \sigma\colon R_\lambda \text{ doesn't exist}\}\ .$$

  (Sandstede 1990)

Spectrum of the linear operator II

- Essential spectrum:

$$\sigma_{\text{ess}} \equiv \sigma \backslash \sigma_{\text{discrete}} \,.$$

- Further:

$$\sigma_{\text{ess}} = \sigma_{\text{continuous}} \cup \sigma_{\text{residual}} \,,$$

  where

$$\sigma_{\text{continuous}} \equiv \{\lambda \in \mathbb{C} \colon R_\lambda \text{ exists, not bdd}\} \,.$$

Linear stability and nonlinear stability

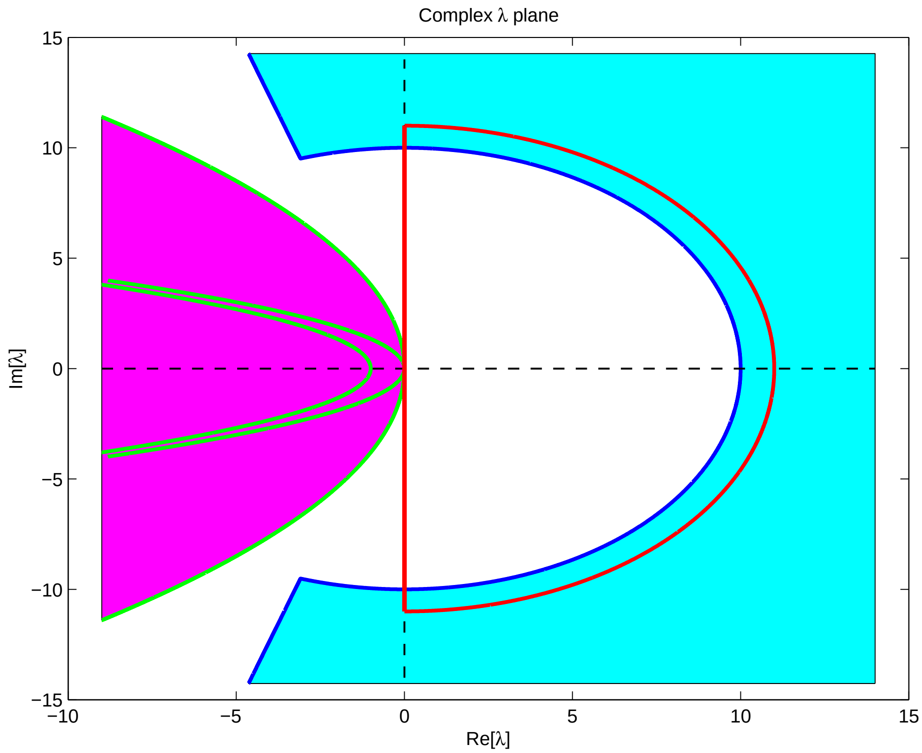
If

- 0 is a simple eigenvalue $(\partial_\xi U_c(\xi))$;
- σ strictly in left-half λ-plane;
- L is sectorial;

then linear stability $\implies$ orbital stability (Henry 1981).

(Relaxed for Fitzhugh–Nagumo systems; Evans 1972).

Spectrum structure



4 The Evans function

Reformulated version

$$Y' = A(\xi, \lambda)\, Y \,,$$

with domain $\mathbb{L}^2(\mathbb{R})$.

Limiting systems

$$A_{\pm}(\lambda) = \lim_{\xi \to \pm\infty} A(\xi, \lambda)$$

- Assume A_- has a k-dimensional unstable manifold
- A_+ has an $(2n-k)$-dimensional stable manifold
- Look for intersection under the "evolution" of the BVP

Wronskian

$$\begin{aligned} D(\lambda) &= \mathrm{e}^{-\int_0^\xi \mathrm{Tr}\, A(x,\lambda)\mathrm{d}x} \cdot \det \left(Y_1^-(\xi;\lambda) \cdots Y_k^-(\xi;\lambda) \;\; Y_{k+1}^+(\xi;\lambda) \cdots Y_{2n}^+(\xi;\lambda) \right) \\ &= \mathrm{e}^{-\int_0^\xi \mathrm{Tr}\, A(x,\lambda)\mathrm{d}x} \cdot \left(Y_1^- \wedge \cdots \wedge Y_k^- \wedge Y_{k+1}^+ \wedge \cdots \wedge Y_{2n}^+ \right) \\ &\equiv \mathrm{e}^{-\int_0^\xi \mathrm{Tr}\, A(x,\lambda)\mathrm{d}x} \cdot \left(U_-(\xi;\lambda) \wedge U_+(\xi;\lambda) \right) \end{aligned}$$

(Prefactor ensures ξ-independence, from Abel's theorem.)

Properties of the Evans function

(Evans 1975; Alexander, Gardner & Jones 1990)

- Zeros correspond to eigenvalues
- Order of the zero corresponds to algebraic multiplicity
- Analytic to the right of the essential spectrum
- Can use argument principle to determine number of zeros in the right half plane

Via transmission coefficients

(Evans 1975; Jones 1984, Swinton 1992)

- $Y_i^{\pm} \sim \eta_i^{\pm} \mathrm{e}^{\mu_i^{\pm}\xi}$ as $\xi \to \pm\infty$, $i = 1, \ldots, 2n$.

$$
\begin{aligned}
Y_1^{-}(+\infty) &\sim b_{1,1}\eta_1 \mathrm{e}^{\mu_1^{+}\xi} + \cdots + b_{1,2n}\eta_1 \mathrm{e}^{\mu_{2n}^{+}\xi}\,,\\
&\vdots\\
Y_k^{-}(+\infty) &\sim b_{k,1}\eta_1 \mathrm{e}^{\mu_1^{+}\xi} + \cdots + b_{k,2n}\eta_1 \mathrm{e}^{\mu_{2n}^{+}\xi}\,.
\end{aligned}
$$

- $c_1 Y_1^{-} + c_2 Y_2^{-} + \cdots c_k Y_k^{-} \to$

$$
\tilde{D}(\lambda) \equiv \det \begin{pmatrix} (Y_1^{+}(\xi;\lambda))^{\dagger} Y_1^{-}(\xi;\lambda) & \cdots & (Y_k^{+}(\xi;\lambda))^{\dagger} Y_1^{-}(\xi;\lambda) \\ \vdots & & \vdots \\ (Y_1^{+}(\xi;\lambda))^{\dagger} Y_k^{-}(\xi;\lambda) & \cdots & (Y_k^{+}(\xi;\lambda))^{\dagger} Y_k^{-}(\xi;\lambda) \end{pmatrix}\,.
$$

- Equivalent to Evans function up to an analytic multiplicative term (Bridges & Derks 1999).
- $k \times k$ submatrix of Dyson S-matrix (from radiation theories of Tomonaga, Schwinger & Feynman).

Asymptotic behaviour $|\lambda| \to \infty$

- Rescale $z = \sqrt{|\lambda|}\,\xi \implies$

$$Y' = \tilde{A}(z,\lambda)\,Y\,,$$

  where

$$\tilde{A}(z,\lambda) = \begin{pmatrix} O & I \\ B^{-1}\big(I\mathrm{e}^{\mathrm{i}\arg\lambda} - DF\big(U_c(z/\sqrt{|\lambda|})\big)/|\lambda|\big) & -c\,B^{-1}/\sqrt{|\lambda|} \end{pmatrix}.$$

- And so as $|\lambda| \to \infty$

$$\tilde{A}(z,\lambda) \to A_\infty = \begin{pmatrix} O & I \\ \mathrm{e}^{\mathrm{i}\arg\lambda}B^{-1} & O \end{pmatrix}.$$

- For example, as $\lambda \to \mathrm{i}\infty$

$$D(\mathrm{i}y) \to -4\mathrm{i}\sqrt{\delta}\,.$$

5 Key landmarks

J.W. Evans 1972(I,II,III), 1975(IV)

- Nerve axon equations (Hodgkin–Huxley $\rightarrow$ Fitzhugh-Nagumo):

$$u_t = u_{xx} + f(u, V)\,,$$
$$V_t = F(u, V)\,.$$

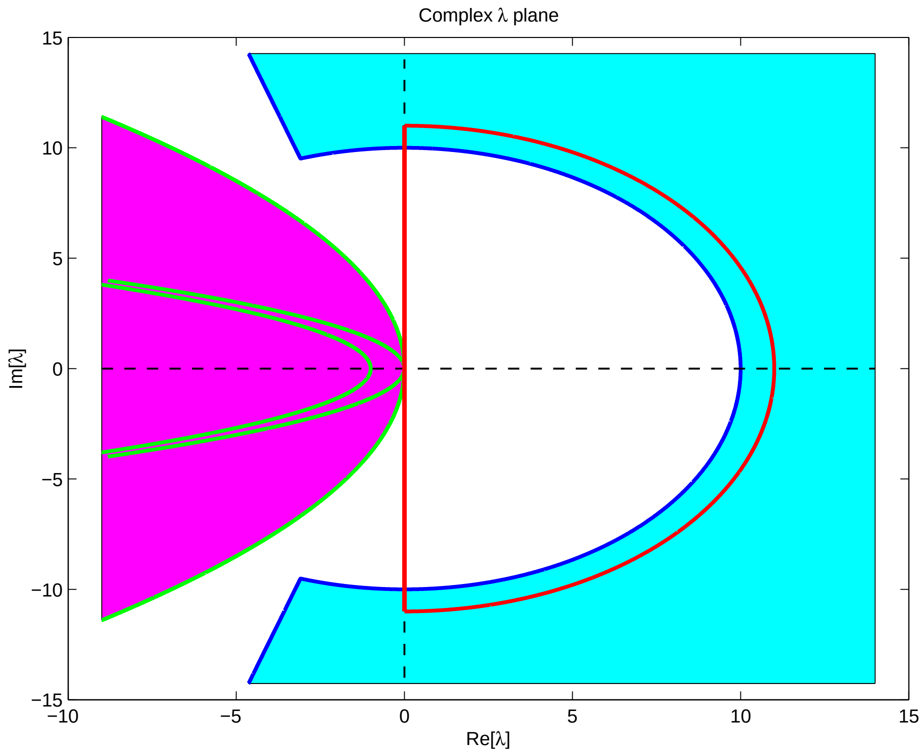
- Construction of $D(\lambda)$ and analytical determination of stability via $D'(0)$ and $D(\infty)$.

Evans & Feroe 1977

- Numerical construction of $D(\lambda)$.
- Introduction of winding number

$$\mathcal{W}(\Gamma) \equiv \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{D'(\lambda)}{D(\lambda)}\, \mathrm{d}\lambda\,.$$

Spectrum structure



C.K.R.T. Jones 1984

- Nerve axon equations (Fitzhugh-Nagumo):

$$u_t = u_{xx} + f(u) - w\,,$$
$$w_t = \epsilon(u - \gamma w)\,,$$

  in asymptotic state $\epsilon \ll 1$, $\gamma \ll 1$;

- Maximum principles difficult to apply.
- Calculated winding number in asymptotic limit, establishing stability.

Alexander, Gardner & Jones 1990

The *bees knees* of Evans function papers, introducing:

- generalization to semi-linear parabolic systems;
- projection spaces as an analytic tool;
- topological invariant Chern number with property

  $$\mathcal{W}(D(\Gamma)) = c_1(\mathcal{E}(\Gamma)) = \#\text{ evals inside }\Gamma\,,$$

  where the augmented unstable bundle

  $$\begin{aligned} \mathcal{E}(\Gamma) &= \mathcal{E}_{\mathrm{S}} \oplus \mathcal{E}_{\mathrm{F}}\,, \\ c_1(\mathcal{E}(\Gamma)) &= c_1(\mathcal{E}_{\mathrm{S}}) + c_1(\mathcal{E}_{\mathrm{F}})\,; \end{aligned}$$

- rigorous treatment of singular perturbation problems, computed the invariant for the reduced fast and slow systems (easy).

Terman 1990

- Stability of *planar waves* to combustion model system in the high activation energy limit.
- Asymptotic expansion of the Evans function in transverse wavenumber and Lewis number.
- Derived the asymptotic neutral stability curves that had hitherto been only established numerically.

Pego & Weinstein 1992

- Stability of solitary waves for generalized

  1. KdV: $\partial_t u + \partial_x f(u) + \partial_x^3 u = 0$
  2. BBM: $\partial_t u + \partial_x u + \partial_x f(u) - \partial_t \partial_x^2 u = 0$
  3. Boussinesq: $\partial_t^2 u - \partial_x^2 u + \partial_x^2 f(u) - \partial_t^2 \partial_x^2 u = 0$

- Each admits a solitary wave, e.g. with $f(u) = u^{p+1}/(p+1)$:

$$u_c(x) = \alpha\, \mathrm{sech}^{2/p}(\gamma\, x)\,.$$

- Establish, if normalize $D(\lambda) \to 1$ as $|\lambda| \to \infty$:

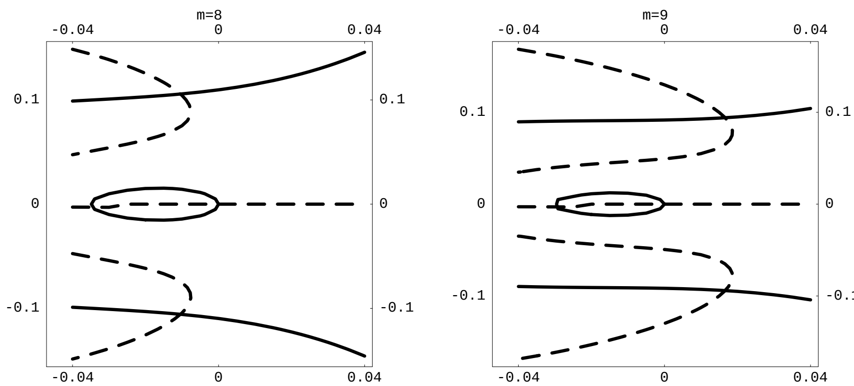
  1. $D'(0) = 0$;
  2. $\mathrm{sgn}\, D''(0) = \mathrm{sgn}\, \dfrac{\mathrm{d}}{\mathrm{d}c} \displaystyle\int_{-\infty}^{\infty} \tfrac{1}{2} u_c^2(x)\, \mathrm{d}x.$

- $\Longrightarrow$ u_c unstable when $p > 4$.

6 Conclusion

- The Evans function stability method provides a versatile method for the location of the discrete spectrum associated with planar travelling waves.
- Better control over accuracy, convergence and asymptotics of discrete spectrum.
- Stability of travelling waves with full two dimensional structure?
- Resonance poles (Chang, Demekhin & Kopelevich, 1996).

7 Example



8 Next week: Numerical construction

- Integrating along unstable manifolds.
- Projection spaces & exterior product representation.
- Projection methods?
- What is the best integrator?
- Preserving Grassmannians.
- Precomputation.
- Scalar reaction-diffusion problems.