

# Sharp condition number estimates for the symmetric 2-Lagrange multiplier method

Stephen W. Drury\* and Sébastien Loisel†

**Abstract** Domain decomposition methods are used to find the numerical solution of large boundary value problems in parallel. In optimized domain decomposition methods, one solves a Robin subproblem on each subdomain, where the Robin parameter  $a$  must be tuned (or optimized) for good performance. We show that the 2-Lagrange multiplier method can be analyzed using matrix analytical techniques and we produce sharp condition number estimates.

## 1 Introduction.

Consider the model problem

$$\Delta u = f \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \quad (1)$$

where  $\Omega$  is the domain,  $f$  is a given forcing and  $u \in H_0^1(\Omega)$  is the unknown solution. In the present paper, we describe a symmetric 2-Lagrange multiplier (S2LM) domain decomposition method to solve elliptic problems such as (1). When we discretize (1) using e.g. piecewise linear finite elements, we obtain a linear system of the form

$$\mathbf{A}\mathbf{u} = \mathbf{f}, \quad (2)$$

where  $\mathbf{u} \in \mathbb{R}^n$  is the finite element coefficient vector of the approximation to the solution  $u$  of (1).

---

\* Dept. of Mathematics, McGill University, 805 Sherbrooke Street West, Montreal, Quebec, Canada, H3A 2K6, [drury@math.mcgill.ca](mailto:drury@math.mcgill.ca)

† Dept. of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom, [S.Loisel@hw.ac.uk](mailto:S.Loisel@hw.ac.uk)

We now consider the domain decomposition [Toselli and Widlund, 2005]  $\Omega = \Gamma \cup \Omega_1 \cup \dots \cup \Omega_p$ , where  $\Omega_1, \dots, \Omega_p$  are the (open, disjoint) “subdomains” and  $\Gamma = \Omega \cap \bigcup_{k=1}^p \partial\Omega_k$  is the “artificial interface”. We introduce the “local problems”

$$\begin{cases} \Delta u_k = f & \text{in } \Omega_k, \quad (\text{PDE}) \\ u_k = 0 & \text{on } \partial\Omega_k \cap \partial\Omega, \quad (\text{natural b.c.}) \\ (a + D_\nu)u_k = \lambda_k & \text{on } \partial\Omega_k \cap \Gamma, \quad (\text{artificial b.c.}) \end{cases} \quad (3)$$

where  $a > 0$  is the Robin tuning parameter and  $k = 1, \dots, p$  and  $D_\nu$  denotes the directional derivative in the outwards pointing normal  $\nu$  of  $\partial\Omega_k$ . The interface  $\Gamma$  is artificial in that it is not a natural part of the “physical problem” (1) but instead is introduced purely for the purpose of calculation.

We again discretize the systems (3) using a finite element method. The Robin b.c. in (3) gives rise to a mass matrix on the interface  $\Gamma \cap \partial\Omega_k$ , which is spectrally equivalent to  $aI$ . Hence, after a suitable “mild” change of basis, we obtain the discrete system

$$\begin{bmatrix} A_{IIk} & A_{I\Gamma k} \\ A_{\Gamma Ik} & A_{\Gamma\Gamma k} + aI \end{bmatrix} \begin{bmatrix} \mathbf{u}_{Ik} \\ \mathbf{u}_{\Gamma k} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{Ik} \\ \mathbf{f}_{\Gamma k} \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{\lambda}_k \end{bmatrix}. \quad (4)$$

The FETI-2LM algorithm was introduced in [Farhat et al., 2000] for cases without cross-points, while the general case including cross points was introduced and analyzed in [Loisel, 2011a]. The method consists of finding the value of  $\boldsymbol{\lambda} = [\boldsymbol{\lambda}_1^T, \dots, \boldsymbol{\lambda}_p^T]^T$  which yields solutions  $\mathbf{u}_1, \dots, \mathbf{u}_p$  to (4) in such a way that  $\mathbf{u}_1, \dots, \mathbf{u}_p$  meet continuously across  $\Gamma$  and glue together into the unique solution  $\mathbf{u}$  of (2).

The main result of the present paper is a new estimate the condition number of FETI-2LM algorithms using matrix analytical techniques. This new idea produces sharp condition number estimates with much more straightforward proof techniques than the techniques used in [Loisel, 2011a] (where the estimates are not sharp). As a result, the present paper is a logical follow-up to [Loisel, 2011a].

The present paper focuses on 1-level algorithms which are known not to scale. Scalable algorithms are considered in [Loisel, 2011b] and [Drury and Loisel, 2011].

Our paper is organized as follows. In Section 2, we give the symmetric 2-Lagrange multiplier method for general domains with cross points. In Section 3, we give spectral estimates including our main result on the condition number of the symmetric 2-Lagrange multiplier system. in Section 4, we verify this Theorem with some numerical experiments.

## 2 The symmetric 2-Lagrange multiplier method.

We now describe the 2-Lagrange multiplier method that we analyze in the present paper. Consider the local problems (4) and eliminate the interior degrees of freedom to obtain the relation

$$a \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_p \end{bmatrix}^{\mathbf{u}_G} = \begin{bmatrix} \overbrace{a(S_1 + aI)^{-1}}^Q & & \\ & \ddots & \\ & & a(S_1 + aI)^{-1} \end{bmatrix} \left( \begin{bmatrix} \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_p \end{bmatrix}^{\mathbf{g}} + \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{bmatrix}^{\lambda} \right), \quad (5)$$

where

$$S_k = A_{\Gamma\Gamma k} - A_{\Gamma I k} A_{II k}^{-1} A_{I\Gamma k} \quad \text{and} \quad \mathbf{g}_k = \mathbf{f}_{\Gamma k} - A_{\Gamma I k} A_{II k}^{-1} \mathbf{f}_{I k}$$

are the ‘‘Dirichlet-to-Neumann maps’’ and ‘‘accumulated right-hand-sides’’.

The matrices  $S_k$  are symmetric and semidefinite. Since  $Q = a(S + aI)^{-1}$ , we find that the spectrum  $\sigma(Q)$  is contained in the set  $[\epsilon, 1 - \epsilon] \cup \{1\}$  for some  $\epsilon > 0$ . The eigenvalue 1 of  $Q$  comes from the kernel of  $S$  and hence the kernel of  $Q - I$  is spanned by the indicating functions of the subdomains that ‘‘float’’. We define  $E$  to be the orthogonal projection onto the kernel of  $Q - I$ .

### 2.1 Relations between (4) and (2) and continuity.

We define the boolean restriction matrix  $R_k$  by selecting rows of the  $n \times n$  identity matrix corresponding to those vertices of  $\Omega$  that are in  $\bar{\Omega}_k \cap \Omega$ . As a result, from a finite element coefficient vector  $\mathbf{v}$  corresponding to a finite element function  $v \in H_0^1(\Omega)$ , we can define a finite element coefficient vector  $\mathbf{v}_k = R_k \mathbf{v}$ , which corresponds to a finite element function  $v \in H^1(\Omega_k) \cap H_0^1(\Omega)$ , which is obtained by restricting  $v$  to  $\Omega_k$ .

The identity  $\int_{\Omega} = \sum_{k=1}^p \int_{\Omega_k}$  induces the following relations between (4) and (2):

$$A = \sum_{k=1}^p R_k^T \begin{bmatrix} A_{II k} & A_{I\Gamma k} \\ A_{\Gamma I k} & A_{\Gamma\Gamma k} \end{bmatrix} R_k \quad \text{and} \quad \mathbf{f} = \sum_{k=1}^p \mathbf{f}_k. \quad (6)$$

Each interface vertex  $\mathbf{x}_i \in \Gamma$  is adjacent to  $m_i \geq 2$  subdomains. As a result, the ‘‘many-sided trace’’  $\mathbf{u}_G$  defined by (5) contains  $m_i$  entries corresponding to  $\mathbf{x}_i$ , one per subdomain adjacent to  $\mathbf{x}_i$ . We define the orthogonal projection matrix  $K$  which averages function values for each interface vertex  $\mathbf{x}_i$ . A many-sided trace  $\mathbf{u}_G$  corresponds to local functions  $\mathbf{u}_1, \dots, \mathbf{u}_p$  that

meet continuously across  $\Gamma$  if and only if

$$K\mathbf{u}_G = \mathbf{u}_G. \quad (7)$$

## 2.2 A problem in $\lambda$ .

The **symmetric 2-Lagrange multiplier** (S2LM) system is given by

$$(Q - K)\lambda = -Q\mathbf{g}. \quad (8)$$

We further let  $E$  be the orthogonal projection onto the kernel of  $Q - I$ .

**Lemma 1.** *Assume that  $\|EK\| < 1$ . The problem (2) is equivalent to (8).*

*Proof.* In order to solve (2) using local problems (4), one should find Robin boundary values  $\lambda_1, \dots, \lambda_p$  which result in local solutions  $\mathbf{u}_1, \dots, \mathbf{u}_p$  that meet continuously across  $\Gamma$ . As a result, we impose the condition (7), which we multiply by  $a > 0$  and convert to an expression in  $\lambda$  using (5) to obtain  $Ka(S + aI)^{-1}(\lambda + \mathbf{g}) = a(S + aI)^{-1}(\lambda + \mathbf{g})$  or

$$(I - K)Q\lambda = (K - I)Q\mathbf{g} \quad (9)$$

With this continuity condition, there is clearly a unique  $\mathbf{u}$  which restricts to the  $\mathbf{u}_j$ :

$$\mathbf{u}_j = R_j\mathbf{u}, \quad j = 1, \dots, p. \quad (10)$$

Imposing continuity is not sufficient, we must also ensure that the “fluxes” match. Indeed, if we impose on the solution  $\mathbf{u}$  of (10) that the equation (2) should hold, one obtains

$$\mathbf{f} = A\mathbf{u} \stackrel{(6)}{=} \sum_{j=1}^p R_{\Gamma_j}^T A_{N_j} R_{\Gamma_j} \mathbf{u} \stackrel{(10)}{=} \sum_{j=1}^p R_{\Gamma_j}^T A_{N_j} \mathbf{u}_j \quad (11)$$

$$\stackrel{(4),(6)}{=} \mathbf{f} - \sum_{j=1}^p R_j^T \begin{pmatrix} 0 \\ \lambda_j - a\mathbf{u}_{\Gamma_j} \end{pmatrix} \quad (12)$$

Canceling the  $\mathbf{f}$  terms on each side and multiplying by  $K$ , we obtain  $K\lambda - Ka\mathbf{u}_G = 0$ . Using (5), we obtain

$$K(Q - I)\lambda = -KQ\mathbf{g}. \quad (13)$$

We add (9) and (13) to obtain (8).

To see that the solution of (8) is unique, observe that the ranges of  $E$  and  $K$  intersect trivially by the hypothesis that  $\|EK\| < 1$ . As a result, the

eigenspace of  $Q$  of eigenvalue 1 intersects trivially with the range of  $K$  and  $Q - K$  is nonsingular.  $\square$

We will further discuss the choice of the parameter  $a$  in Section 3.1.

### 3 Spectral estimates.

If we use GMRES or MINRES on the symmetric indefinite system (8), the residual norm can be estimated as a function of the condition number of  $Q - K$ , cf. [Driscoll et al., 1998]. In order to estimate the condition number of  $Q - K$ , we begin by giving a canonical form for the pair of projections  $E$  and  $K$ .

**Lemma 2.** *Let  $E$  and  $K$  be orthogonal projections. There is a choice of orthonormal basis that block diagonalizes  $E$  and  $K$  simultaneously and such that the blocks  $E_k$  and  $K_k$  of  $E$  and  $K$  satisfy*

$$E_k \in \left\{ 0, 1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} \quad \text{and} \quad K_k \in \left\{ 0, 1, \begin{bmatrix} c_k^2 & c_k s_k \\ c_k s_k & s_k^2 \end{bmatrix} \right\}, \quad (14)$$

where  $c_k = \cos \theta_k > 0$ ,  $s_k = \sin \theta_k > 0$  and  $\theta_k \in (0, \pi/2)$  is a “principal angle” relating  $E$  and  $K$ .

The canonical form (14) can be obtained from the CS decomposition [Davis and Kahan, 1969] by starting from  $E = \text{diag}(I, 0)$  and picking orthonormal bases for the range and kernel of  $K$ . Due to space constraints, we omit this argument.

We also give a technical lemma which describes the spectrum of a sum of certain symmetric matrices.

**Lemma 3.** *Let  $X, Y$  be symmetric matrices of dimensions  $m \times m$ . Let  $0 < y_{\min} < y_{\max}$  and assume that  $|\sigma(Y)| \subset [y_{\min}, y_{\max}]$ . Denote by  $\rho(X)$  the spectral radius of  $X$  and assume that  $\rho(X) < y_{\min}$ . Then,*

$$|\sigma(X + Y)| \subset [y_{\min} - \rho(X), y_{\max} + \rho(X)]. \quad (15)$$

*Proof.* This follows from a Theorem of Weyl [Horn and Johnson, 1990, Theorem 4.3.1, pp 181–182].  $\square$

#### 3.1 Condition number of $Q - K$ .

We now come to our main result.

**Theorem 1.** *Let  $\epsilon > 0$ . Assume that  $\sigma(Q) \subset [\epsilon, 1 - \epsilon] \cup \{1\}$ . Let  $E, K$  be orthogonal projections and assume that  $\|EK\| < 1$ . Then we have the sharp estimates*

$$|\sigma(Q - K)| \subset \left[ \frac{\epsilon + \sqrt{(1 + \epsilon)^2 - 4\|EK\|^2\epsilon} - 1}{2}, 1 \right], \quad \text{and} \quad (16)$$

$$\kappa(Q - K) \leq \frac{2}{\epsilon + \sqrt{(1 + \epsilon)^2 - 4\|EK\|^2\epsilon} - 1} = O((1 - \|EK\|)^{-1}\epsilon^{-1}). \quad (17)$$

*Proof.* Let  $X = Q - \frac{1}{2}I - \epsilon E$  and  $Y = \frac{1}{2}I + \epsilon E - K$ . Then,  $Q - K = X + Y$  and we are in a position to use Lemma 3. We now estimate the spectral properties of  $X$  and  $Y$ .

**Spectral properties of  $X$ :** Recall that  $E$  projects onto the eigenspace of  $Q$  with eigenvalue 1. As a result, after some orthonormal change of basis, we find that  $Q = \text{diag}(Q_0, I)$  and  $E = \text{diag}(0, I)$  and hence

$$\rho(X) \leq \frac{1}{2} - \epsilon. \quad (18)$$

**Spectral properties of  $Y$ :** Lemma 2 shows that  $E$  and  $K$  block diagonalize simultaneously and  $Y$  is also block diagonal in the same basis. Using (14), we find that the  $k$ th block  $Y_k$  of  $Y$  is given by

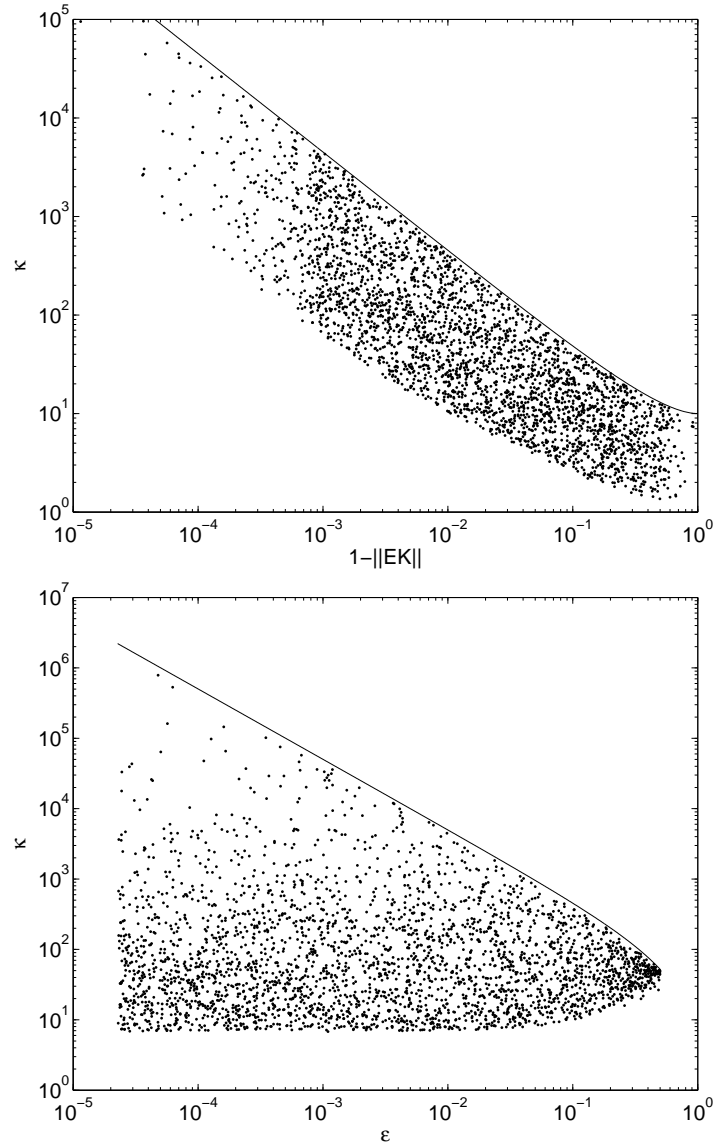
$$Y_k = \begin{cases} \frac{1}{2} & \text{if } E_k = K_k = 0, \\ -\frac{1}{2} & \text{if } E_k = 0, K_k = 1, \\ \frac{1}{2} + \epsilon & \text{if } E_k = 1, K_k = 0, \\ \begin{bmatrix} \frac{1}{2} + \epsilon - c_k^2 & -c_k s_k \\ -c_k s_k & \frac{1}{2} - s_k^2 \end{bmatrix} & \end{cases}; \quad (19)$$

where the case  $E_k = K_k = 1$  is excluded by the hypothesis that  $\|EK\| < 1$ . As a result, the eigenvalues of  $Y_k$  are in the set  $\{\pm\frac{1}{2}, \frac{1}{2} + \epsilon, \lambda_{\pm}(c_k^2)\}$ , where

$$\lambda_{\pm}(c_k^2) = \frac{\epsilon \pm \sqrt{(1 + \epsilon)^2 - 4c_k^2\epsilon}}{2}. \quad (20)$$

Note that  $\|EK\| = \sqrt{\rho(EKE)} = c_k$  and that the functions  $\lambda_{\pm}(c_k^2)$  are monotonic in  $c_k^2$ . Hence, we find the following bounds for the modulus of an eigenvalue of  $Y$ :

$$|\sigma(Y)| \subset \left[ \frac{\overbrace{\sqrt{(1 + \epsilon)^2 - 4\|EK\|^2\epsilon} - \epsilon}^{y_{\min}}}{2}, \overbrace{\frac{1}{2} + \epsilon}^{y_{\max}} \right]. \quad (21)$$



**Fig. 1** Comparing random  $Q - K$  (points) versus the estimate (17) (solid). Top:  $\epsilon = 0.1$ , varying  $\|EK\|$ , 3000 repetitions. Bottom:  $\|EK\| = 0.99$ , varying  $\epsilon$ , 3000 repetitions.

Combining (15), (18) and (21) gives (16).

The sharpness of the estimate is shown by considering the example  $Q = \text{diag}(1, 1 - \epsilon)$  and  $K = \begin{bmatrix} c^2 & c\sqrt{1-c^2} \\ c\sqrt{1-c^2} & 1-c^2 \end{bmatrix}$  for  $c = 0$  and  $c = \|EK\|$ .  $\square$

In view of Theorem 1, the Robin parameter  $a$  should be chosen so as to make  $\epsilon$  as large as possible. This occurs precisely when  $a$  is the geometric mean of the extremal positive eigenvalues of  $S$ . More details can be found in [Loisel, 2011a].

## 4 Numerical verification.

We verify numerically the validity of Theorem 1 by generating random  $5 \times 5$  matrices  $Q$  and  $E$  as follows. We set  $Q = \text{diag}(\epsilon, q, 1 - \epsilon, 1, 1)$  where  $q$  is chosen randomly between  $\epsilon$  and  $1 - \epsilon$ . We generate randomly a 2-dimensional space and set  $K$  to be the orthogonal projection onto that space. We compare the resulting condition number  $\kappa = \kappa(Q - K)$  against (17), cf. Fig. 1.

## References

- Chandler Davis and W. M. Kahan. Some new bounds on perturbation of subspaces. *Bulletin of the American Mathematical Society*, pages 863–868, 1969.
- Tobin A. Driscoll, Kim-Chuan Toh, and Lloyd N. Trefethen. From potential theory to matrix iterations in six steps. *SIAM Review*, pages 547–578, 1998.
- Stephen W. Drury and Sébastien Loisel. The performance of optimized Schwarz and 2-Lagrange multiplier preconditioners for GMRES. *Manuscript*, 2011.
- Charbel Farhat, Antonini Macedo, Michel Lesoinne, Francois-Xavier Roux, Frédéric Magoulès, and Armel de La Bourdonnaie. Two-level domain decomposition methods with lagrange multipliers for the fast iterative solution of acoustic scattering problems. *Computer Methods in Applied Mechanics and Engineering*, 184:213–239, 2000.
- Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, 1990.
- Sébastien Loisel. Condition number estimates for the nonoverlapping optimized Schwarz method and the 2-Lagrange multiplier method for general domains and cross points. *Submitted to SIAM Journal on Numerical Analysis*, 2011a.
- Sébastien Loisel. Condition number estimates and weak scaling for 2-level 2-Lagrange multiplier methods for general domains and cross points. *Submitted*, 2011b.
- Andrea Toselli and Olof B. Widlund. *Domain Decomposition Methods – Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer Berlin Heidelberg, 2005.