Schwarz Preconditioner for the Stochastic Finite Element Method

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1 Introduction

The intrusive polynomial chaos approach for uncertainty quantification in numerous engineering problems constitutes a computationally challenging task. Indeed, the spectral stochastic finite element method (SSFEM) leads to a large-scale deterministic linear system for the polynomial chaos coefficients of the solution process. The size of the linear system grows exponentially with the spatial problem size, dimension and order of the stochastic expansion. When the underlying physical problem is already large, domain decomposition techniques are a natural way to split the problem into a set of smaller subproblems and solve them concurrently on multiprocessor computers. Domain decomposition methods are categorized into overlapping and non-overlapping techniques. FETI-DP and BDDC domain decomposition techniques for the SSFEM have recently proposed in Subber and Sarkar [2013, 2014]. In this paper, we formulate overlapping domain decomposition (Schwarz method) for the solution of the large-scale linear system in the SSFEM. In the Schwarz preconditioner, the global vertices of the physical domain are split into (preferably, but not necessarily overlapping) subsets which constitute the local subdomains. Based on these subsets, restriction operators are defined to extract the local polynomial choose coefficients of the solution process from the global one. The restriction operators associated with the local vertices are then used to construct the local stochastic stiffness matrix of each subdomain as block of the global stiffness matrix. Consequently, stochastic Dirichlet problems corresponding to the local stiffness matrices can be solved on each subdomain concurrently. The solution of these local Dirichlet problems are used to defined the stochastic Schwarz preconditioner. It turns out that the one-level stochastic Schwarz preconditioner can be viewed as a parallel generalization of the block-diagonal mean based preconditioner Powell and Elman [2009], whereby the associated deterministic problems are solved in parallel using the deterministic Schwarz preconditioner. A coarse grid correction providing a mechanism to propagate information across the subdomains globally is supplied to the one-level Schwarz preconditioner. This global exchange of information leads to a scalable performance for large number of subdomains. For the numerical illustrations, a two dimensional elliptic SPDE with spatially varying random coefficients is considered. Numerical scalability of the

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algorithm is investigated with respect to dimension and order of the stochastic expansion, strength of the input uncertainty and the geometric parameters.

2 Mathematical Formulations

We consider the case of finite dimensional noise in a suitable probability space $(\Theta, \Sigma, \mathcal{P})$ Babuska and Chatzipantelidis [2002]. That is we assume that there exist a finite set of independent and identically distributed random variables $\boldsymbol{\xi}(\theta) = \{\xi_1(\theta), \xi_2(\theta), \cdots, \xi_M(\theta)\}$ with joint probability density function $p(\boldsymbol{\xi}) = p_1(\xi_1)p_2(\xi_2)\cdots p_M(\xi_M)$ can be used to paramatrized the input uncertainty. Subsequently, we consider the following stochastic boundary value problem: Find a random function $u(\mathbf{x}, \theta) : \Omega \times \Gamma \rightarrow \mathbb{R}$ such that:

$$\begin{aligned} -\nabla \cdot (\kappa(\mathbf{x}, \xi(\theta)) \nabla u(\mathbf{x}, \xi(\theta))) &= f(\mathbf{x}), \text{ in } \quad \Omega \times \Gamma, \\ u(\mathbf{x}, \xi(\theta)) &= 0, \qquad \text{on } \partial\Omega \times \Gamma, \end{aligned} \tag{1}$$

where $(\Omega \subset \mathbb{R}^d, d = 1, 2, 3)$ denotes a bounded domain with Lipschitz boundary $\partial\Omega$ and $\Gamma = \Gamma_1 \times \Gamma_2 \cdots \times \Gamma_M \subset \mathbb{R}^M$ is the support of the joint probability density function $p(\boldsymbol{\xi})$ of the random vector $\boldsymbol{\xi}(\theta)$. Here we assume that the input uncertainty $\kappa(\mathbf{x}, \boldsymbol{\xi}(\theta)) : \Omega \times \Gamma \rightarrow \mathbb{R}$ is a $\mathcal{P}$-almost surely bounded and strictly positive random field, that is

$$0 < \kappa_{min} \leq \kappa(\mathbf{x}, \boldsymbol{\xi}(\theta)) \leq \kappa_{max} < +\infty, \quad \text{a.e. in} \quad \Omega \times \Gamma. \tag{2}$$

According to the Doob-Dynkin lemma the solution to the stochastic problem can be represented by a finite number of random variables as $u(\mathbf{x}, \theta) = u\left(\mathbf{x}, \xi_1(\theta), \xi_2(\theta), \cdots, \xi_M(\theta)\right)$ Babuska and Chatzipantelidis [2002]. Subsequently, the weak form of the stochastic boundary value problem Eq.(1), can be stated as: Find $u(\mathbf{x}, \boldsymbol{\xi}) \in V$ such that for all $v \in V$

$$\begin{aligned} &\int_\Gamma \left(\int_\Omega \kappa_M(\mathbf{x}, \boldsymbol{\xi}) \nabla u(\mathbf{x}, \boldsymbol{\xi}) \cdot \nabla v(\mathbf{x}, \boldsymbol{\xi}) \mathrm{d}\mathbf{x} \right) p(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi} = \\ &\quad \int_\Gamma \left(\int_\Omega f(\mathbf{x}) v(\mathbf{x}, \boldsymbol{\xi}) \mathrm{d}\mathbf{x} \right) p(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi}, \quad \forall u, v \in V, \end{aligned} \tag{3}$$

where the tensor product function space $V = H_0^1(\Omega) \otimes L^2(\Gamma)$ is defined as

$$V = \{v(\mathbf{x}, \boldsymbol{\xi}(\theta)) : \Omega \times \Gamma \rightarrow \mathbb{R} \mid \|v\|_V^2 < \infty\} \subset H_0^1(\Omega) \otimes L^2(\Gamma), \tag{4}$$

here $H_0^1(\Omega)$ and $L^2(\Gamma)$ represent the deterministic Hilbert space and the space of second-order random variables,respectively. The energy norm $\| \cdot \|_V^2$ is given by

$$\|v(\mathbf{x}, \boldsymbol{\xi}(\theta))\|_V^2 = \int_\Gamma \left(\int_\Omega \kappa(\mathbf{x}, \boldsymbol{\xi}) |\nabla v(\mathbf{x}, \boldsymbol{\xi})|^2 \mathrm{d}\mathbf{x} \right) p(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi}, \tag{5}$$

3 Stochastic Process Representation

Let $\kappa_0(\mathbf{x})$ and $C_{\kappa\kappa}(\mathbf{x}_1, \mathbf{x}_2)$ denote the mean and covariance function of the input uncertainty, then the Karhunen-Loeve expansion (KLE) can be used represent $\kappa(\mathbf{x}, \boldsymbol{\xi})$ as

$$\kappa(\mathbf{x}, \theta) = \sum_{i=0}^{M} \kappa_i(\mathbf{x})\xi_i(\theta), \tag{6}$$

where $\xi_0(\theta) = 1$ and $\kappa_i(\mathbf{x}) = \sigma\sqrt{\lambda_i}\phi_i(\mathbf{x}), i \geq 1$, here σ denotes the standard deviation of the input process and λ_i and $\phi_i(\mathbf{x})$ are the eigenpairs of the covariance kernel and can be obtained from the solution of the following integral equation

$$\int_{\Omega} C_{\kappa\kappa}(\mathbf{x}_1, \mathbf{x}_2)\phi_i(\mathbf{x}_1)\mathrm{d}\mathbf{x}_1 = \lambda_i\phi_i(\mathbf{x}_2), \tag{7}$$

The solution process (with *a priori* unknown mean and covariance function) can be approximated using the polynomial chaos expansion (PCE) as

$$u(\mathbf{x}, \theta) = \sum_{j=0}^{N} u_j(\mathbf{x})\Psi_j(\boldsymbol{\xi}), \tag{8}$$

where $N+1$ denote the total number of terms in PCE and $u_j(\mathbf{x})$ are the deterministic PC coefficients to be determine and $\Psi_j(\boldsymbol{\xi})$ are a set of multivariate orthogonal random polynomials with the following properties

$$\langle\Psi_0\rangle = \int_{\Gamma} \Psi_0(\boldsymbol{\xi})p(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = 1, \quad \langle\Psi_j\rangle = 0, j > 0, \quad \text{and} \quad \langle\Psi_j\Psi_k\rangle = \delta_{jk}\langle\Psi_j^2\rangle.$$

4 The Stochastic Finite Element Discretization

For the spatial discretization, let $\mathcal{T}_h$ denotes the triangulation of the physical domain Ω with a maximum element size h, and let the associated finite element space $\mathcal{X}_h \subset H_0^1(\Omega)$ be spanned by the traditional nodal basis functions $\{\phi_l(\mathbf{x})\}_{l=1}^{L}$. Further, for the stochastic discretization, let $\mathcal{Y}_p \subset L_2(\Gamma)$ be a finite dimensional space spanned by the polynomial chaos basis functions $\{\Psi_j(\boldsymbol{\xi})\}_{j=0}^{N}$ in the random variables $\boldsymbol{\xi}$. Thus, the approximate SSFEM solution u_{hp} in the discrete tensor product space $\mathcal{X}_h \otimes \mathcal{Y}_p \subset H_0^1(\Omega) \otimes L_2(\Gamma)$ can be expressed as

$$u_{hp}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{j=0}^{N}\sum_{l=1}^{L} u_{jl}\phi_l(\mathbf{x})\Psi_j(\boldsymbol{\xi}). \tag{9}$$

Using KLE of the input uncertainty Eq.(6) and the discrete SSFEM solution Eq.(9), we can translate the stochastic weak form defined in Eq.(3) into the following coupled set of deterministic linear system

$$\sum_{j=0}^{N}\sum_{i=0}^{M}\sum_{l=1}^{L} u_{jl}\left(\int_{\Gamma}\boldsymbol{\xi}_i\Psi_j(\boldsymbol{\xi})\Psi_k(\boldsymbol{\xi})p(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}\right)\left(\int_{\Omega}\kappa_i(\mathbf{x})\nabla\phi_l(\mathbf{x})\cdot\nabla\phi_m(\mathbf{x})\mathrm{d}\mathbf{x}\right)=$$
$$\int_{\Gamma}\left(\int_{\Omega}f(\mathbf{x})\phi_m(\mathbf{x})\mathrm{d}\mathbf{x}\right)\Psi_k(\boldsymbol{\xi})p(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi},\quad m=1,\cdots,L,\quad k=0,\cdots,N \tag{10}$$

The linear system arising from Eq.(10) can be expressed as follows

$$\sum_{i=0}^{M}\mathbf{A}^{(i)}\mathbf{U}\mathbf{C}^{(i)}=\mathbf{F}, \tag{11}$$

where we define

$$\mathbf{A}^{(i)}_{lm}=\int_{\Omega}\kappa_i\nabla\phi_l\cdot\nabla\phi_m\mathrm{d}\mathbf{x},\quad \mathbf{C}^{(i)}_{jk}=\int_{\Gamma}\boldsymbol{\xi}_i\Psi_j(\boldsymbol{\xi})\Psi_k(\boldsymbol{\xi})p(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}. \tag{12}$$

$$F_{mk}=\int_{\Gamma}\left(\int_{\Omega}f(\mathbf{x})\phi_m(\mathbf{x})\mathrm{d}\mathbf{x}\right)\Psi_k(\boldsymbol{\xi})p(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}. \tag{13}$$

Eq.(11) can be vectorized by taking the vec$(\cdot)$ operator for the both sides leading to the following concise form

$$\mathcal{A}\mathcal{U}=\mathcal{F}, \tag{14}$$

where

$$\mathcal{A}=\sum_{i=0}^{M}\mathbf{C}^{(i)}\otimes\mathbf{A}^{(i)},\quad \mathcal{U}=\mathrm{vec}(\mathbf{U})\quad\text{and}\quad \mathcal{F}=\mathrm{vec}(\mathbf{F}). \tag{15}$$

Note that $\mathbf{U}=[\mathbf{u}_0,\cdots,\mathbf{u}_N]\in\mathbb{R}^{n\times(N+1)}$ and thus, $\mathrm{vec}(\mathbf{U})=\left\{\begin{matrix}\mathbf{u}_0\\ \vdots\\ \mathbf{u}_N\end{matrix}\right\}\in R^{n(N+1)\times 1}$.

5 Schwarz Preconditioner for Stochastic PDEs

In the Schwarz preconditioner for the stochastic problem, the physical domain Ω is partitioned into a number of overlapping subdomain $\{\Omega_s, 1\le s\le S\}$ by splitting the vertices of the computational mesh. For each subdomain $\Omega_s\subset\Omega$, let $\mathbf{R}_s$ be a restriction matrix of size $n_s\times n$ (where n_s and n are the size of the subdomain and global unknowns) to extract the local nodal values from the global unknowns vector as

$$\mathbf{U}_s=\mathbf{R}_s\mathbf{U}, \tag{16}$$

applying the vec(·) operator to Eq.(16), leads to

$$\mathrm{vec}(\mathbf{U}_{\mathrm{s}}) = (\mathbf{I} \otimes \mathbf{R}_{\mathrm{s}})\mathrm{vec}(\mathbf{U}), \tag{17}$$

here $\mathbf{I}$ is $(N+1)\times(N+1)$ identity matrix. Let $\mathcal{U}_s = \mathrm{vec}(\mathbf{U}_{\mathrm{s}})$ and $\mathcal{R}_s = (\mathbf{I}\otimes\mathbf{R}_s)$ denote the stochastic subdomain nodal values and the stochastic restriction matrix, then Eq.(17) becomes

$$\mathcal{U}_s = \mathcal{R}_s \mathcal{U}, \tag{18}$$

which can be expanded as

$$\begin{Bmatrix} \mathbf{u}_s^0 \\ \mathbf{u}_s^1 \\ \vdots \\ \mathbf{u}_s^N \end{Bmatrix} = \begin{bmatrix} \mathbf{R}_s & & & \\ & \mathbf{R}_s & & \\ & & \ddots & \\ & & & \mathbf{R}_s \end{bmatrix} \begin{Bmatrix} \mathbf{u}^0 \\ \mathbf{u}^1 \\ \vdots \\ \mathbf{u}^N \end{Bmatrix} \tag{19}$$

Consequently, the the stochastic stiffness matrix for subdomain Ω_s can be defined as a block extracted from the global stiffness matrix $\mathcal{A}$ as

$$\mathcal{A}_s = \mathcal{R}_s \mathcal{A} \mathcal{R}_s^T, \tag{20}$$

$$= (\mathbf{I} \otimes \mathbf{R}_s) \left(\sum_{i=0}^{M} \mathbf{C}^{(i)} \otimes \mathbf{A}^{(i)} \right) (\mathbf{I} \otimes \mathbf{R}_s^T), \tag{21}$$

$$= \sum_{i=0}^{M} \mathbf{C}^{(i)} \otimes \mathbf{A}_s^{(i)}, \tag{22}$$

where we define $\mathbf{A}_s^{(i)} = \mathbf{R}_s \mathbf{A}^{(i)} \mathbf{R}_s^T$ as the subdomain stiffness matrix corresponding to each KLE coefficient $\{\kappa_i(\mathbf{x})\}_{i=0}^{M}$. Note that this local stiffness matrix can be viewed as the stiffness matrix of the following local Dirichlet problem:

$$\begin{aligned} \nabla \cdot (\kappa_i(\mathbf{x}) \nabla u(\mathbf{x})) &= f(\mathbf{x}), \text{ in } \quad \Omega_s, \\ u(\mathbf{x}) &= g(\mathbf{x}), \text{ on } \partial\Omega_s. \end{aligned} \tag{23}$$

Next, we define the one-level stochastic Schwarz preconditioner as a direct sum of the solution of the local stochastic Dirichlet problems as

$$\mathcal{M}^{-1} = \sum_{s=1}^{S} \mathcal{R}_s^T \mathcal{A}_s^{-1} \mathcal{R}_s. \tag{24}$$

The local stochastic Dirichlet problems can be performed in parallel and therefore there is no need to construct the preconditioner explicitly. Some properties of the stochastic Schwarz preconditioner are summarized next

Lemma 1. *The stochastic Schwarz preconditioner is symmetric positive definite.*

Proof. Since the subdomain stiffness matrix $\mathcal{A}_s$, is constructed as a block submatrix from the global stiffness matrix $\mathcal{A}$, the former inherits the property of being symmetric positive definite matrix from the latter and thus the Schwarz preconditioner is also symmetric positive definite.

Remark 1. The stochastic Schwarz preconditioner has the same structure as the one-level stochastic Neumann-Neumann preconditioner Subber and Sarkar [2010]. Unlike the stochastic Neumann-Neumann preconditioner however, the local problems in the Schwarz preconditioner are always solvable.

Lemma 2. *The stochastic Schwarz preconditioner based on the mean properties is given as*

$$\mathcal{M}_0^{-1} = [\mathbf{C}^{(0)}]^{-1} \otimes \sum_{s=1}^{S} \mathbf{R}_s^T [\mathbf{A}_s^{(0)}]^{-1} \mathbf{R}_s, \tag{25}$$

where the subdomain mean stiffness matrix $\mathbf{A}_s^{(0)} = \mathbf{R}_s \mathbf{A}^{(0)} \mathbf{R}_s^T$ *and* $\mathbf{C}^{(0)} = \delta_{ij} \langle \Psi_i^2 \rangle$.

Proof. Using Eq.(22), the one-level stochastic Schwarz preconditioner in Eq.(24) can be rewritten as

$$\mathcal{M}^{-1} = \sum_{s=1}^{S} (\mathbf{I} \otimes \mathbf{R}_s^T) \left(\sum_{i=0}^{M} \mathbf{C}^{(i)} \otimes \mathbf{A}_s^{(i)} \right)^{-1} (\mathbf{I} \otimes \mathbf{R}_s), \tag{26}$$

for the mean properties $i = 0$, and thus

$$\mathcal{M}^{-1} = [\mathbf{C}^{(0)}]^{-1} \otimes \sum_{s=1}^{S} \mathbf{R}_s^T [\mathbf{A}_s^{(0)}]^{-1} \mathbf{R}_s. \tag{27}$$

Proposition 1. *The one-level stochastic Schwarz preconditioner based on the mean properties is a generalization of the block-diagonal mean based preconditioner Powell and Elman [2009] whereby the associated deterministic problem is solved in parallel using the deterministic Schwarz preconditioner.*

Proof. For one subdomain $S = 1$ and normalized PC basis functions, $\mathbf{C}^{(0)} = \mathbf{I}$, the mean-based Schwarz preconditioner defined in Eq.(25) becomes

$$\mathcal{M}_0^{-1} = \mathbf{I} \otimes [\mathbf{A}^{(0)}]^{-1}. \tag{28}$$

6 Coarse Grid Correction

Domain decomposition preconditioners can achieve a scalable performance provided that they are equipped with a coarse grid correction for global communication. To define a coarse problem for the stochastic Schwarz preconditioner, let $\mathbf{R}_0^T \in \mathbb{R}^{n_i \times n_0}$ be an interpolation matrix defined as

$$\mathbf{R}_0^T = \begin{bmatrix} \psi_1(\mathbf{x}_1) & \psi_2(\mathbf{x}_1) & \cdots & \psi_{n_0}(\mathbf{x}_1) \\ \psi_1(\mathbf{x}_2) & \psi_2(\mathbf{x}_2) & \cdots & \psi_{n_0}(\mathbf{x}_2) \\ \vdots & \vdots & \cdots & \vdots \\ \psi_1(\mathbf{x}_{n_i}) & \psi_2(\mathbf{x}_{n_i}) & \cdots & \psi_{n_0}(\mathbf{x}_{n_i}) \end{bmatrix} \tag{29}$$

where $\{\psi_i(\mathbf{x})\}_{i=1}^{n_0}$ is a set of linear basis functions, here n_0 denotes the dimension of the coarse space and $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{n_i})$ are the coordinates of the nodal points of the fine mesh. The corresponding *stochastic coarse space interpolation operator* can be defined as

$$\mathcal{R}_0 = \mathbf{I} \otimes \mathbf{R}_0, \tag{30}$$

and thus the coarse grid correction for the stochastic problem can be obtained as

$$\mathcal{A}_0 = \mathcal{R}_0^T \mathcal{A} \mathcal{R}_0. \tag{31}$$

According, the two-level stochastic Schwarz preconditioner can be defined by adding the coarse grid correction to the one-level preconditioner in Eq.(24) leading to

$$\mathcal{M}^{-1} = \mathcal{R}_0^T \mathcal{A}_0^{-1} \mathcal{R}_0 + \sum_{s=1}^{S} \mathcal{R}_s^T \mathcal{A}_s^{-1} \mathcal{R}_s. \tag{32}$$

Theorem 1. *There exists a positive constant C that is independent of the geometric parameters (i.e. mesh size h, subdomain size H and the overlap distance δ) and the stochastic parameters (i.e. strength of randomness σ, dimension M and order p of the stochastic expansion), such that*

$$cond(\mathcal{M}^{-1}\mathcal{A}) \leq C(d+1)^2 \left(\frac{\kappa_{max}}{\kappa_{min}}\right)^2 \frac{H}{\delta}. \tag{33}$$

Proof. See Loisel and Subber [2014]

7 Numerical Results

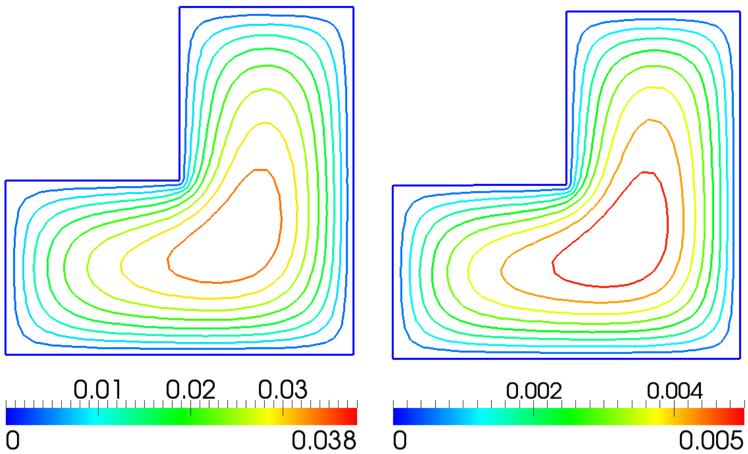
For the numerical illustrations, we consider the following elliptic SPDE

$$\begin{aligned} -\nabla \cdot (\kappa(\mathbf{x}, \theta) \nabla u(\mathbf{x}, \theta)) &= f(\mathbf{x}), \text{ in } \quad \Omega \times \Theta, \\ u(\mathbf{x}, \theta) &= 0, \qquad \text{on } \partial\Omega \times \Theta, \end{aligned} \tag{34}$$

where $f(\mathbf{x})$ denotes the source term taken as unity. The diffusivity coefficient $\kappa(\mathbf{x}, \theta)$ is modelled as a random field with an invariant mean and the following exponential covariance function

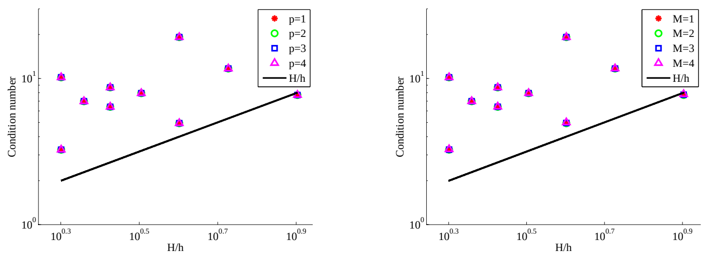
$$C_{\kappa\kappa}(\mathbf{x}, \mathbf{y}) = \sigma^2 \exp\left(\frac{-|x_1 - y_1|}{b_1} + \frac{-|x_2 - y_2|}{b_2}\right), \tag{35}$$

where σ^2 is the variance of the process, b_1 and b_2 are the correlation lengths. Fig.(1(a)) and Fig.(1(b)) show the mean and standard deviation of the solution process. In Fig.(2(a)) and Fig.(2(b)), we show the condition number growth of the Schwarz preconditioner for fixed number of random variables $M = 2$ and fixed order $p = 2$, respectively, while increasing the global problem size by adding more subdomains with fixed problem size per subdomain. Table(1) and Table(2) show the condition number and iterations count of the preconditioned conjugate gradient solver equipped with Schwarz preconditioner with respect to dimension and order and coefficient of variation (CoV), respectively.



(a) mean (b) st. deviation

Fig. 1 The mean and standard deviation of the solution process



(a) $M = 2$ (b) $p = 2$

Fig. 2 Condition number growth with respect to fixed problem size per subdomain: $M = 2$

Table 1 Condition number and iterations count with respect to M and p

M	p	*cond*	*iter*
1	1	10.1642	17
	2	10.1706	19
	3	10.1725	19
	4	10.1733	19
2	1	10.1781	19
	2	10.1834	19
	3	10.1861	19
	4	10.1876	19
3	1	10.1785	19
	2	10.1842	19
	3	10.1873	19
	4	10.1892	19
4	1	10.1816	19
	2	10.1887	20
	3	10.1926	20
	4	10.1951	20

Table 2 Condition number and iterations count with respect to the CoV

$\frac{\sigma}{\mu}$	p	*cond*	*iter*
0.2	1	10.1760	19
	2	10.1812	19
	3	10.1841	19
	4	10.1860	19
0.3	1	10.1816	19
	2	10.1887	20
	3	10.1926	20
	4	10.1951	20
0.4	1	10.1871	19
	2	10.1959	20
	3	10.2006	20
	4	10.2035	20
0.5	1	10.1925	20
	2	10.2030	20
	3	10.2085	20
	4	10.2122	20

8 Conclusion

Two-level Schwarz preconditioner is proposed for the solution of the large-scale linear system arising from the stochastic finite element discretization. The proposed algorithm demonstrates a scalable performance with respect to both the stochastic and geometric parameters.

References

I. Babuska and P. Chatzipantelidis. On solving elliptic stochastic partial differential equations. *CMAME*, 191:4093–4122, 2002.

S. Loisel and W. Subber. Schwarz preconditioners for stochastic PDEs. *CMAME*, 2014. Under Review.

C. Powell and H. Elman. Block-diagonal preconditioning for spectral stochastic finite-element systems. *IMA-JNA*, 29:350–375, 2009.

W. Subber and A. Sarkar. Domain decomposition of stochastic PDEs: A novel preconditioner and its parallel performance. In *HPCS*, volume 5976, pages 251–268. Springer, 2010.

W. Subber and A. Sarkar. Dual-primal domain decomposition method for uncertainty quantification. *CMAME*, 266(0):112 – 124, 2013.

W. Subber and A. Sarkar. A domain decomposition method of stochastic PDEs: An iterative solution techniques using a two-level scalable preconditioner. *JCP*, 257, Part A(0):298 – 317, 2014.