The explicit solution to a sequential switching problem with non-smooth data

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Abstract

We consider the problem faced by a decision maker who can switch between two random payoff flows. Each of these payoff flows is an additive functional of a general one-dimensional Itô diffusion. There are no bounds on the number or on the frequency of the times at which the decision maker can switch, but each switching incurs a cost, which may depend on the underlying diffusion. The objective of the decision maker is to select a sequence of switching times that maximises the associated expected discounted payoff flow. In this context, we develop and study a model in the presence of assumptions that involve minimal smoothness requirements from the running payoff and switching cost functions, but which guarantee that the optimal strategies have relatively simple forms. In particular, we derive a complete and explicit characterisation of the decision maker’s optimal tactics, which can take qualitatively different forms, depending on the problem data.

Keywords: optimal switching, sequential entry and exit decisions, stochastic impulse control, system of variational inequalities

2000 Mathematics Subject Classifications: 93E20, (49K45, 60G40, 91B28, 91B70)

1 Introduction

The origins of the problem that we study are located in economics. Indeed, consider a manager who lives in an economy that is driven by a one-dimensional Itô diffusion. This manager can switch, at a cost, between two investment modes that are associated with different payoff flows and are dependent on the state of the economy. One of these investment
modes is preferable when the economic environment is poor, while the other one is preferable when the economic environment is positive. The manager has an infinite time horizon and wishes to maximise their expected discounted payoff flow by switching between the two investment modes. For instance, the manager may be switching between an asset with stochastic price dynamics and a bank account, or may be the operator of a production facility that can be shut down when it is not sufficiently profitable.

To fix ideas, we assume that the economy is driven by the one-dimensional Itô diffusion

\[ dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t, \quad X_0 = x \in I, \]

where \( W \) is a standard one-dimensional Brownian motion, and \( I = ]\alpha, \beta[ \) is a given interval. In particular, we consider a stochastic system that can be operated in two modes, say “open” and “closed”. We use the controlled finite variation process \( Z \) that takes values in \( \{0, 1\} \) to keep track of the system’s operating mode over time. Indeed, if \( Z_t = 1 \) (resp., \( Z_t = 0 \)), then the system is in its open (resp., closed) operating mode at time \( t \), while the jumps of \( Z \) occur at the sequence of times \( (T_n) \) when the decision maker switches the system between its two operating modes. Assuming that the system is initially in operating mode \( z \in \{0, 1\} \), the decision maker’s objective is to select a switching strategy \( Z_{z,x} \) that maximises the performance criterion

\[
\tilde{J}(Z_{z,x}) = \mathbb{E}_x \left[ \int_0^\infty e^{-\Lambda_t} Z_t \, dA_t^h + \int_0^\infty e^{-\Lambda_t} (1 - Z_t) \, dA_t^c \right]
- \sum_{n=1}^{\infty} \mathbb{E}_x \left[ e^{-\Lambda_{T_n}} \left[ g_o(X_{T_n}) 1_{\{\Delta Z_{T_n} = 1\}} + g_c(X_{T_n}) 1_{\{\Delta Z_{T_n} = -1\}} \right] 1_{\{T_n < \infty\}} \right].
\]

The additive functionals \( A^h_o \) and \( A^h_c \) model the running payoff flows that the system yields while it is operated in its open and in its closed operating modes, respectively, the functions \( g_o \) and \( g_c \) provide the costs of switching the system from its closed to its open operating mode and vice versa, while the state-dependent discounting rate \( \Lambda \) is defined by

\[
\Lambda_t = \int_0^t r(X_s) \, ds,
\]

for some function \( r > 0 \). The precise definition of the additive functionals \( A^h_o \) and \( A^h_c \), which are parametrised by the measures \( h_o \) and \( h_c \), is given by (10) below. At this point, it is worth observing that, if the measures \( h_o \) and \( h_c \) are absolutely continuous with respect to the Lebesgue measure, i.e., if \( h_o(dx) = \hat{h}_o(x) \, dx \) and \( h_c(dx) = \hat{h}_c(x) \, dx \), for some functions \( \hat{h}_o \) and \( \hat{h}_c \), then the performance index \( \tilde{J} \) defined by (2) takes the form

\[
\tilde{J}(Z_{z,x}) = \mathbb{E}_x \left[ \int_0^\infty e^{-\Lambda_t} Z_t \hat{h}_o(X_t) \, dt + \int_0^\infty e^{-\Lambda_t} (1 - Z_t) \hat{h}_c(X_t) \, dt \right]
- \sum_{n=1}^{\infty} \mathbb{E}_x \left[ e^{-\Lambda_{T_n}} \left[ g_o(X_{T_n}) 1_{\{\Delta Z_{T_n} = 1\}} + g_c(X_{T_n}) 1_{\{\Delta Z_{T_n} = -1\}} \right] 1_{\{T_n < \infty\}} \right],
\]
which is more familiar in the stochastic control literature (see also Remark 1).

Problems involving sequential entry and exit decisions have attracted considerable interest in the literature, particularly, in relation to the management of commodity production facilities. Following Brennan and Schwartz [BS85], Dixit and Pindyck [DP94], and Trigeorgis [T96], who were the first to address this type of a decision problem in the economics literature, Brekke and Øksendal [BO94], Bronstein and Zervos [BZ06], Carmona and Ludkovski [CL05], Duckworth and Zervos [DZ01], Guo and Pham [GP05], Guo and Tomecek [GT07], Hamadène and Jeanblanc [HJ07], Lumley and Zervos [LZ01], Ly Vath and Pham [LVP01], Porchet, Warin and Touzi [PTW06], and Zervos [Z03], provide an incomplete, alphabetically ordered list of authors who have studied a number of related models by means of rigorous mathematics. The contributions of these authors range from explicit solutions to characterisations of the associated value functions in terms of classical as well as viscosity solutions of the corresponding Hamilton-Jacobi-Bellman (HJB) equations, as well as in terms of backward stochastic differential equation characterisations of the optimal strategies.

The paper is organised as follows. Section 2, which is composed by four parts, is mostly concerned with the problem formulation. In Section 2.1, we discuss some of the notation that we use throughout the paper, in Section 2.2, we develop our assumptions on the data of the underlying Itô diffusion defined by (1), in Section 2.3, we review a number of results regarding the solvability of a second order linear ODE on which our analysis relies, while, in Section 2.4, we complete the formulation of the control problem that we solve. Section 3 is concerned with the well-posedness of our optimisation problem. In Section 4, we study a number of implications stemming from our Assumption 5 in Section 2.4. Indeed, Assumption 5 plays a central role in our analysis in the sense that it is this assumption that implies a relatively simple structure of the optimal strategies. In Section 5, we prove a verification theorem, which does not rely on Assumption 5. Finally, in Section 6 we develop the explicit solution of our control problem.

2 Problem formulation, assumptions and preliminary results

2.1 Notation

We denote by $\mathcal{I}$ a given open interval with left endpoint $\alpha \geq -\infty$ and right endpoint $\beta \leq \infty$, and by $\mathcal{B}(\mathcal{I})$ the Borel $\sigma$-algebra on $\mathcal{I}$. Given a point $c \in \mathcal{I}$, we adopt the convention $]c, c[ = ]c, c] = [c, c[ = \emptyset$.

Given a $\sigma$-finite measure $\mu$ on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ we denote by $\mu^+$ and $\mu^-$ the unique measures on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ resulting from the Radon decomposition of $\mu$, so that $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$, where $|\mu|$ is the total variation measure of $\mu$. We denote by supp $\mu$ the support of $\mu$, and by

$$\mu''(dx) = \mu^+_{ac}(x) \, dx + \mu^+_{s}(dx)$$

the Lebesgue decomposition of $\mu$ into the measure $\mu^+_{ac}(x) \, dx$, which is absolutely continuous with respect to the Lebesgue measure, and the measure $\mu^+_{s}(dx)$, which is mutually singular.
with the Lebesgue measure. Also, we say that a measure \( \mu \) has full-support if \( \text{supp} \mu = \mathcal{I} \), and that it is non-atomic if \( \mu(\{c\}) = 0 \), for all \( c \in \mathcal{I} \).

Recalling that a function \( f : \mathcal{I} \to \mathbb{R} \) is the difference of two convex functions if and only if its second distributional derivative is a measure, we denote by \( f'_- \) and by \( f'_+ \) left-hand side and the right-hand side first derivatives of \( f \), respectively, which both are functions of finite variation, and by \( f'' \) the measure on \((\mathcal{I}, \mathcal{B}(\mathcal{I}))\) that identifies with the second distributional derivative of \( f \).

### 2.2 The underlying Itô diffusion

We assume that the data of the one-dimensional Itô diffusion given by (1) in the introduction satisfy the following assumption.

**Assumption 1** The functions \( b, \sigma : \mathcal{I} \to \mathbb{R} \) are \( \mathcal{B}(\mathcal{I}) \)-measurable, 

\[
\sigma^2(x) > 0, \quad \text{for all } x \in \mathcal{I},
\]

and

\[
\int_{\alpha}^{\beta} \frac{1 + |b(s)|}{\sigma^2(s)} \, ds < \infty \quad \text{and} \quad \sup_{s \in [\alpha, \beta]} \sigma^2(s) < \infty, \quad \text{for all } \alpha < \alpha < \beta < \beta.
\]

\[\square\]

With reference to Karatzas and Shreve [KS91, Section 5.5.C], the conditions appearing in this assumption are sufficient for the SDE (1) to have a weak solution \( S_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X) \) that is unique in the sense of probability law up to a possible explosion time. In particular, given \( c \in \mathcal{I} \), the scale function \( p_c \) and the speed measure \( m \), given by

\[
p_c(x) = \int_c^x \exp \left( -2 \int_c^s \frac{b(u)}{\sigma^2(u)} \, du \right) \, ds, \quad \text{for } x \in \mathcal{I},
\]

\[
m(dx) = \frac{2}{\sigma^2(x)p'(x)} \, dx,
\]

which characterise one-dimensional diffusions, are well-defined. We also assume that the solution of (1) in non-explosive, i.e., the hitting time of the boundary \( \{\alpha, \beta\} \) of the interval \( \mathcal{I} \) is infinite with probability 1 (see Karatzas and Shreve [KS91, Theorem 5.5.29] for appropriate necessary and sufficient analytic conditions).

**Assumption 2** The solution of (1) is non-explosive.

\[\square\]

Relative to the discounting rate \( \Lambda \) defined by (3), we make the following assumption.

**Assumption 3** The function \( r : \mathcal{I} \to [0, \infty] \) is \( \mathcal{B}(\mathcal{I}) \)-measurable and locally bounded. Also, there exists \( r_0 > 0 \) such that \( r(x) \geq r_0 \), for all \( x \in \mathcal{I} \).

\[\square\]
2.3 The solution of an associated ODE

In the presence of Assumptions 1, 2 and 3, the general solution of the second-order linear homogeneous ODE

\[ \frac{1}{2} \sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) = 0, \quad x \in \mathcal{I}, \]

exists in the classical sense and is given by

\[ w(x) = A\phi(x) + B\psi(x), \]

for some constants \( A, B \in \mathbb{R} \). The functions \( \phi \) and \( \psi \) are \( C^1 \), their first derivatives are absolutely continuous functions,

\[
0 < \phi(x) \quad \text{and} \quad \phi'(x) < 0, \quad \text{for all } x \in \mathcal{I}, \\
0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0, \quad \text{for all } x \in \mathcal{I},
\]

and

\[
\lim_{x \downarrow \alpha} \phi(x) = \lim_{x \uparrow \beta} \psi(x) = \infty. \tag{6}
\]

In this context, \( \phi \) and \( \psi \) are unique, modulo multiplicative constants, and the scale function \( p_c \) admits the expression

\[
p'_c(x) = \frac{\phi(x)\psi'(x) - \phi'(x)\psi(x)}{\phi(c)\psi'(c) - \phi'(c)\psi(c)} = \frac{W(x)}{W(c)}, \quad \text{for all } x, c \in \mathcal{I},
\]

where \( W > 0 \) is the Wronskian of the functions \( \phi \) and \( \psi \). Also, given any points \( x_1 < x_2 \) in \( \mathcal{I} \) and weak solutions \( S_{x_1}, S_{x_2} \) of the SDE (1), the functions \( \phi \) and \( \psi \) satisfy

\[
\phi(x_2) = \phi(x_1)E_{x_2}[e^{-\Lambda_{x_1}}] \quad \text{and} \quad \psi(x_1) = \psi(x_2)E_{x_1}[e^{-\Lambda_{x_2}}]. \tag{7}
\]

Here, as well as in the rest of the paper, we denote by \( \tau_{\gamma} \), where \( \gamma \) is any point in \( \mathcal{I} \), the first hitting time of \( \{\gamma\} \), which is defined by

\[
\tau_{\gamma} = \{ t \geq 0 \mid X_t = \gamma \}.
\]

All of these claims are standard, and can be found in various forms in several references, including Feller [F52], Breiman [B68], Itô and McKean [IM74], Karlin and Taylor [KT81], Rogers and Williams [RW00], and Borodin and Salminen [BS02].

To proceed further, we consider the solvability of the non-homogeneous ODE

\[
\mathcal{L}R_{\mu} + \mu = 0, \tag{8}
\]

where \( \mu \) is a measure on \( (\mathcal{I}, \mathcal{B}(\mathcal{I})) \) and the operator \( \mathcal{L} \) is defined by

\[
\mathcal{L}f(dx) = \frac{1}{2} \sigma^2(x)f''(dx) + b(x)f'(x)dx - r(x)f(x)dx \tag{9}
\]

on the space of all functions \( f : \mathcal{I} \rightarrow \mathbb{R} \) that are differences of two convex functions (see also Section 2.1 above). Also, we recall Definition 2.5 from Johnson and Zervos [JZ07].
**Definition 1** The space $\mathcal{I}_{\phi,\psi}$ of $(\phi,\psi)$-integrable measures is defined to be the set of all $\sigma$-finite measures $\mu$ on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ such that

$$
\int_{[\alpha,\gamma[} \Psi(s) |\mu|(ds) + \int_{[\gamma,\beta]} \Phi(s) |\mu|(ds) < \infty, \text{ for all } \gamma \in \mathcal{I},
$$

where the functions $\Phi$ and $\Psi$ are defined by

$$
\Phi(x) = \frac{\phi(x)}{\sigma^2(x)W(x)} \text{ and } \Psi(x) = \frac{\psi(x)}{\sigma^2(x)W(x)}.
$$

□

For the rest of this section, we fix a weak solution $S_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ of (1), and we consider the linear functional $\mu \mapsto A^\mu$ mapping the family of all $\sigma$-finite measures $\mu$ into the set of finite variation, continuous processes $A^\mu$ defined by

$$
A_t^\mu = \int_{-\infty}^{t} \frac{L_y^\mu}{\sigma^2(y)} \mu(dy), \quad (10)
$$

where $L_y^\mu$ is the local-time process of $X$ at $y \in \mathcal{I}$. It is straightforward to see that the total variation process $|A^\mu|$ of $A^\mu$ is given by $|A^\mu| = A|\mu|$, and that

$$
\text{if } \mu \text{ is a positive measure, then } A^\mu \text{ is an increasing process,} \quad (11)
$$

because $L_y^\mu$ is an increasing process, for all $y \in \mathcal{I}$. Also,

$$
\int_0^\infty 1_{\Gamma}(t) \ dA_t^{\mu} = 0, \text{ for all countable sets } \Gamma \subset \mathcal{I}, \quad (12)
$$

because $A^\mu$ has continuous sample paths.

The following results, which we will need, have been established by Johnson and Zervos [JZ07]. A measure $\mu$ belongs to $\mathcal{I}_{\phi,\psi}$ if and only if

$$
\mathbb{E}_x \left[ \int_0^\infty e^{-\Lambda t} \ dA_t^{\mu} \right] < \infty. \quad (13)
$$

Given any $\mu \in \mathcal{I}_{\phi,\psi}$, the function $R_{\mu}$ defined by

$$
R_{\mu}(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-\Lambda t} \ dA_t^{\mu} \right] \quad (14)
$$

admits the analytic representation

$$
R_{\mu}(x) = \phi(x) \int_{[\alpha,x]} \Psi(s) \mu(ds) + \psi(x) \int_{[x,\beta]} \Phi(s) \mu(ds), \quad (15)
$$

6
and satisfies the ODE (8) as well as
\[
\lim_{x \uparrow \alpha} \frac{R_{\mu}(x)}{\phi(x)} = \lim_{x \downarrow \beta} \frac{R_{\mu}(x)}{\psi(x)} = 0.
\]

(16)

Given any \((\mathcal{F}_t)\)-stopping time \(\upsilon\),
\[
\mathbb{E}_x \left[ e^{-\Lambda_\upsilon} |R_{\mu}(X_\upsilon)| \mathbf{1}_{\{\upsilon < \infty\}} \right] < \infty,
\]

(17)

and \(R_{\mu}\) satisfies Dynkin’s formula, i.e., given any \((\mathcal{F}_t)\)-stopping times \(\upsilon_1 < \upsilon_2\),
\[
\mathbb{E}_x \left[ e^{-\Lambda_{\upsilon_2}} R_{\mu}(X_{\upsilon_2}) \mathbf{1}_{\{\upsilon_2 < \infty\}} \right] = \mathbb{E}_x \left[ e^{-\Lambda_{\upsilon_1}} R_{\mu}(X_{\upsilon_1}) \mathbf{1}_{\{\upsilon_1 < \infty\}} \right] + \mathbb{E}_x \left[ \int_{\upsilon_1}^{\upsilon_2} e^{-\Lambda_t} dA_t^{-\mu} \right],
\]

(18)
as well as the strong transversality condition, i.e., given any sequence of \((\mathcal{F}_t)\)-stopping times \((\upsilon_n)\) such that \(\lim_{n \to \infty} \upsilon_n = \infty\),
\[
\lim_{n \to \infty} \mathbb{E}_x \left[ e^{-\Lambda_{\upsilon_n}} |R_{\mu}(X_{\upsilon_n})| \mathbf{1}_{\{\upsilon_n < \infty\}} \right] = 0,
\]

(19)

Furthermore,
\[
(R_{\mu})'_+ (x) \phi(x) - R_{\mu}(x) \phi'(x) = -\mathcal{W}(x) \int_{[x, \beta]} \Phi(s) \mathcal{L} R_{\mu}(ds),
\]

(20)
\[
(R_{\mu})'_- (x) \phi(x) - R_{\mu}(x) \phi'(x) = -\mathcal{W}(x) \int_{[\alpha, x]} \Phi(s) \mathcal{L} R_{\mu}(ds),
\]

(21)
\[
(R_{\mu})'_+ (x) \psi(x) - R_{\mu}(x) \psi'(x) = \mathcal{W}(x) \int_{[\alpha, x]} \Psi(s) \mathcal{L} R_{\mu}(ds),
\]

(22)
\[
(R_{\mu})'_- (x) \psi(x) - R_{\mu}(x) \psi'(x) = \mathcal{W}(x) \int_{[\alpha, x]} \Psi(s) \mathcal{L} R_{\mu}(ds).
\]

(23)

At this point, we should also note that, if \(\mu\) is absolutely continuous with respect to the Lebesgue measure, i.e., if \(\mu(dx) = \mu_{ac}(x) dx\) in the Lebesgue decomposition (5), then given any \((\mathcal{F}_t)\)-stopping times \(\upsilon_1 < \upsilon_2\),
\[
\mathbb{E}_x \left[ \int_{\upsilon_1}^{\upsilon_2} e^{-\Lambda_t} dA_t^{\mu} \right] = \mathbb{E}_x \left[ \int_{\upsilon_1}^{\upsilon_2} e^{-\Lambda_t} \mu_{ac}(X_t) dt \right],
\]

(24)

which is essentially a consequence of the so-called occupation times formula.

### 2.4 The objective of the optimisation problem

We adopt a weak formulation of the the optimal control problem that we solve.

**Definition 2** Given an initial condition \((z, x) \in \{0, 1\} \times \mathcal{I}\), an admissible *switching* strategy, is any collection \(Z_{z,x} = (S_x, Z, T_n)\) such that
$(I)$ $S_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, X, W)$ is a weak solution of the SDE (1),

$(II)$ $Z$ is an $(\mathcal{F}_t)$-adapted, finite variation, càglàd process with values in $\{0, 1\}$, and such that $Z_0 = z$, and

$(III)$ $(T_n)$ is the sequence of $(\mathcal{F}_t)$-stopping times at which the jumps of $Z$ occur, which can be defined recursively by

$$T_1 = \inf\{t > 0 \mid Z_t \neq z\} \quad \text{and} \quad T_{j+1} = \inf\{t > T_j \mid Z_t \neq Z_{T_j}\}, \quad \text{for} \ j = 1, 2, \ldots,$$

with the usual convention that $\inf \emptyset = \infty$.

We denote by $A_{z,x}$ the set of all admissible strategies.

With each admissible switching strategy, we associate the performance criterion

$$J(Z_{z,x}) = \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} Z_t \, dA_t^h \right] - \sum_{n=1}^\infty \mathbb{E}_x \left[ e^{-\lambda T_n} \left( g_o(X_{T_n})1_{\{\Delta Z_{T_n} = 1\}} + g_c(X_{T_n})1_{\{\Delta Z_{T_n} = -1\}} \right) 1_{\{T_n < \infty\}} \right].$$

(26)

The objective of the control problem is to maximise $J(Z_{z,x})$ over all admissible $Z_{z,x}$. Accordingly, we define the value function $v$ by

$$v(z, x) = \sup_{Z_{z,x} \in A_{z,x}} J(Z_{z,x}), \quad \text{for} \ z \in \{0, 1\} \ \text{and} \ x \in I.$$

To ensure that our optimisation problem is well-posed, we make the following assumption. It is worth observing that among the other conditions, (27) has a simple economic interpretation because it excludes the possibility of generating arbitrarily high profits by rapidly switching between the system’s two operating modes.

**Assumption 4** Each of the functions $g_c, g_o : I \to \mathbb{R}$ is the difference of two convex functions, and

$$g_c(x) + g_o(x) > 0, \quad \text{for all} \ x \in I.$$

(27)

The measures $L_{g_c}, L_{g_o}$ and $h$ are $(\phi, \psi)$-integrable,

$$g_c = R_{-L_{g_c}} \quad \text{and} \quad g_o = R_{-L_{g_o}},$$

where $R_{-L_{g_c}}$ and $R_{-L_{g_o}}$ are defined as in (14)-(15).

**Remark 1** The structure of the performance criterion defined by (26) involves a running payoff flow only when the system is in its open operating mode. We have chosen this setting instead of the apparently more general one involving the performance index $\tilde{J}$ defined by (2) in the introduction, only with a view to simplifying the presentation of our results. Indeed, assuming that both of $h_o$ and $h_c$ are $(\phi, \psi)$-integrable, the linearity of the mapping $\mu \mapsto A^\mu$ and (14) implies that

$$\tilde{J}(Z_{z,x}) = \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} \, dA_t^{hc} \right] + J(Z_{z,x}), \quad \text{for all} \ Z_{z,x} \in A_{z,x},$$

for all $Z_{z,x} \in A_{z,x}$. 8
if we let $h = h_o - h_c$, which reveals that the two optimisation problems are equivalent. Furthermore, it is worth noting at this point that the expression (4) of $\tilde{\mathcal{A}}^2$. 

Furthermore, it is worth noting at this point that the expression (4) of $\tilde{\mathcal{A}}^2$. 

Our final assumption below ensures that the optimal strategies of the optimisation problem that we study admit an explicit characterisation.

**Assumption 5** The measure $\mathcal{L}(R_h + g_c)$ satisfies one of the following mutually exclusive conditions:

**A1.** $|\mathcal{L}(R_h + g_c)|(I) = 0$;

**A2.** $|\mathcal{L}(R_h + g_c)|(I) > 0$, and $-\mathcal{L}(R_h + g_c)$ is a positive measure;

**A3.** $|\mathcal{L}(R_h + g_c)|(I) > 0$, and $\mathcal{L}(R_h + g_c)$ is a positive measure;

**A4.** $|\mathcal{L}(R_h + g_c)|(I) > 0$, supp$[\mathcal{L}(R_h + g_c)]^+ \neq \emptyset$, supp$[\mathcal{L}(R_h + g_c)]^- \neq \emptyset$, and there exists a point $\tilde{a} \in I$ such that

$$\text{supp}[\mathcal{L}(R_h + g_c)]^+ \subseteq ]\alpha, \tilde{a}] \quad \text{and} \quad \text{supp}[\mathcal{L}(R_h + g_c)]^- \subseteq ]\tilde{a}, \beta[. \quad (28)$$

Similarly, the measure $\mathcal{L}(R_h - g_o)$ satisfies one of the mutually exclusive conditions:

**B1.** $|\mathcal{L}(R_h - g_o)|(I) = 0$;

**B2.** $|\mathcal{L}(R_h - g_o)|(I) > 0$, and $\mathcal{L}(R_h - g_o)$ is a positive measure;

**B3.** $|\mathcal{L}(R_h - g_o)|(I) > 0$, and $-\mathcal{L}(R_h - g_o)$ is a positive measure;

**B4.** $|\mathcal{L}(R_h - g_o)|(I) > 0$, supp$[\mathcal{L}(R_h - g_o)]^+ \neq \emptyset$, supp$[\mathcal{L}(R_h - g_o)]^- \neq \emptyset$, and there exists a point $\tilde{b} \in I$ such that

$$\text{supp}[\mathcal{L}(R_h - g_o)]^+ \subseteq ]\alpha, \tilde{b}] \quad \text{and} \quad \text{supp}[\mathcal{L}(R_h - g_o)]^- \subseteq ]\tilde{b}, \beta[. \quad (29)$$

The coupling of these conditions is subject to a number of constraints resulting from (27) in Assumption 4, which we discuss in Section 4 below. Furthermore, if conditions **A4** and **B4** simultaneously hold, then

$$\tilde{a} < \tilde{b}, \quad (30)$$

$$\int_{]\alpha, \tilde{a}]} \Psi(s) \mathcal{L}(g_c + g_o)(ds) \leq 0, \quad \text{for all } u \in ]\alpha, \tilde{a}[, \quad (31)$$

and

$$\int_{]u, \beta]} \Phi(s) \mathcal{L}(g_c + g_o)(ds) \leq 0, \quad \text{for all } u \in ]\tilde{b}, \beta[, \quad (32)$$

□
3 Well-posedness of the optimisation problem

The following result establishes that the performance criterion of the optimisation problem that we study is well-posed for every choice of an admissible switching strategy as well as a convergence result that we will need.

Lemma 1 In the presence of Assumptions 1–4, the performance index $J(Z_{z,x})$ is well defined and takes values in $[-\infty, \infty]$ for all initial conditions $(z, x) \in \{0, 1\} \times I$ and every admissible switching strategy $Z_{z,x} \in A_{z,x}$. Furthermore,

$$J(Z_{z,x}) = \lim_{n \to \infty} \left\{ \mathbb{E}_x \left[ \int_0^{T_n} e^{-\Lambda_t} Z_t dA_t^h \right] - \sum_{j=1}^{n-1} \mathbb{E}_x \left[ e^{-\Lambda_{T_j}} \left[ g_0(X_{T_j})1_{\{\Delta Z_{T_j}=1\}} + g_c(X_{T_j})1_{\{\Delta Z_{T_j}=-1\}} \right] 1_{(T_j, \infty)} \right] \right\}. \quad (33)$$

Proof. We fix any initial condition $(z, x) \in \{0, 1\} \times I$ and any admissible switching strategy $Z_{z,x} \in A_{z,x}$, and we note that $\lim_{n \to \infty} T_n = \infty$, $P$-a.s., because $Z$ is a finite variation process whose jumps all have size 1. Recalling that the total variation process $|A^h|$ of $A^h$ is equal to $A^{[h]}$, we note that

$$\left| \int_0^{T_n} e^{-\Lambda_t} Z_t dA_t^h \right| \leq \int_0^{T_n} e^{-\Lambda_t} Z_t d|A_t^h| = \int_0^{T_n} e^{-\Lambda_t} Z_t dA_t^{[h]} \leq \int_0^{\infty} e^{-\Lambda_t} Z_t dA_t^{[h]}.$$ 

The last term in these inequalities has finite expectation thanks to the assumption that the measure $h$ is $(\phi, \psi)$-integrable and (13). However, this observation and the dominated convergence theorem imply that

$$\mathbb{E}_x \left[ \int_0^{\infty} e^{-\Lambda_t} Z_t dA_t^{[h]} \right] = \lim_{n \to \infty} \mathbb{E}_x \left[ \int_0^{T_n} e^{-\Lambda_t} Z_t dA_t^h \right] \in \mathbb{R}. \quad (34)$$

Similarly, we can see that the assumption that the measure $\mathcal{L}g_c$ is $(\phi, \psi)$-integrable implies that

$$\mathbb{E}_x \left[ \int_0^{\infty} e^{-\Lambda_t} Z_t dA_t^{-[\mathcal{L}g_c]} \right] = \lim_{n \to \infty} \mathbb{E}_x \left[ \int_0^{T_n} e^{-\Lambda_t} Z_t dA_t^{-[\mathcal{L}g_c]} \right] \in \mathbb{R}. \quad (35)$$

Now, let us assume that $z = 1$. Using Dynkin’s formula (18), we can calculate

$$\sum_{j=1}^{2n-1} \mathbb{E}_x \left[ e^{-\Lambda_{T_j}} \left[ g_0(X_{T_j})1_{\{\Delta Z_{T_j}=1\}} + g_c(X_{T_j})1_{\{\Delta Z_{T_j}=-1\}} \right] 1_{(T_j, \infty)} \right]$$

$$= \sum_{j=0}^{n-1} \mathbb{E}_x \left[ g_0(X_{T_{2j}})1_{(T_{2j}, \infty)} \right] + \sum_{j=0}^{n-1} \mathbb{E}_x \left[ g_c(X_{T_{2j+1}})1_{(T_{2j+1}, \infty)} \right]$$

$$= \sum_{j=1}^{n-1} \mathbb{E}_x \left[ e^{-\Lambda_{T_{2j}}} \left[ g_0(X_{T_{2j}}) + g_c(X_{T_{2j}}) \right] 1_{(T_{2j}, \infty)} \right]$$

$$+ g_c(x) + \mathbb{E}_x \left[ \int_0^{T_{2n}} e^{-\Lambda_t} Z_t dA_t^{-[\mathcal{L}g_c]} \right]. \quad (36)$$
However, this observation, the limit
\[
\lim_{n \to \infty} E_x \left[ g_o(X_{T_n})1_{\{T_n < \infty\}} \right] = 0,
\]
which follows from the strong transversality condition (19), the fact that \( \lim_{n \to \infty} T_n = \infty \), (35), and (27) in Assumption 4 imply that
\[
\sum_{n=1}^{\infty} E_x \left[ e^{-\Lambda T_n} \left[ g_o(X_{T_n})1_{\{\Delta Z_{T_n} = 1\}} + g_c(X_{T_n})1_{\{\Delta Z_{T_n} = -1\}} \right] 1_{\{T_n < \infty\}} \right] = \lim_{n \to \infty} \sum_{j=1}^{n} E_x \left[ e^{-\Lambda T_j} \left[ g_o(X_{T_j})1_{\{\Delta Z_{T_j} = 1\}} + g_c(X_{T_j})1_{\{\Delta Z_{T_j} = -1\}} \right] 1_{\{T_j < \infty\}} \right] \in [-\infty, \infty].
\]
However, this limit and (34) imply the well-posedness of the problem as well as (33). Finally, the analysis of the case when \( z = 0 \) follows similar steps. \( \square \)

4 Ramifications of our assumptions

We now consider the functions \( (R_h + g_c)/\phi \) and \( (R_h - g_o)/\psi \), which will play a fundamental role in the solution of our problem, and we make the following observations. First, we note that (16) and Assumption 4 imply that
\[
\lim_{x \downarrow \alpha} \frac{(R_h + g_c)(x)}{\phi(x)} = 0 \quad \text{and} \quad \lim_{x \uparrow \beta} \frac{(R_h - g_o)(x)}{\psi(x)} = 0. \tag{36}
\]
Also, using (20)–(23), we can calculate
\[
- \left( \frac{R_h + g_c}{\phi} \right)'_+ (x) = \frac{W(x)}{\phi^2(x)} \int_{[x, \beta]} \Phi(s) \mathcal{L}(R_h + g_c)(ds) \tag{37}
\]
and
\[
\left( \frac{R_h - g_o}{\psi} \right)'_+ (x) = \frac{W(x)}{\psi^2(x)} \int_{[\alpha, x]} \Psi(s) \mathcal{L}(R_h - g_o)(ds). \tag{38}
\]
Combining these expressions with Assumption 5, we can see that the functions \(- (R_h + g_c)/\phi\) and \( (R_h - g_o)/\psi \) have the same general form: both of these functions may be first increasing and then decreasing as \( x \) increases from \( \alpha \) to \( \beta \). In particular, we note that (36), (37) and Assumption 5 imply that we can have one of the following cases:

- In Case A1 of Assumption 5,
\[
-R_h(x) - g_c(x) = 0, \quad \text{for all} \ x \in \mathcal{I}. \tag{39}
\]
• In Case A2 of Assumption 5,
\[
\int_{[\alpha, \beta]} \Phi(s) \mathcal{L}(R_h + g_c)(ds) \leq 0, \quad \text{for all } x \in \mathcal{I},
\]
(40)

\[-R_h(x) - g_c(x) < 0, \quad \text{for all } x \in \mathcal{I}, \quad \text{and } \quad -\frac{R_h + g_c}{\phi} \text{ is decreasing.} \tag{41}
\]

• In Case A3 of Assumption 5,
\[
\int_{[\alpha, \beta]} \Phi(s) \mathcal{L}(R_h + g_c)(ds) \geq 0, \quad \text{for all } x \in \mathcal{I},
\]
(42)

\[-R_h(x) - g_c(x) > 0, \quad \text{for all } x \in \mathcal{I}, \quad \text{and } \quad -\frac{R_h + g_c}{\phi} \text{ is increasing.} \tag{43}
\]

• In Case A4 of Assumption 5, we can have one of the following possibilities:

A41.\[
\int_{[\alpha, \beta]} \Phi(s) \mathcal{L}(R_h + g_c)(ds) \leq 0, \quad \text{for all } x \in \mathcal{I},
\]
(44)

\[-R_h(x) - g_c(x) < 0, \quad \text{for all } x \in \mathcal{I}, \quad \text{and } \quad -\frac{R_h + g_c}{\phi} \text{ is decreasing;} \tag{45}
\]

A42. there exists a point \(a^* \in ]\alpha, \tilde{a}]\) such that
\[
\int_{[\alpha, \beta]} \Phi(s) \mathcal{L}(R_h + g_c)(ds) \begin{cases} 0, & \text{if } x \in ]\alpha, a^*[, \\ \leq 0, & \text{if } x \in ]a^*, \beta[, \\ \end{cases}
\]
(46)

\[-R_h(x) - g_c(x) \begin{cases} = 0, & \text{for } x \in ]\alpha, a^*[, \\ < 0, & \text{for } x \in ]a^*, \beta[, \\ \end{cases} \quad \text{and } \quad -\frac{R_h + g_c}{\phi} \text{ is decreasing;} \tag{47}
\]

A43. there exists a point \(a^* \in ]\alpha, \tilde{a}]\) such that
\[
\int_{[\alpha, \beta]} \Phi(s) \mathcal{L}(R_h + g_c)(ds) \begin{cases} \geq 0, & \text{if } x \in ]\alpha, a^*[, \\ \leq 0, & \text{if } x \in ]a^*, \beta[, \\ \end{cases}
\]
(48)

\[-\frac{R_h + g_c}{\phi} \left\{ \begin{array}{ll}
\text{positive and increasing in }]\alpha, a^*[], \\
\text{decreasing in }]a^*, \beta[. \\
\end{array} \right.
\]
(49)

\[-R_h(a^*) - g_c(a^*) > 0 \quad \text{and} \quad -R_h(x) - g_c(x) < -R_h(a^*) - g_c(a^*), \quad \text{for all } x \in ]a^*, \beta[. \tag{50}
\]

Similarly, we can see that (36), (38) and Assumption 5 imply that we can have one of the following cases:
• In Case B1 of Assumption 5,

\[ R_h(x) - g_o(x) = 0, \quad \text{for all } x \in \mathcal{I}. \]  \hspace{1cm} (51)

• In Case B2 of Assumption 5,

\[ \int_{[a,x]} \Psi(s) \mathcal{L}(R_h - g_o)(ds) \geq 0, \quad \text{for all } x \in \mathcal{I}, \]  \hspace{1cm} (52)

\[ R_h(x) - g_o(x) < 0, \quad \text{for all } x \in \mathcal{I}, \quad \text{and } \frac{R_h - g_o}{\psi} \text{ is increasing.} \]  \hspace{1cm} (53)

• In Case B3 of Assumption 5,

\[ \int_{[a,x]} \Psi(s) \mathcal{L}(R_h - g_o)(ds) \leq 0 \quad \text{for all } x \in \mathcal{I}, \]  \hspace{1cm} (54)

\[ R_h(x) - g_o(x) > 0, \quad \text{for all } x \in \mathcal{I}, \quad \text{and } \frac{R_h - g_o}{\psi} \text{ is decreasing.} \]  \hspace{1cm} (55)

• In Case B4 of Assumption 5, we can have have one of the following possibilities:

B41.

\[ \int_{[a,x]} \Psi(s) \mathcal{L}(R_h - g_o)(ds) \geq 0 \quad \text{for all } x \in \mathcal{I}, \]  \hspace{1cm} (56)

\[ R_h(x) - g_o(x) < 0, \quad \text{for all } x \in \mathcal{I}, \quad \text{and } \frac{R_h - g_o}{\psi} \text{ is increasing;} \]  \hspace{1cm} (57)

B42. there exists a point \( b^* \in [\bar{b}, \beta] \) such that

\[ \int_{[a,x]} \Psi(s) \mathcal{L}(R_h - g_o)(ds) \begin{cases} \geq 0, & \text{if } x \in [\alpha, b^*[, \\ = 0, & \text{if } x \in ]b^*, \beta[, \end{cases} \]  \hspace{1cm} (58)

\[ R_h(x) - g_o(x) \begin{cases} < 0, & \text{for } x \in [\alpha, b^*[, \\ = 0, & \text{for } x \in ]b^*, \beta[, \end{cases} \quad \text{and } \frac{R_h - g_o}{\psi} \text{ is increasing;} \]  \hspace{1cm} (59)

B43. there exists a point \( b^* \in [\bar{b}, \beta] \) such that

\[ \int_{[a,x]} \Psi(s) \mathcal{L}(R_h - g_o)(ds) \begin{cases} \geq 0, & \text{if } x \in [\alpha, b^*[, \\ \leq 0, & \text{if } x \in ]b^*, \beta[, \end{cases} \]  \hspace{1cm} (60)

\[ \frac{R_h - g_o}{\psi} \text{ is increasing in } [\alpha, b^*[, \quad \text{positive and decreasing in } ]b^*, \beta[, \]  \hspace{1cm} (61)

\[ R_h(b^*) - g_o(b^*) > 0 \quad \text{and} \quad R_h(x) - g_o(x) < R_h(b^*) - g_o(b^*), \quad \text{for all } x \in [\alpha, b^*.} \]  \hspace{1cm} (62)
To proceed further, we consider the cases $A_{42}$, $A_{43}$, $B_{41}$, $B_{42}$ and $B_{43}$, and the inequality
\[
\frac{R_h(a^*) + g_c(a^*)}{\phi(a^*)} < \lim_{x \to \beta} \frac{R_h(x) - g_o(x)}{\phi(x)},
\]
as well as the cases $A_{41}$, $A_{42}$, $A_{43}$, $B_{42}$ and $B_{43}$, and the inequality
\[
\lim_{x \to \alpha} \frac{R_h(x) + g_c(x)}{\psi(x)} < \frac{R_h(b^*) - g_o(b^*)}{\psi(b^*)}.
\]
A comparison of (45), (47) and (50) with (57), (59) and (62) reveals that
\[
(63) \text{ and (64) both are true in cases } A_{42} - B_{43}, A_{43} - B_{42}, A_{43} - B_{43},
\]
(63) may or may not be true in case $A_{43} - B_{41}$, and
\[
(64) \text{ may or may not be true in case } A_{41} - B_{43}.
\]

Now, we note that (27) in Assumption 4 is equivalent to
\[-(R_h + g_c)(x) < -(R_h - g_o)(x), \text{ for all } x \in I,
\]which implies that there exists no $x \in I$ such that $-(R_h + g_c)(x)$ and $(R_h - g_o)(x)$ both are non-negative. However, this observation implies that none of the pairs $A_{11}$–$B_{11}$, $A_{11}$–$B_{33}$, $A_{11}$–$B_{42}$, $A_{11}$–$B_{43}$, $A_{33}$–$B_{11}$, $A_{42}$–$B_{11}$, $A_{43}$–$B_{11}$, $A_{33}$–$B_{33}$, $A_{33}$–$B_{42}$, $A_{33}$–$B_{43}$, $A_{42}$–$B_{33}$, $A_{43}$–$B_{33}$ can occur.

We can summarise this discussion by observing that our assumptions can all be satisfied only if the problem data is such that a pair in Table 1 occurs. We have organised the various pairs appearing in this table in six groups that correspond to the six possible forms that an optimal switching strategy can take.

<table>
<thead>
<tr>
<th>Group</th>
<th>A1–B2</th>
<th>A1–B41</th>
<th>A2–B41</th>
<th>A2–B42</th>
<th>A41–B42</th>
<th>A2–B2</th>
<th>A41–B41</th>
<th>A42–B42</th>
<th>A2–B43</th>
<th>A41–B43 if (64) is false</th>
<th>B2–A43</th>
<th>A42–B43</th>
<th>A43–B42</th>
<th>A43–B43</th>
<th>A41–B43 if (64) is true</th>
<th>B41–A43 if (63) is true</th>
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<tbody>
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</table>

It is straightforward to check that, when the measures $L(R_h + g_c)$ and $L(R_h - g_o)$ have full support (see Section 2.1), none of the cases $A_{11}$, $B_{11}$, $A_{42}$ or $B_{42}$ can occur. In particular, the inequalities (40), (42), (44), (48), (52), (54), (56) and (60) all are strict. In this context, we can see that our assumptions result in a classification of the problem data in the six mutually exclusive groups of Table 2.
<table>
<thead>
<tr>
<th>Group</th>
<th>NA</th>
<th>A2–B2</th>
<th>A2–B41</th>
<th>A41–B2</th>
<th>A41–B41</th>
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<td>A2–B3</td>
<td>A41–B3</td>
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<tr>
<td>Group C</td>
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<td>B2–A3</td>
<td>B41–A3</td>
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<td>Group WC</td>
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<td>B2–A43</td>
<td>B41–A43</td>
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</tbody>
</table>

Table 2

For future reference, we note that

in Cases A42 and A43, \[ \int_{[a^*,\beta]} \Phi(s) L(R_h + g_c)(ds) \leq 0 \leq \int_{[a^*,\beta]} \Phi(s) L(R_h + g_c)(ds), \quad (68) \]

while,

in Cases B42 and B43, \[ \int_{[a,b^*]} \Psi(s) L(R_h - g_o)(ds) \leq 0 \leq \int_{[a,b^*]} \Psi(s) L(R_h - g_o)(ds). \quad (69) \]

5 A verification theorem

In view of the existing theory on similar stochastic control problems, we expect that the value function \( v \) identifies with a classical solution \( w \) of the HJB equation

\[
\max \left\{ \frac{1}{2} \sigma^2(x) w_{xx}(z, x) + b(x) w_x(z, x) - r(x) w(z, x) + zh(x), \\
w(1 - z, x) - w(z, x) - zg_c(x) - (1 - z)g_o(x) \right\} = 0. \quad (70)
\]

when the problem data are smooth functions. In the case that we consider in this paper, we do not assume that the problem data are smooth, so, we have to consider generalised solutions of (70).

**Definition 3** A function \( w : \{0, 1\} \times \mathcal{I} \rightarrow \mathbb{R} \) is a solution of the HJB equation (70) if \( w(z, \cdot) \) is the difference of two convex functions,

\[
-\mathcal{L} w(z, \cdot) - zh \text{ is a positive measure on } (\mathcal{I}, \mathcal{B}(\mathcal{I})), \quad (71)
\]

\[
w(1 - z, x) - w(z, x) - zg_c(x) - (1 - z)g_o(x) \leq 0, \quad \text{for all } x \in \mathcal{I}, \quad (72)
\]

and

\[
\mathcal{L} w(0, \cdot)(\mathcal{C}_c) = \mathcal{L} w(1, \cdot)(\mathcal{C}_o) + h(\mathcal{C}_o) = 0, \quad (73)
\]
where the operator $\mathcal{L}$ is defined by (9), and $\mathcal{C}_c$ and $\mathcal{C}_o$ are the open sets defined by
\[
\mathcal{C}_c = \{ x \in \mathcal{I} \mid w(0, x) > w(1, x) - g_o(x) \}, \tag{74}
\]
\[
\mathcal{C}_o = \{ x \in \mathcal{I} \mid w(1, x) > w(0, x) - g_c(x) \}. \tag{75}
\]

In the context of this definition, we can make the following observations that are informed by the existing literature in the area and link the four components composing (70) to optimal decision tactics. The sets $\mathcal{C}_c$ and $\mathcal{C}_o$ are the so-called “continuation” regions associated with the system in its closed and its open operating modes, respectively. For instance, the decision maker should take no action if the system is in its closed mode and the state process $X$ assumes values inside $\mathcal{C}_c$. In view of (73) and Section 2.3, a solution of the HJB equation (70) should be given by
\[
w(0, x) = A_c \phi(x) + B_c \psi(x)
\]
and
\[
w(1, x) = A_o \phi(x) + B_o \psi(x) + R_h(x),
\]
for some constants $A_c, B_c, A_o, B_o \in \mathbb{R}$, which may depend on the relative location of $x$ in $\mathcal{I}$. On the other hand, the sets $\mathcal{I} \setminus \mathcal{C}_c$ and $\mathcal{I} \setminus \mathcal{C}_o$ characterise the part of the state space in which the decision maker should take action. In particular, if, at any given time $t$, the system is in its closed (resp., open) mode and $X_t \in \mathcal{I} \setminus \mathcal{C}_c$ (resp., $X_t \in \mathcal{I} \setminus \mathcal{C}_o$), then the system’s controller should switch the system from its closed (resp., open) mode to its open (resp., closed) one.

The following result is concerned with sufficient conditions for a solution of (70) to identify with the value function of our control problem.

**Theorem 2** Consider the stochastic control problem formulated in Section 2, and suppose that Assumptions 1–4 hold. If the function $w : \{0, 1\} \times \mathcal{I} \mapsto \mathbb{R}$ satisfies the HJB equation (70) in the sense of Definition 3,
\[
\text{the measures } \mathcal{L}w(0, \cdot) \text{ and } \mathcal{L}w(1, \cdot) \text{ are } (\phi, \psi)\text{-integrable}, \tag{76}
\]
and
\[
|w(z, \cdot)| \leq C (1 + |R_h| + |g_o| + |g_c|), \tag{77}
\]
for some constant $C > 0$, for $z = 0, 1$, then $w \leq v$. Furthermore, if there exists an admissible strategy $Z^\ast_{z,x} \in \mathcal{A}_{z,x}$ such that the random sets
\[
\{ t \geq 0 \mid Z^\ast_t = 0 \text{ and } X^\ast_t \in \mathcal{I} \setminus \mathcal{C}_c \} \text{ and } \{ t \geq 0 \mid Z^\ast_t = 1 \text{ and } X^\ast_t \in \mathcal{I} \setminus \mathcal{C}_o \} \tag{78}
\]
both are countable, $P$-a.s.,
\[
\{ t \geq 0 \mid \Delta Z^\ast_t = 1 \} \subseteq \mathcal{I} \setminus \mathcal{C}_c \text{ and } \{ t \geq 0 \mid \Delta Z^\ast_t = -1 \} \subseteq \mathcal{I} \setminus \mathcal{C}_o, \tag{79}
\]
then $w = v$ and $Z^\ast_{z,x}$ is an optimal strategy.
Proof. We first note that, in view of (6), (16) and Assumption 4, we can see that (77) implies that
\[
\lim_{x \to a} \frac{w(z, x)}{\phi(x)} = 0 = \lim_{x \to b} \frac{w(z, x)}{\psi(x)}, \tag{80}
\]
Now, we fix any initial condition \( x \in \mathcal{I} \) and any weak solution \( \mathbb{S}_x \) of the SDE (1), and we consider any strictly increasing sequence \( (\alpha_m) \) and any strictly decreasing sequence \( (\beta_n) \) such that
\[
\alpha_1 < x < \beta_1, \quad \lim_{m \to \infty} \alpha_m = \alpha \quad \text{and} \quad \lim_{n \to \infty} \beta_n = \infty.
\]
Given \( z = 0, 1 \), the locally bounded function \( w(z, \cdot) \) plainly satisfies the ODE (8) with \( \mu = -\mathcal{L}w(z, \cdot) \). In view of Johnson and Zervos [JZ07, Lemma 5], it follows that, given any \( (\mathcal{F}_t) \)-stopping time \( \nu \),
\[
\mathbb{E}_x \left[ e^{-\Lambda \tau_{\alpha_m} \land \beta_n} w(z, X_{\tau_{\alpha_m} \land \beta_n}) \right] \equiv \mathbb{E}_x \left[ e^{-\Lambda \tau_{\alpha_m}} w(z, X_\nu) \mathbb{1}_{\{v < \tau_{\alpha_m} \land \beta_n\}} \right]
\]
\[
+ \mathbb{E}_x \left[ e^{-\Lambda \tau_{\alpha_m} \land \beta_n} w(z, X_{\tau_{\alpha_m} \land \beta_n}) \mathbb{1}_{\{\tau_{\alpha_m} \land \beta_n \leq v\}} \right] = w(x) - \mathbb{E}_x \left[ \int_{\tau_{\alpha_m} \land \beta_n \land \nu} e^{-\Lambda t} dA_t w(z, \cdot) \right]. \tag{81}
\]
Using (7) and (80), we can see that
\[
\lim_{m \to \infty} \|w(z, \alpha_m)\|_\mathbb{E}_x \left[ e^{-\Lambda \tau_{\alpha_m}} \mathbb{1}_{\{\tau_{\alpha_m} < \beta_n \land \nu\}} \right] \leq \lim_{m \to \infty} \|w(z, \alpha_m)\|_\mathbb{E}_x \left[ e^{-\Lambda \tau_{\alpha_m}} \right] = \lim_{m \to \infty} \frac{\|w(z, \alpha_m)\|_\phi(x)}{\phi(\alpha_m)} = 0,
\]
and that
\[
\lim_{n \to \infty} \|w(z, \beta_n)\|_\mathbb{E}_x \left[ e^{-\Lambda \tau_{\beta_n}} \mathbb{1}_{\{\tau_{\beta_n} \leq \tau_{\alpha_m} \land \nu\}} \right] \leq \lim_{n \to \infty} \frac{\|w(z, \beta_n)\|_\psi(x)}{\psi(\beta_n)} = 0.
\]
In light of these calculations, we can see that
\[
\lim_{m,n \to \infty} \mathbb{E}_x \left[ e^{-\Lambda \tau_{\alpha_m} \land \beta_n} w(z, X_{\tau_{\alpha_m} \land \beta_n}) \mathbb{1}_{\{\tau_{\alpha_m} \land \beta_n \leq \nu\}} \right] = 0. \tag{82}
\]
Also, (77), Assumption 4, (17) and the dominated convergence theorem imply that
\[
\lim_{m \to \infty} \mathbb{E}_x \left[ e^{-\Lambda \nu} w(z, X_\nu) \mathbb{1}_{\{v < \tau_{\alpha_m} \land \beta_n\}} \right] = \mathbb{E}_x \left[ e^{-\Lambda \nu} w(z, X_\nu) \mathbb{1}_{\{v < \infty\}} \right], \tag{83}
\]
while, (76), (13) and the dominated convergence theorem imply that
\[
\lim_{m,n \to \infty} \mathbb{E}_x \left[ \int_0^{\tau_{\alpha_m} \land \beta_n \land \nu} e^{-\Lambda t} dA_t \mathcal{L}w(z, \cdot) \right] = \mathbb{E}_x \left[ \int_0^{\nu} e^{-\Lambda t} dA_t \mathcal{L}w(z, \cdot) \mathbb{1}_{\{v < \infty\}} \right]. \tag{84}
\]
However, (81)–(84) imply that
\[
\mathbb{E}_x \left[ e^{-\Lambda \nu} w(z, X_\nu) \mathbb{1}_{\{v < \infty\}} \right] = w(z, x) - \mathbb{E}_x \left[ \int_0^{\nu} e^{-\Lambda t} dA_t \mathcal{L}w(z, \cdot) \mathbb{1}_{\{v < \infty\}} \right]. \tag{85}
\]
To proceed further, we assume that the system is in its open operating mode at time 0, i.e., that \( z = 1 \); the analysis of the case associated with \( z = 0 \) follows exactly the same steps. In particular, we consider any admissible switching strategy \( Z_{1,x} \in A_{1,x} \), and we recall that the jumps of the associated switching process \( Z \) occur at the times composing the sequence \( (T_n, n \geq 1) \) defined by (25) in Definition 2. For notational simplicity, we define \( T_0 = 0 \), and we note that \( 0 = T_0 \leq T_1 < T_2 < \cdots \). Iterating (85), we calculate

\[
\mathbb{E}_x \left[ e^{-\Lambda T_{2n}} w(0, X_{T_{2n}}) 1_{\{T_{2n} < \infty\}} \right] = w(1, x) + \sum_{j=0}^{n-1} \mathbb{E}_x \left[ e^{-\Lambda T_{2j+1}} \left[ w(0, X_{T_{2j+1}}) - w(1, X_{T_{2j+1}}) \right] 1_{\{T_{2j+1} < \infty\}} \right]
\]

\[
+ \sum_{j=1}^{n-1} \mathbb{E}_x \left[ e^{-\Lambda T_{2j}} \left[ w(1, X_{T_{2j}}) - w(0, X_{T_{2j}}) \right] 1_{\{T_{2j} < \infty\}} \right]
\]

\[
- \sum_{j=0}^{n-1} \mathbb{E}_x \left[ \int_{T_{2j}}^{T_{2j+1}} e^{-\Lambda t} dA_t \right] - \sum_{j=0}^{n-1} \mathbb{E}_x \left[ \int_{T_{2j+1}}^{T_{2j+2}} e^{-\Lambda t} dA_t \right].
\]

Adding the term

\[
\mathbb{E}_x \left[ \int_0^{T_{2n}} e^{-\Lambda t} Z_t A_t^h \right] - \sum_{j=1}^{2n-1} \mathbb{E}_x \left[ e^{-\Lambda T_j} \left[ g_o(X_{T_j}) 1_{\{\Delta Z_{T_j} = 1\}} + g_c(X_{T_j}) 1_{\{\Delta Z_{T_j} = -1\}} \right] 1_{\{T_j < \infty\}} \right]
\]

\[
= - \sum_{j=0}^{n-1} \mathbb{E}_x \left[ \int_{T_{2j}}^{T_{2j+1}} e^{-\Lambda t} dA_t^h \right]
\]

\[
- \sum_{j=0}^{n-1} \mathbb{E}_x \left[ e^{-\Lambda T_{2j+1}} g_c(X_{T_{2j+1}}) 1_{\{T_{2j+1} < \infty\}} \right] - \sum_{j=1}^{n-1} \mathbb{E}_x \left[ e^{-\Lambda T_{2j}} g_o(X_{T_{2j}}) 1_{\{T_{2j} < \infty\}} \right]
\]

on both sides of (86), we obtain

\[
\mathbb{E}_x \left[ \int_0^{T_{2n}} e^{-\Lambda t} Z_t A_t^h \right] - \sum_{j=1}^{2n-1} \mathbb{E}_x \left[ e^{-\Lambda T_j} \left[ g_o(X_{T_j}) 1_{\{\Delta Z_{T_j} = 1\}} + g_c(X_{T_j}) 1_{\{\Delta Z_{T_j} = -1\}} \right] 1_{\{T_j < \infty\}} \right]
\]

\[
= w(1, x) - \mathbb{E}_x \left[ e^{-\Lambda T_{2n}} w(0, X_{T_{2n}}) 1_{\{T_{2n} < \infty\}} \right]
\]

\[
- \sum_{j=0}^{n-1} \mathbb{E}_x \left[ \int_{T_{2j}}^{T_{2j+1}} e^{-\Lambda t} dA_t^{-\mathcal{L}(1,-)} h \right] - \sum_{j=0}^{n-1} \mathbb{E}_x \left[ \int_{T_{2j+1}}^{T_{2j+2}} e^{-\Lambda t} dA_t^{-\mathcal{L}(0,-)} \right]
\]

\[
+ \sum_{j=0}^{n-1} \mathbb{E}_x \left[ e^{-\Lambda T_{2j+1}} \left[ w(0, X_{T_{2j+1}}) - w(1, X_{T_{2j+1}}) - g_c(X_{T_{2j+1}}) \right] 1_{\{T_{2j+1} < \infty\}} \right]
\]

\[
+ \sum_{j=1}^{n-1} \mathbb{E}_x \left[ e^{-\Lambda T_{2j}} \left[ w(1, X_{T_{2j}}) - w(0, X_{T_{2j}}) - g_o(X_{T_{2j}}) \right] 1_{\{T_{2j} < \infty\}} \right].
\]
In view of (33) and the fact that $w$ satisfies the HJB equation (70) in the sense of Definition 3, it follows that

$$\mathbb{E}_x \left[ \int_0^{T_{2n}} e^{-\Lambda t} Z_t dA_t^b \right] - \sum_{j=1}^{2n-1} \mathbb{E}_x \left[ e^{-\Lambda T_j} \left[ g_o(X_{T_j}) 1_{\{\Delta Z_{T_j} = 1\}} + g_c(X_{T_j}) 1_{\{\Delta Z_{T_j} = -1\}} \right] 1_{\{T_j < \infty\}} \right] \leq w(1, x) - \mathbb{E}_x \left[ e^{-\Lambda T_{2n}} w(0, X_{T_{2n}}) 1_{\{T_{2n} < \infty\}} \right].$$

(87)

In view of the fact that

$$\lim_{n \to \infty} \mathbb{E}_x \left[ e^{-\Lambda T_{2n}} w(0, X_{T_{2n}}) 1_{\{T_{2n} < \infty\}} \right] = 0,$$

which follows from (77), Assumptions 3–4 and the strong transversality condition (19), we can pass to the limit as $n \to \infty$ in (87) to obtain the inequality $J(Z_{1,x}) \leq w(1, x)$, which implies that $v(1, x) \leq w(1, x)$.

Now, let $Z_{1,x}^*$ be an admissible switching strategy that is characterised by the properties discussed in the relevant part of the theorem’s statement. Recalling (12), we can see that, in this case, (87) holds with equality. In view of (33), we can then pass to the limit as $n \to \infty$ to obtain $J(Z_{1,x}^*) = w(1, x)$, which, combined with the inequality $v(1, x) \leq w(1, x)$ that we have established above, implies that $v(1, x) = w(1, x)$, and that $Z_{1,x}^*$ is optimal.

□

6 The solution to the control problem

We now solve our control problem by constructing an explicit solution to the HJB equation (70) that satisfies the requirements of Theorem 2. To this end, we consider the various qualitatively different forms that the optimal switching strategy may take, guided by the discussion following Definition 3 and by condition (80) that is required by the verification theorem proved in the previous section.

To start with, the optimal strategy could involve no switchings, that is, it might be optimal to always leave the system in its original operating mode. In this case, the choice

$$w(0, \cdot) = 0 \quad \text{and} \quad w(1, \cdot) = R_h,$$

(88)

should provide the required solution of (70). A second possibility arises when it is optimal to irreversibly switch the system to its open operating mode, in which case,

$$w(0, \cdot) = w(1, \cdot) - g_o \quad \text{and} \quad w(1, \cdot) = R_h,$$

(89)

should satisfy (70). Similarly, it might be optimal to irreversibly switch the system to its closed operating mode, which is associated with a solution of (70) of the form

$$w(0, \cdot) = 0 \quad \text{and} \quad w(1, \cdot) = -g_c.$$

(90)

The following result, the proof of which we develop in the appendix, is concerned with conditions under which (88)–(90) indeed provide a solution of the HJB equation (70).
Lemma 3 In the presence of Assumptions 1–5, the following statements are true:

(I) The function \( w \) given by (88) satisfies the HJB equation (70) in the sense of Definition 3 if and only if the problem data are such that any of the pairs in Group NA of Table 1 occurs.

(II) The function \( w \) given by (89) satisfies the HJB equation (70) in the sense of Definition 3 if the problem data are such that any of the pairs in Group O of Table 1 occurs.

(III) The function \( w \) given by (90) satisfies the HJB equation (70) in the sense of Definition 3 if the problem data are such that any of the pairs in Group C of Table 1 occurs.

Furthermore, if the measures \( L(R_h + g_c) \) and \( L(R_h - g_o) \) both have full-support, then each of the above statements is true if and only if the problem data is such that one of the pairs in the corresponding groups of Table 2 occurs.

At this point, we should note that, in the general case, statements (II) and (III) in this lemma involve only sufficient conditions. For instance, a quick inspection of this result’s proof in the appendix reveals that the strategy of irreversibly switching the system to its open operating mode is also optimal if either of the pairs \( A2-B1 \) or \( A41-B1 \), which appear in Group NA of Table 1, occurs. A similar comment also applies to Lemmas 4 and 5 below.

Departing from the consideration of strategies that have the simple structures considered above, the next possibility that arises is when it is optimal to wait before permanently switching the system to its open operating mode. In this case, we look for a point \( b_o \in \mathcal{I} \) such that, if the system is in its closed operating mode at time 0, then it is optimal to wait as long as the state process assumes values in the interval \( [\alpha, b_o] \), and permanently switch the system to its open operating mode as soon as the state process hits the interval \( [b_o, \beta] \).

In this case, we look for a solution of the HJB equation (70) of the form given by

\[
w(0, x) = \begin{cases} 
B \psi(x), & \text{if } x \in [\alpha, b_o], \\
R_h(x) - g_o(x), & \text{if } x \in [b_o, \beta],
\end{cases} \quad \text{and} \quad w(1, x) = R_h,
\]

for some constant \( B \). To determine the parameter \( B \) and the free-boundary point \( b_o \), we conjecture that the inequalities

\[
B \psi(b_o) = R_h(b_o) - g_o(b_o)
\]

and

\[
(R_h - g_o)_-(b_o) \leq B \psi'(b_o) \leq (R_h - g_o)_+(b_o),
\]

should hold. Indeed, these inequalities are the generalisation of the so-called “principle of smooth fit” that is appropriate for the analysis of our problem. Solving (92) for \( B \) and substituting for it into (93), we can see that the point \( b_o \) should satisfy the inequalities

\[
\psi(b_o)(R_h - g_o)_-(b_o) - \psi'(b_o)(R_h - g_o)(b_o) \leq 0,
\]

\[
\psi(b_o)(R_h - g_o)_+(b_o) - \psi'(b_o)(R_h - g_o)(b_o) \geq 0,
\]

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which are equivalent to
\[
\left( \frac{R_h - g_o}{\psi} \right)'(b_o) \leq 0 \leq \left( \frac{R_h - g_o}{\psi} \right)'(b_o).
\] (94)

These inequalities can hold if and only if \(b_o\) is a local maximum of the function \((R_h - g_o)/\psi\). In view of (51)–(62), we can see that, beyond the cases covered by the previous lemma, this possibility can arise only in the context of B42 or B43, in which cases, the choice \(b_o = b^*\) is an appropriate one.

Similarly, it might be optimal to wait before irreversibly switching the system to its closed operating mode, which is associated with a solution of the HJB equation (70) of the form given by
\[
w(0, x) = 0 \quad \text{and} \quad w(1, x) = \begin{cases}
-g_c(x), & \text{if } x \in ]\alpha, a_c[, \\
A\phi(x) + R_h(x), & \text{if } x \in ]a_c, \beta[, 
\end{cases}
\] (95)

for some constant \(A\) and free-boundary point \(a_c\). Following the same reasoning as above, we can see that this case can be optimal if the problem data is such that either A42 or A43 is satisfied,
\[
a_c = a^* \quad \text{and} \quad A = -\frac{R_h(a_c) + g_c(a_c)}{\phi(a_c)}.
\] (96)

The following result is concerned with conditions under which these strategies are indeed associated with solutions of the HJB equation (70).

**Lemma 4** In the presence of Assumptions 1–5, the following statements are true:

(I) The function \(w\) given by (91), where the constant \(B\) is given by (92) and \(b_o\) satisfies (94), is well-defined and satisfies the HJB equation (70) in the sense of Definition 3 if the problem data are such that any of the cases in Group WO of Table 1 occurs.

(II) The function \(w\) given by (95), where the constants \(A\) and \(a_c\) satisfy (96), is well-defined and satisfies the HJB equation (70) in the sense of Definition 3 if the problem data are such that any of the cases in Group WC of Table 1 occurs.

Furthermore, if the measures \(\mathcal{L}(R_h + g_c)\) and \(\mathcal{L}(R_h - g_o)\) both have full-support, then each of the above statements is true if and only if the problem data is such that one of the pairs in the corresponding groups of Table 2 occurs.

The final possibility that arises is when it is optimal to sequentially switch the the system from its open operating mode to its closed one, and vice versa. In this case, we postulate that the value function of our control problem identifies with a solution \(w\) to the HJB equation (70) that has the form given by the expressions
\[
w(0, x) = \begin{cases}
B\psi(x), & \text{if } x \in ]\alpha, b_o[, \\
A\phi(x) + R_h(x) - g_o(x), & \text{if } x \in ]b_o, \beta[, 
\end{cases}
\] (97)

\[
w(1, x) = \begin{cases}
B\psi(x) - g_c(x), & \text{if } x \in ]\alpha, a_c[, \\
A\phi(x) + R_h(x), & \text{if } x \in ]a_c, \beta[, 
\end{cases}
\] (98)
for some constants $A$, $B$ and free-boundary points $a_c$, $b_o$ such that $\alpha < a_c < b_o < \beta$. To determine these variables, we conjecture that the inequalities
\begin{align}
A\phi(a_c) + R_h(a_c) &= B\psi(a_c) - g_c(a_c), \quad (99) \\
A\phi'(a_c) + (R_h)'_-(a_c) &\leq B\psi'(a_c) - (g_c)'_-(a_c), \quad (100) \\
A\phi'(a_c) + (R_h)'_+(a_c) &\geq B\psi'(a_c) - (g_c)'_+(a_c) \quad (101)
\end{align}
should hold at $a_c$, and the inequalities
\begin{align}
B\psi(b_o) &= A\phi(b_o) + R_h(b_o) - g_o(b_o), \quad (102) \\
B\psi'(b_o) &\leq A\phi'(b_o) + (R_h)'_-(b_o) - (g_o)'_-(b_o), \quad (103) \\
B\psi'(b_o) &\geq A\phi'(b_o) + (R_h)'_+(b_o) - (g_o)'_+(b_o) \quad (104)
\end{align}
should hold at $b_o$. An inspection of (99)--(104) reveals that, when the functions $R_h$, $g_c$ and $g_o$ are $C^1$, these inequalities all hold as equalities. Indeed, in this case, (99)--(104) reduce to the system of four equations that would be suggested by the so-called “principle of smooth fit”, which would require that the functions $w(0, \cdot)$ and $w(1, \cdot)$ should be $C^1$ at the free boundary points $a_c$ and $b_o$, respectively.

To proceed further, we note that (99) and (102) are equivalent to
\begin{align}
A &= \left( \frac{R_h(b_o) - g_o(b_o)}{\psi(b_o)} - \frac{R_h(a_c) + g_c(a_c)}{\psi(a_c)} \right) \left( \frac{\phi(a_c)}{\psi(a_c)} - \frac{\phi(b_o)}{\psi(b_o)} \right)^{-1}, \quad (105) \\
B &= \left( \frac{R_h(b_o) - g_o(b_o)}{\phi(b_o)} - \frac{R_h(a_c) + g_c(a_c)}{\phi(a_c)} \right) \left( \frac{\psi(b_o)}{\phi(b_o)} - \frac{\psi(a_c)}{\phi(a_c)} \right)^{-1}. \quad (106)
\end{align}
Furthermore, in view of the identities (20)--(23), we can see that (99)--(101) imply the system of inequalities
\begin{align}
\int_{[a_c, \beta]} \Phi(s) \mathcal{L}(R_h + g_c)(ds) &\leq -B \leq \int_{[a_c, \beta]} \Phi(s) \mathcal{L}(R_h + g_c)(ds), \quad (107) \\
-\int_{[\alpha, a_c]} \Psi(s) \mathcal{L}(R_h + g_c)(ds) &\leq -A \leq -\int_{[\alpha, a_c]} \Psi(s) \mathcal{L}(R_h + g_c)(ds), \quad (108)
\end{align}
while (102)--(104) imply the system of inequalities
\begin{align}
-\int_{[b_o, \beta]} \Phi(s) \mathcal{L}(R_h - g_o)(ds) &\leq B \leq -\int_{[b_o, \beta]} \Phi(s) \mathcal{L}(R_h - g_o)(ds), \quad (109) \\
\int_{[\alpha, b_o]} \Psi(s) \mathcal{L}(R_h - g_o)(ds) &\leq A \leq \int_{[\alpha, b_o]} \Psi(s) \mathcal{L}(R_h - g_o)(ds). \quad (110)
\end{align}
It follows that the free boundary points $a_c < b_o$ should satisfy the system of inequalities
\begin{align}
q_\phi(a_c, b_o) &\leq 0 \leq q_\phi^c(a_c, b_o), \quad (111) \\
q_\psi(a_c, b_o) &\leq 0 \leq q_\psi^c(a_c, b_o), \quad (112)
\end{align}
where

\[ q^O_\phi(u, v) = - \int_{[u, \beta]} \Phi(s) \mathcal{L}(R_h + g_c)(ds) + \int_{[v, \beta]} \Phi(s) \mathcal{L}(R_h - g_o)(ds), \tag{113} \]

\[ q^C_\phi(u, v) = - \int_{[u, \beta]} \Phi(s) \mathcal{L}(R_h + g_c)(ds) + \int_{[v, \beta]} \Phi(s) \mathcal{L}(R_h - g_o)(ds), \tag{114} \]

\[ q^C_\psi(u, v) = - \int_{[\alpha, u]} \Psi(s) \mathcal{L}(R_h + g_c)(ds) + \int_{[\alpha, v]} \Psi(s) \mathcal{L}(R_h - g_o)(ds) \tag{115} \]

and

\[ q^C_\psi(u, v) = - \int_{[\alpha, u]} \Psi(s) \mathcal{L}(R_h + g_c)(ds) + \int_{[\alpha, v]} \Psi(s) \mathcal{L}(R_h - g_o)(ds). \tag{116} \]

The following result is concerned with conditions under which there exist points \( a_c < b_o \) in \( I \) that satisfy (111)–(112) and the corresponding function \( w \) defined by (97)–(98) satisfies the HJB equation (70).

**Lemma 5** Suppose that Assumptions 1–5 hold true. There exist points \( a_c < b_o \) in \( I \) satisfying the system of inequalities (111)–(112) if the problem data is such that any of the cases in Group S of Table 1 occurs. Also, if the measures \( \mathcal{L}(R_h + g_c) \) and \( \mathcal{L}(R_h - g_o) \) both have full-support, then there exist points \( a_c < b_o \) in \( I \) satisfying the system of inequalities (111)–(112) if and only if the problem data is such that any of the cases in Group S of Table 2 occurs. In either case, the function \( w \) defined by (97)–(98) with \( A \) and \( B \) given by (105) and (106), respectively, satisfies the HJB equation (70) in the sense of Definition 3.

We can now establish our main result.

**Theorem 6** Consider the optimal sequential switching problem formulated in Section 2, and suppose that its data satisfy Assumptions 1–5. We have the following six cases that are differentiated by the various pairings listed in Table 1 (or Table 2 if the measures \( \mathcal{L}(R_h + g_c) \) and \( \mathcal{L}(R_h - g_o) \) have full-support):

(a) if the problem data is such that any of the cases in Group NA of Table 1 (or Table 2) occurs, then the value function \( v \) identifies with the function \( w \) given by (88);

(b) if the problem data is such that any of the cases in Group O of Table 1 (or Table 2) occurs, then the value function \( v \) identifies with the function \( w \) given by (89);

(c) if the problem data is such that any of the cases in Group C of Table 1 (or Table 2) occurs, then the value function \( v \) identifies with the function \( w \) given by (90);

(d) if the problem data is such that any of the cases in Group WO of Table 1 (or Table 2) occurs, then the value function \( v \) identifies with the function \( w \) given by (91), where the constant \( B \) is given by (92) and \( b_o \) satisfies (94);
(e) If the problem data is such that any of the cases in Group WC of Table 1 (or Table 2) occurs, then the value function \( v \) identifies with the function \( w \) given by (95), where the constants \( A \) and \( a_c \) satisfy (96);

(f) If the problem data is such that any of the cases in Group S of Table 1 (or Table 2) occurs, then the value function \( v \) identifies with the function \( w \) given by (97)–(98), where the points \( a_c < b_o \) satisfy the system of inequalities (111)–(112) and \( A, B \) are given by (105), (106).

Optimal switching strategies associated with each of these cases are constructed in the proof below.

Proof. In view of Lemmas 3, 4 and 5, the function \( w \) associated with each case satisfies the HJB equation (70) in the sense of Definition 3. Also, it is straightforward to check that, in all cases, \( w \) satisfies (76) and (77) in the verification Theorem 2. In view of these observations, we only need to construct a switching strategy \( Z^* \) that possesses the properties required by Theorem 2. To this end, we fix any initial condition \( (z, x) \in \{0, 1\} \times I \) and any weak solution \( S^*_{z,x} \) of the SDE (1), and we only discuss the construction of the switching process \( Z^* \), the jumps of which occur at the times composing the sequence \( (T^*_n) \), in what follows.

In Case (a), the sets \( C_c \) and \( C_o \) defined by (74) and (75) in Definition 3 are given by \( C_c = C_o = I \), and the switching process \( Z^* \equiv z \), which is associated with \( T^*_n = \infty \), for all \( n \geq 1 \), is the required one because both of the sets in (78) are empty and the inclusions in (79) are plainly true.

In Case (b), \( C_c = \emptyset \) and \( C_o = I \), and the switching process \( Z^* \) given by

\[
Z^*_t = z 1_{\{0\}}(t) + 1_{[0,\infty)}(t)
\]

is optimal because the sets in (78) contain at most one element, while (79) plainly holds.

In Case (d), \( I \setminus C_c = [b_o, \beta[, I \setminus C_o = \emptyset \), and the switching process \( Z^* \equiv \beta \), which is associated with \( T^*_n = \infty \), for all \( n \geq 1 \), is the required one because both of the sets in (78) are empty and the inclusions in (79) are plainly true.

In Case (b), \( C_c = \emptyset \) and \( C_o = I \), and the switching process \( Z^* \) given by

\[
Z^*_t = z 1_{[0,T^*_1]}(t) + 1_{[T^*_1,\infty)}(t),
\]

where \( T^*_1 = \inf\{t \geq 0 \mid X_t \geq b_o\} \), has all of the required properties.

The constructions that are appropriate for Cases (c) and (e) are mirror images of the constructions associated with Cases (b) and (d) above, respectively.

Finally, in Case (f), \( I \setminus C_c = [b_o, \beta[ \) and \( I \setminus C_o = ]\alpha, a_c] \). If \( z = 1 \), then the switching process \( Z^* \) given by

\[
Z^*_t = 1_{\{0\}}(t) + \sum_{j=0}^{\infty} 1_{[T^*_{2j+1},T^*_{2j}]}(t),
\]

where the \( (\mathcal{F}_t) \)-stopping times \( T^*_n \), \( n \geq 1 \) are defined recursively by

\[
T^*_{2n+1} = \inf\{t \geq T^*_n \mid X_t \leq a_c\}, \quad n = 0, 1, 2, \ldots,
\]
\[
T^*_{2n} = \inf\{t \geq T^*_{2n-1} \mid X_t \geq b_o\}, \quad n = 1, 2, \ldots,
\]

where we have set \( T^*_0 = 0 \), provides an optimal choice because the sets in (78) both are countable, while the inclusions in (79) both hold. If \( z = 0 \), then optimal switching process \( Z^* \) can be constructed in a similar fashion. \( \square \)
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Appendix: proofs of results in Section 6

Proof of Lemma 3. By construction, the function \( w \) given by (88) satisfies the HJB equation (70) if and only if

\[
\begin{align*}
\quad w(1, \cdot) - w(0, \cdot) - g_o & \leq 0 \quad \text{and} \quad w(0, \cdot) - w(1, \cdot) - g_c \leq 0,
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
R_h - g_o & \leq 0 \quad \text{and} \quad -R_h - g_c \leq 0.
\end{align*}
\]

However, a simple inspection of the lists in Section 4 reveals that these inequalities hold true if and only if the problem data is in the corresponding statement of the lemma.

To establish part (II) of the lemma, we note that the function \( w \) defined by (89) satisfies (70) if and only if

\[
\begin{align*}
-L(R_h - g_o) \quad \text{is a positive measure} \quad (117)
\end{align*}
\]

and

\[
\begin{align*}
w(0, \cdot) - w(1, \cdot) - g_c \leq 0. \quad (118)
\end{align*}
\]

Inequality (118) is plainly equivalent to \(-g_o - g_c \leq 0\), which is true by assumption. On the other hand, inequality (117) holds if and only in the problem data is such that \(B_1\) or \(B_3\) is satisfied, which gives rise to the cases in Group O of Table 1. Furthermore, if the measures \(L(R_h + g_c)\) and \(L(R_h - g_o)\) have full-support, then (117) holds if and only if the problem data is such that one of the pairs \(A_2 - B_3\) or \(A_41 - B_3\) occurs.

Finally, we can use symmetric arguments to prove all claims associated with part (III) of the lemma. \(\square\)

Proof of Lemma 4. We have already observed that, beyond the cases considered in Lemma 3, a point \(b_o \in I\) satisfies (94) only if \(B_42\) or \(B_43\) is combined with one of \(A_2\) or \(A_43\), in which cases, the choice \(b_o = b^*\) satisfies (94). Also, a point \(a_c\) satisfies the corresponding free-boundary inequalities only if \(A_42\) or \(A_43\) is combined with one of \(B_2\) or \(B_43\), in which cases \(a_c = a^*\) is an appropriate choice. Furthermore, if the measure \(L(R_h - g_o)\) (resp., \(L(R_h + g_c)\)) has full-support, then all relevant cases in Lemma 3 as well as \(B_42\) (resp., \(A_42\) are out of the picture, so a point \(b_o \in I\) (resp., \(a_c \in I\)) can satisfy the
required free-boundary inequalities if and only if $B_{43}$ (resp., $A_{43}$) holds true and $b_o = b^*$ (resp., $a_o = a^*$).

To proceed further, let us assume that the problem data is such that $B_{42}$ or $B_{43}$ is combined with one of $A_{2}$ or $A_{43}$. By construction, the function $w$ given by (94), which is relevant to this case, will satisfy the HJB equation (70) if and only if

$$\text{supp}[L w(0, \cdot)]^+ \cap [b^*, \beta[ = \emptyset,$$  \tag{119}

$$w(0, x) - w(1, x) - g_c(x) \leq 0, \text{ for all } x \in \mathcal{I},$$  \tag{120}

$$w(1, x) - w(0, x) - g_o(x) \leq 0, \text{ for all } x \in ]\alpha, b^*[.$$  \tag{121}

Noting that the restriction of the measure $L w(0, \cdot)$ in $[b^*, \beta[$ is equal to the restriction of the measure $L(R_h - g_o)$ in $[b^*, \beta[$, we can see that (119) follows immediately from (29) in Assumption 5 and the fact that $b < b^*$. For $x \geq b^*$, (120) is equivalent to $-g_c(x) - g_o(x) \leq 0$, which is true by Assumption 27. Furthermore, we can use the definition (91) of $w$ and the expression for $B$ provided by (92) with $b_o = b^*$, to verify that (120), for $x < b^*$, and (121) are equivalent to

$$\frac{R_h(x) + g_c(x)}{\psi(x)} \geq \frac{R_h(b^*) - g_o(b^*)}{\psi(b^*)} = \frac{R_h(x) - g_o(x)}{\psi(x)}, \text{ for all } x \in ]\alpha, b^*[.$$  \tag{122}

The second of these inequalities follows immediately from the fact that the function $(R_h - g_o)/\psi$ is increasing in $]\alpha, b^*[\text{ (see (59) and (61))}. With a view to the first one, we use (22) to calculate

$$\left(\left(\frac{R_h + g_c}{\psi}\right)\right)'(x) = \frac{W(x)}{\psi^2(x)} \int_{]\alpha, x]} \Psi(s) L(R_h + g_c)(ds).$$

When the problem data are such that case $A_{2}$ in Assumption 5 holds, this calculation implies that the function $(R_h + g_c)/\psi$ is decreasing in $\mathcal{I}$, which combined with the inequality

$$\frac{R_h(b^*) + g_c(b^*)}{\psi(b^*)} \geq \frac{R_h(b^*) - g_o(b^*)}{\psi(b^*)}$$  \tag{123}

which follows from (27) in Assumption 4, we can see that the first of the two inequalities in (122) is satisfied. On the other hand, when the problem data is such that case $A_{4}$ in Assumption 5 holds, this calculation implies that there exists a point $\gamma \in ]\bar{a}, \beta[$ such that the function $(R_h + g_c)/\psi$ is increasing in $]\alpha, \gamma[$, is decreasing in $]\gamma, \beta[. However, this observation and inequality (123) imply that the first of the inequalities in (122) is satisfied if and only if

$$\lim_{x \downarrow \alpha} \frac{R_h(x) + g_c(x)}{\psi(x)} \geq \frac{R_h(b^*) - g_o(b^*)}{\psi(b^*)},$$

i.e., if and only if (64) is not true. In light of (65), we can see that this can be the case only if

Finally, the proof of part (II) of the lemma follows arguments that are symmetric to the ones we have developed above to establish part (I).

\[\square\]
Proof of Lemma 5. We start by assuming that the problem data is such that \( A4 \) and \( B4 \) in Assumption 5 are satisfied. Given any \( v \in \mathcal{I} \), we can see that

\[
q^o_c(u_2, v) - q^o_c(u_1, v) = \int_{[u_1, u_2]} \Phi(s) \mathcal{L}(R_h + g_v)(ds) \begin{cases} 
\geq 0, & \text{if } u_1 < u_2 < \tilde{a}, \\
\leq 0, & \text{if } \tilde{a} < u_1 < u_2,
\end{cases}
\]

the inequalities following because of (29) in Assumption 5. In view of this calculation and a similar one with \( q^o_c \), we can see that, given any \( v \in \mathcal{I} \),

the functions \( u \mapsto q^o_c(u, v) \) and \( u \mapsto q^c_o(u, v) \) are

\[
\begin{align*}
\text{increasing in } & ]\alpha, \tilde{a}[, \\
\text{decreasing in } & ]\tilde{a}, \beta[.
\end{align*}
\]

(124)

In the same way, we can see that, given any \( u \in \mathcal{I} \),

the functions \( v \mapsto q^o_c(u, v) \) and \( v \mapsto q^c_o(u, v) \) are

\[
\begin{align*}
\text{decreasing in } & ]\alpha, \tilde{b}[, \\
\text{increasing in } & ]\tilde{b}, \beta[.
\end{align*}
\]

(125)

given any \( v \in \mathcal{I} \),

the functions \( u \mapsto q^o_c(u, v) \) and \( u \mapsto q^c_o(u, v) \) are

\[
\begin{align*}
\text{decreasing in } & ]\alpha, \tilde{a}[, \\
\text{increasing in } & ]\tilde{a}, \beta[.
\end{align*}
\]

(126)

and, given any \( u \in \mathcal{I} \),

the functions \( v \mapsto q^o_c(u, v) \) and \( v \mapsto q^c_o(u, v) \) are

\[
\begin{align*}
\text{increasing in } & ]\alpha, \tilde{b}[, \\
\text{decreasing in } & ]\tilde{b}, \beta[.
\end{align*}
\]

(127)

We can also see that (28) and (29) in Assumption 5 imply that

\[
q^c_o(u, v) - q^o_c(u, v) = \Psi(u) \mathcal{L}(R_h + g_v)(\{u\}) + \Psi(v) \mathcal{L}(R_h - g_v)(\{v\}) 
\leq 0, \quad \text{for all } u < \tilde{a} \leq \tilde{b} < v,
\]

(128)

and that

\[
q^o_c(u, v) - q^c_o(u, v) \leq 0, \quad \text{for all } u < \tilde{a} \leq \tilde{b} < v.
\]

(129)

We next assume that \( b^* < \beta \), and we observe that (69) implies that

\[
\lim_{u \downarrow \alpha} q^c_o(u, b^*) \leq 0 \leq \lim_{u \downarrow \alpha} q^o_c(u, b^*),
\]

(130)

the first of which inequalities and (126) imply that

\[
q^c_o(u, b^*) \leq 0, \quad \text{for all } u \in ]\alpha, \tilde{a}].
\]

(131)
Also, the definition (116) of $q^c_\psi$ and (31) in Assumption 5 imply that

$$q^c_\psi(u, v) = -\int_{[a,u]} \Psi(s) \mathcal{L}(g_c + g_o)(ds) + \int_{[u,v]} \Psi(s) \mathcal{L}(R_h - g_o)(ds) > 0, \quad \text{for all } u < \tilde{a} \text{ and } v \in ]u, b[. \tag{132}$$

Now, (131), (132), (127) and the right-continuity of $v \mapsto q^c_\psi(u, v)$ imply that, given any $u \in ]\alpha, \tilde{a}[,$ there exists a point $v_{u,c} \in [\tilde{b}, b^*]$ such that

$$q^c_\psi(u, v) \begin{cases} > 0, & \text{for all } v \in [\tilde{b}, v_{u,c}], \\ \leq 0, & \text{for all } v \in [v_{u,c}, b^*]. \end{cases} \tag{133}$$

On the other hand, (132), (128) and the left-continuity of $v \mapsto q^c_\psi(u, v)$ imply that, given any $u \in ]\alpha, \tilde{a}[,$ there exists a point $v_{u,c} \in [v_{u,c}, b^*]$ such that

$$q^c_\psi(u, v) \begin{cases} \geq 0, & \text{for all } v \in [\tilde{b}, v_{u,c}], \\ < 0, & \text{for all } v \in [v_{u,c}, b^*]. \end{cases} \tag{134}$$

Combining this result with (133), we can see that, given any $u \in ]\alpha, \tilde{a}[,$ $q^c_\psi(u, v) \leq 0 \leq q^c_\psi(u, v)$, for all $v \in [v_{\psi,c}, v_{u,c}].$ In particular, if we set $l_\psi(u) = v_{u,c},$ then we obtain a function $l_\psi : [\alpha, \tilde{a}] \to [\tilde{b}, b^*]$ such that

$$q^c_\psi(u, l_\psi(u)) \leq 0 \leq q^c_\psi(u, l_\psi(u)), \quad \text{for all } u \in ]\alpha, \tilde{a}[. \tag{135}$$

Furthermore,

$$l_\psi \text{ is left-continuous and decreasing, and } \lim_{u \uparrow \alpha} l_\psi(u) = b^*. \tag{136}$$

Indeed, the left-continuity of $l_\psi$ follows from (134), the definition $l_\psi(u) = v_{u,c}$ and the left-continuity of $v \mapsto q^c_\psi(u, v).$ Also, the fact that $l_\psi$ is decreasing follows from an inspection of (126) and (127), while the limit in (136) is a simple consequence of (130).

In general, the function $l_\psi$ can have jumps as well as intervals of constancy. A careful consideration of the definitions (115), (116) of $q^c_\psi,$ $q^c_\psi,$ and the arguments above also reveal that

if $\mathcal{L}(R_h + g_c),$ $\mathcal{L}(R_h - g_o)$ have full support, then $l_\psi$ is strictly decreasing,

and that

if $\mathcal{L}(R_h + g_c),$ $\mathcal{L}(R_h - g_o)$ are non-atomic, then $l_\psi$ is continuous,

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□

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References


