Productive Corecursion in Logic Programming

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Outline

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   - Overview of motivation
   - Background knowledge for understanding motivation
   - Problem description

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   - Loop detection rule review
   - Productivity guarantee

3. Conclusion

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1 Motivation
   • Overview of motivation
   • Background knowledge for understanding motivation
   • Problem description

2 Productive Corecursion
   • Loop detection rule review
   • Productivity guarantee

3 Conclusion

4 Future Work & Implementation
non-terminating SLD derivation

(Jaffar & Stuckey 86)

greatest fixed point semantics
non-terminating SLD derivation

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greatest fixed point semantics

(van Emden & Abdallah 85)

productivity
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CoLP

implementation

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✓: sound

✓: sound

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✓: sound
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✓: sound

✗: not sound
Definition (Syntax of definite clause logic) (Lloyd 87)
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Term: = Constant | Variable | Functor \(<\text{List of Terms}>\)
Definition (Syntax of definite clause logic) (Lloyd 87)

Term := Constant | Variable | Functor (<List of Terms>)
Definite clause := Term ← Set of Terms
Definition (Syntax of definite clause logic) (Lloyd 87)

Term := Constant | Variable | Functor (<List of Terms>)
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Definition (Syntax of definite clause logic) (Lloyd 87)

Term: := Constant | Variable | Functor (<List of Terms>)
Definite clause: := Term ← Set of Terms
Goal clause: := List of Terms
Program: := Set of Definite clauses
Definition (Fixed point semantics) (Lloyd 87)

Given a logic program,
Definition (Fixed point semantics) (Lloyd 87)

Given a logic program,

the least fixed point is the smallest set closed forward under the program.
**Definition (Fixed point semantics) (Lloyd 87)**

Given a logic program,

- the **least fixed point** is the smallest set closed forward under the program.
- the **greatest fixed point** is the largest set closed backward under the program.
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Given a logic program,

the least fixed point is the smallest set closed forward under the program.

the greatest fixed point is the largest set closed backward under the program.

**Example**

```prolog
nat(0)  
nat(s(X)) ← nat(X)
```
Definition (Fixed point semantics) (Lloyd 87)

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Given a logic program,

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Example

nat(0)
nat(s(X)) ← nat(X)

The least fixed point is \{nat(0), nat(s(0)), nat(s(s(0))), \ldots \}.
Definition (Fixed point semantics) (Lloyd 87)

Given a logic program,

the least fixed point is the smallest set closed forward under the program.

the greatest fixed point is the largest set closed backward under the program.

Example

\[
\begin{align*}
nat(0) \\
nat(s(X)) & \leftarrow nat(X)
\end{align*}
\]

The least fixed point is \(\{nat(0), nat(s(0)), nat(s(s(0))), \ldots\}\).

The greatest fixed point is \(\{nat(0), nat(s(0)), nat(s(s(0))), \ldots\} \cup \{nat(s(s(\ldots))))\}\).
**Definition (Fixed point semantics) (Lloyd 87)**

Given a logic program,

- **the least fixed point** is the smallest set closed forward under the program.
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**Example**

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nat(s(X)) ← nat(X)

The least fixed point is \{nat(0), nat(s(0)), nat(s(s(0))), \ldots \}.

The greatest fixed point is \{nat(0), nat(s(0)), nat(s(s(0))), \ldots \} \cup \{nat(s(s(\ldots ))))\}.

Formulae computed by non-terminating derivations are in greatest fixed points. (Jaffar & Stuckey 86; van Emden & Abdallah 85)
Definition (Productivity)
(LP: van Emden & Abdallah 86; Komendantskaya et al. 16; FP: Sijtsma 89; Endrullis et al. 08)

A productive non-terminating derivation does useful computations while looping rather than just looping.

Example
\[ \text{nat}(X) \leftarrow \text{nat}(X) \]

has non-productive derivation:
\[ \text{nat}(X) \leftarrow \text{nat}(X) \leftarrow \text{nat}(X) \leftarrow \ldots \]

Example
\[ \text{nat}(s(X)) \leftarrow \text{nat}(X) \]

computes the first limit ordinal \( \text{nat}(s(s(\ldots ))) \):
\[ \begin{align*}
\text{nat}(X) & \leftarrow s(\text{nat}(X)) \\
\text{nat}(X) & \leftarrow s(s(\text{nat}(X))) \\
\& \ldots
\end{align*} \]

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Example

\[ \text{nat}(X) \leftarrow \text{nat}(X) \]

has non-productive derivation:

\[ \begin{array}{c}
\text{nat}(X) \\
\downarrow \\
\text{nat}(X) \\
\downarrow \\
\text{nat}(X) \\
\vdots 
\end{array} \]
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\vdots
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nat(X) \leftarrow nat(X)
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\begin{array}{c}
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\downarrow \\
nat(X) \\
\downarrow \\
nat(X) \\
\vdots
\end{array}
\]

Example

\[
nat(s(X)) \leftarrow nat(X)
\]

computes the first limit ordinal
\[
nat(s(s(\ldots)))
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nat(X) ← nat(X)
has non-productive derivation:

\[
\begin{array}{l}
nat(X) \\
\downarrow \\
nat(X) \\
\downarrow \\
nat(X) \\
\vdots
\end{array}
\]

Example

nat(s(X)) ← nat(X)
computes the first limit ordinal

\[
\begin{array}{l}
nat(X) \\
\downarrow \\
x \mapsto s(x_2) \\
nat(x_2) \\
\downarrow \\
x_2 \mapsto s(x_3) \\
nat(x_3) \\
\vdots
\end{array}
\]
Now consider finite implementation of non-terminating SLD derivations.
Since regular formulae have cyclic derivations, finding a cycle (loop) is suffice for knowing the whole derivation.

Definition (Gupta et al. 07)
CoLP = SLD resolution + loop detection rule.

Definition (Loop detection rule) (Gupta et al. 07)
A goal succeeds if it unifies with its ancestor goal.

Theorem (Coinductive soundness of coLP) (Gupta et al. 07)
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Successful coLP derivations only compute formulae in greatest fixed points.
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Successful coLP derivations only compute formulae in greatest fixed points.
Example (CoLP at work)

nat(s(X))

We compare coLP and SLD derivation for goal nat(X).

SLD derivation (non-terminating)

\[
\begin{align*}
G_0 & : \text{nat}(X) \\
G_1 & : \text{nat}(X^2) \\
& \vdots \\
G_2 & : X^2 \mapsto \rightarrow s(X^2) \\
& \rightarrow s(X^3) \\
& \ldots
\end{align*}
\]

CoLP derivation (terminating)

\[
\begin{align*}
G_0 & : \text{nat}(X) \\
G_1 & : \Box \\
G_2 & : X^2 \mapsto \rightarrow s(X^2) \\
& \rightarrow s(X^3) \\
& \ldots
\end{align*}
\]

SLD derivation computes \( s(s(\ldots)) \) by accumulating:

\[
X_2 \mapsto \rightarrow s(X^2), \quad X^2 \mapsto \rightarrow s(X^3), \ldots
\]

CoLP derivation computes \( s(s(\ldots)) \) by circular binding:

\[
X^2 \mapsto \rightarrow s(X^2)
\]
Motivation
Background knowledge for understanding motivation

Example (CoLP at work)

\[ \text{nat}(s(X)) \leftarrow \text{nat}(X) \text{ defines the first limit ordinal } s(s(\ldots)). \]
Example (CoLP at work)

nat(s(X)) ← nat(X) defines the first limit ordinal s(s(...)). We compare CoLP and SLD derivation for goal nat(X).
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SLD derivation
(non-terminating)
Example (CoLP at work)

\[
\text{nat}(s(X)) \leftarrow \text{nat}(X) \text{ defines the first limit ordinal } s(s(\ldots)). \text{ We compare coLP and SLD derivation for goal } \text{nat}(X).
\]

SLD derivation  
(non-terminating)

\[
\begin{align*}
G_0 & \quad \text{nat}(X) \\
\text{↓} & \quad X \mapsto s(X_2) \\
G_1 & \quad \text{nat}(X_2) \\
\text{↓} & \quad X_2 \mapsto s(X_3) \\
G_2 & \quad \text{nat}(X_3) \\
\vdots & \\
\end{align*}
\]

CoLP derivation  
(terminating)

\[
\begin{align*}
G_0 & \quad \text{nat}(X) \\
\vdots & \quad \text{nat}(X) \\
\end{align*}
\]

SLD derivation computes \(s(s(\ldots))\) by accumulating \(X \mapsto s(X_2), X_2 \mapsto s(X_3), \ldots\)

CoLP derivation computes \(s(s(\ldots))\) by circular binding \(X \mapsto s(X)\)
Example (CoLP at work)

\( \text{nat}(s(X)) \leftarrow \text{nat}(X) \) defines the first limit ordinal \( s(s(\ldots)) \). We compare CoLP and SLD derivation for goal \( \text{nat}(X) \).

**SLD derivation (non-terminating)**

\[
\begin{align*}
G_0 & : \text{nat}(X) \\
G_1 & : \text{nat}(X_2) \\
G_2 & \vdots
\end{align*}
\]

\( X \mapsto \leftarrow s(X_2) \)

\( X_2 \mapsto \leftarrow s(X_3) \)

**CoLP derivation (terminating)**

\[
\begin{align*}
\square & \quad G_1 \quad \text{(unifies)} \\
G_0 & : \text{nat}(X) \\
G_1 & : \text{nat}(X_2) \\
G_2 & \vdots
\end{align*}
\]

SLD derivation computes \( s(s(\ldots)) \) by accumulating \( X \mapsto \leftarrow s(X_2), X_2 \mapsto \leftarrow s(X_3), \ldots \).

CoLP derivation computes \( s(s(\ldots)) \) by circular binding \( X \mapsto \leftarrow s(X) \).
Example (CoLP at work)

\[ \text{nat}(s(X)) \leftarrow \text{nat}(X) \] defines the first limit ordinal \( s(s(\ldots)) \). We compare coLP and SLD derivation for goal \( \text{nat}(X) \).

**SLD derivation** (non-terminating)

\[
\begin{align*}
G_0 & : \text{nat}(X) \\
& \downarrow X \mapsto s(X_2) \\
G_1 & : \text{nat}(X_2) \\
& \downarrow X_2 \mapsto s(X_3) \\
G_2 & : \text{nat}(X_3) \\
& \cdots
\end{align*}
\]

**CoLP derivation** (terminating)

\[
\begin{align*}
G_0 & : \text{nat}(X) \\
& \downarrow X \mapsto s(X_2) \\
G_1 & : \text{nat}(X_2) \\
& \downarrow X_2 \mapsto X \ (G_1 \text{ unifies } G_0) \\
G_2 & : \square
\end{align*}
\]
nat(s(X)) ← nat(X) defines the first limit ordinal s(s(…)). We compare coLP and SLD derivation for goal nat(X).

SLD derivation (non-terminating)

\[
\begin{align*}
G_0 & \quad \text{nat}(X) \\
\downarrow & \quad X \mapsto s(X_2) \\
G_1 & \quad \text{nat}(X_2) \\
\downarrow & \quad X_2 \mapsto s(X_3) \\
G_2 & \quad \text{nat}(X_3) \\
\vdots
\end{align*}
\]

SLD derivation computes s(s(…)) by accumulating

CoLP derivation (terminating)

\[
\begin{align*}
G_0 & \quad \text{nat}(X) \\
\downarrow & \quad X \mapsto s(X_2) \\
G_1 & \quad \text{nat}(X_2) \\
\downarrow & \quad X_2 \mapsto X \ (G_1 \text{ unifies } G_0) \\
G_2 & \quad \square
\end{align*}
\]
Example (CoLP at work)

nat(s(X)) ← nat(X) defines the first limit ordinal s(s(…)). We compare coLP and SLD derivation for goal nat(X).

**SLD derivation (non-terminating)**

\[G_0 \quad \text{nat}(X)\]
\[\downarrow \quad X \mapsto s(X_2)\]
\[G_1 \quad \text{nat}(X_2)\]
\[\downarrow \quad X_2 \mapsto s(X_3)\]
\[\vdots\]
\[G_2 \quad \text{nat}(X_3)\]

**CoLP derivation (terminating)**

\[G_0 \quad \text{nat}(X)\]
\[\downarrow \quad X \mapsto s(X_2)\]
\[G_1 \quad \text{nat}(X_2)\]
\[\downarrow \quad X_2 \mapsto X \quad (G_1 \text{ unifies } G_0)\]
\[G_2 \quad \square\]

SLD derivation computes s(s(…)) by accumulating \(X \mapsto s(X_2)\),
Example (CoLP at work)

nat(s(X)) ← nat(X) defines the first limit ordinal s(s(…)). We compare coLP and SLD derivation for goal nat(X).

SLD derivation
(non-terminating)

\[ G_0 \quad \text{nat}(X) \]
\[ \downarrow \quad X \mapsto s(X_2) \]
\[ G_1 \quad \text{nat}(X_2) \]
\[ \downarrow \quad X_2 \mapsto s(X_3) \]
\[ \text{nat}(X_3) \]
\[ \vdots \]

CoLP derivation
(terminating)

\[ G_0 \quad \text{nat}(X) \]
\[ \downarrow \quad X \mapsto s(X_2) \]
\[ G_1 \quad \text{nat}(X_2) \]
\[ \downarrow \quad X_2 \mapsto X \quad (G_1 \text{ unifies } G_0) \]
\[ G_2 \quad \square \]

SLD derivation computes \( s(s(\ldots)) \) by accumulating \( X \mapsto s(X_2), X_2 \mapsto s(X_3), \ldots \)
Example (CoLP at work)

\( \text{nat}(s(X)) \leftarrow \text{nat}(X) \) defines the first limit ordinal \( s(s(\ldots)) \). We compare coLP and SLD derivation for goal \( \text{nat}(X) \).

**SLD derivation**  
(non-terminating)

\[
\begin{align*}
G_0 & \quad \text{nat}(X) \\
& \quad \downarrow \quad X \mapsto s(X_2) \\
G_1 & \quad \text{nat}(X_2) \\
& \quad \downarrow \quad X_2 \mapsto s(X_3) \\
G_2 & \quad \text{nat}(X_3) \\
& \vdots
\end{align*}
\]

SLD derivation computes \( s(s(\ldots)) \) by accumulating \( X \mapsto s(X_2) \), \( X_2 \mapsto s(X_3) \), \ldots \CoLP derivation computes \( s(s(\ldots)) \) by circular binding

**CoLP derivation**  
(terminating)

\[
\begin{align*}
G_0 & \quad \text{nat}(X) \\
& \quad \downarrow \quad X \mapsto s(X_2) \\
G_1 & \quad \text{nat}(X_2) \\
& \quad \downarrow \quad X_2 \mapsto X \quad (G_1 \text{ unifies } G_0) \\
G_2 & \quad \Box
\end{align*}
\]

SLD derivation computes \( s(s(\ldots)) \) by accumulating \( X \mapsto s(X_2) \), \( X_2 \mapsto s(X_3) \), \ldots CoLP derivation computes \( s(s(\ldots)) \) by circular binding
Example (CoLP at work)

nat(s(X)) ← nat(X) defines the first limit ordinal s(s(\ldots)). We compare CoLP and SLD derivation for goal nat(X).

SLD derivation (non-terminating)

\[
\begin{align*}
G_0 \quad & \text{nat}(X) \\
\downarrow & \quad X \mapsto s(X_2) \\
G_1 \quad & \text{nat}(X_2) \\
\downarrow & \quad X_2 \mapsto s(X_3) \\
\quad & \text{nat}(X_3) \\
G_2 & \quad \ldots
\end{align*}
\]

SLD derivation computes s(s(\ldots)) by accumulating $X \mapsto s(X_2)$, $X_2 \mapsto s(X_3)$, \ldots.

CoLP derivation (terminating)

\[
\begin{align*}
G_0 \quad & \text{nat}(X) \\
\downarrow & \quad X \mapsto s(X_2) \\
G_1 \quad & \text{nat}(X_2) \\
\downarrow & \quad X_2 \mapsto X \quad (G_1 \text{ unifies } G_0) \\
G_2 & \quad \square
\end{align*}
\]

CoLP derivation computes s(s(\ldots)) by circular binding $X \mapsto s(X)$.
However, coLP does not take good care of productivity . . .
Assume some successful coLP derivation that computes an infinite formula.

**Problem 1:** *It is not guaranteed that there exists a corresponding non-terminating SLD derivation.*

e.g. For program $p(f(X),X) \leftarrow p(X,X)$ and goal $p(f(X),X)$, coLP computes $p(f(f(...)),f(f(...)))$ but here is no non-terminating SLD derivation.

**CoLP derivation**

\[
\begin{align*}
G_0 : & \quad p(f(X),X) \\
\downarrow & \\
G_1 : & \quad p(X,X) \\
\downarrow & \quad X \mapsto f(f(...)) \text{ by unifying } G_1 \text{ with } G_0 \\
G_2 : & \quad \Box
\end{align*}
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CoLP derivation

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\begin{align*}
G_0 : & \quad p(f(X),X) \\
\downarrow & \\
G_1 : & \quad p(X,X) \\
\downarrow & \quad X \mapsto f(f(\ldots)) \text{ by unifying } G_1 \text{ with } G_0 \\
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**CoLP derivation**

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\begin{align*}
G_0 : & \quad p(f(X),X) \\
\downarrow \\
G_1 : & \quad p(X,X) \\
\downarrow & \quad X \mapsto f(f(\ldots)) \text{ by unifying } G_1 \text{ with } G_0 \\
G_2 : & \quad \square
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CoLP derivation

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\begin{align*}
G_0 & : \ p(f(X),X) \\
\downarrow \\
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e.g. For program \( p(f(X),X) \leftarrow p(X,X) \) and goal \( p(f(X),X) \), coLP computes \( p(f(f(\ldots)),f(f(\ldots))) \) but here is no non-terminating SLD derivation.

\[
\begin{align*}
\text{CoLP derivation} \\
G_0 : & \quad p(f(X),X) \\
\downarrow \\
G_1 : & \quad p(X,X) \\
\downarrow & \quad X \mapsto f(f(\ldots)) \text{ by unifying } G_1 \text{ with } G_0 \\
G_2 : & \quad \square
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However, coLP does not take good care of productivity . . .
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e.g. For program \( p(f(X),X) \leftarrow p(X,X) \) and goal \( p(f(X),X) \), coLP computes \( p(f(f(\ldots)),f(f(\ldots))) \) but here is no non-terminating SLD derivation.

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Assume some successful coLP derivation that computes an infinite formula. **Problem 2:** *There exists a corresponding non-terminating SLD derivation but it computes a different formula.*

e.g. For program \( q(f(X),Y) \leftarrow q(X,h(Y)) \) and goal \( q(f(X),Y) \), coLP computes \( q(f(f(\ldots)),h(h(\ldots))) \) but the corresponding non-terminating SLD derivation computes \( q(f(f(\ldots)), Y) \).

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\downarrow & X \mapsto f(f(\ldots)), Y \mapsto h(h(\ldots)) \text{ by unifying } G_1 \text{ with } G_0 \\
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Our question

Can we have an implementation of infinite SLD derivation, that is not only sound regarding greatest fixed points, but also sound regarding productivity?

Our answer is affirmative.
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1 Motivation
- Overview of motivation
- Background knowledge for understanding motivation
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2 Productive Corecursion
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3 Conclusion

4 Future Work & Implementation
If two formulae unify \textit{without} occurs check and produce an infinite regular formula,
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If two formulae unify \textit{without} occurs check and produce an infinite regular formula, we should standardize them apart, then try to unify the standardized version \textit{with} occurs check. The following are possible results.

\begin{itemize}
  \item \textbf{Case 1} They do not unify. e.g. \texttt{p(f(X),X)} and \texttt{p(X,X)}; \texttt{p(f(X),X)} and \texttt{p(X,1)}.
  \item \textbf{Case 2} One is a variant or instance of the other. e.g. \texttt{nat(X)} and \texttt{nat(s(X))}; \texttt{nat(X)} and \texttt{nat(s(X))}.
  \item \textbf{Case 3} Otherwise. e.g. \texttt{q(f(X),Y)} and \texttt{q(X,h(Y))}; \texttt{q(f(X),Y)} and \texttt{q(X,1)}.
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To guarantee soundness regarding productivity, we propose using the following loop detection rule:
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**Definition (Our loop detection rule)**

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instead of

**Definition (Loop detection rule) (Gupta et al. 07)**

A goal succeeds if it unifies with its ancestor goal.
We also characterized a class of logic programs whose non-terminating SLD derivations, if any, are guaranteed to be productive.
Definition (Rewriting for LP) (Komendantskaya et al. 15)

Rewriting is a special case of SLD resolution, where the selected subgoal is an instance of the chosen program clause's head.
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Example (Rewriting)

- From goal nat(X) and clause nat(X) ← nat(X), derive nat(X).
- From goal nat(s(X)) and clause nat(Y) ← nat(s(Y)), derive nat(s(s(Y)).
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A logic program that is
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Theorem (our main result: Productivity Semi-decision)
Productivity is semi-decidable for programs characterized above, by SLD resolution combined with our loop detection rule.
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Conclusion

non-terminating SLD derivation

implementation

CoLP
(Gupta et al. 07)

implementation

Our Work

(greatest fixed point semantics)
(Jaffar & Stuckey 86)
(van Emden & Abdallah 85)

(productivity)

✓: sound
✗: not sound
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Streams of natural numbers, e.g. 3 1 4 1 5 9 2 . . . ,
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Example (number streams) (Gupta et al. 07)
Streams of natural numbers, e.g. 3 1 4 1 5 9 2 . . . , are defined by the corecursive clause \( \text{nats}([X|S]) \leftarrow \text{nat}(X), \text{nats}(S) \).
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**Example (increasing stream) (Simon et al. 06)**
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Streams of natural numbers, e.g. 3 1 4 1 5 9 2 . . . , are defined by the corecursive clause
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fibs(X, Y, [X|S]) \leftarrow \text{add}(X, Y, Z), \ fibs(Y, Z, S).
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- Programs that contain existential variables?
- Practical application in
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Implementation is available at
GitHub / coalp / Productive-Corecursion
Thanks!