An alternative derivation of the Vasicek model [1]

Andrew J.G. Cairns
Department of Actuarial Mathematics and Statistics,
Heriot-Watt University,
Edinburgh, EH14 4AS, United Kingdom

E-mail: A.Cairns@ma.hw.ac.uk
WWW: http://www.ma.hw.ac.uk/~andrewc/

Abstract
This note gives a brief description of how the Vasicek model can be derived from the potential approach described by Rogers (1997), Rutkowski (1997) and Flesaker & Hughston (1996). In particular, a relaxation of the supermartingale assumption required in these papers for positive interest allows us to derive the Vasicek (1977) model.

Keywords: Forward measure

[1] Technical Note 99/13

We follow the formulation of Rutkowski (1997).

Suppose that $A_t = \exp(\sigma X(t) - \mu t)$ where $X(t)$ is an Ornstein-Uhlenbeck process under the forward measure P_∞: that is, $dX(t) = -\alpha X(t)dt + d\hat{Z}(t)$ where $\hat{Z}(t)$ is a Brownian motion under P_∞. Hence $X(T)|\mathcal{F}_t \sim N\left(\exp\left(-\alpha(T-t)\right)\ ,\ \left(1-\exp\left(-2\alpha(T-t)\right)\right)/2\alpha\right)$.

Following Rutkowski (1997) (but relaxing the conditions that A_t be a supermartingale) we define

$$
\begin{aligned}
P(t,T) &= \frac{E_{P_\infty}[A_T \mid \mathcal{F}_t]}{A_t} \\
&= \exp\left[-\mu(T-t) + \frac{\sigma^2}{4\alpha}\left(1-e^{-2\alpha(T-t)}\right) - \sigma\left(1-e^{-\alpha(T-t)}\right)X(t)\right] \\
\Rightarrow f(t,T) &= -\frac{\partial}{\partial T}\log P(t,T) \\
&= \mu + \alpha\sigma e^{-\alpha(T-t)}X(t) - \frac{\sigma^2}{2}e^{-2\alpha(T-t)} \\
\Rightarrow r(t) &= \mu + \alpha\sigma X(t) - \frac{\sigma^2}{2}
\end{aligned}
$$

With a few lines of algebra we can also see that

$$
\begin{aligned}
dP(t,T) &= P(t,T)\Bigg[r(t)dt + \frac{\sigma^2}{2}\left(1-e^{-2\alpha(T-t)}\right)dt - \sigma\left(1-e^{-\alpha(T-t)}\right)d\hat{Z}(t) \\
&\qquad\qquad + \frac{\sigma^2}{2}\left(1-e^{-\alpha(T-t)}\right)^2 dt\Bigg] \\
&= P(t,T)\left[r(t)dt - \sigma\left(1-e^{-\alpha(T-t)}\right)d\tilde{Z}(t)\right] \\
\text{where } d\tilde{Z}(t) &= d\hat{Z}(t) - \sigma dt
\end{aligned}
$$

With an appropriate change from the measure P_∞ to what we will call the risk-neutral measure Q, $\tilde{Z}(t)$ is a Brownian motion under Q.

Since we have $r(t) = \mu + \alpha\sigma X(t) - \frac{\sigma^2}{2}$ we thus find that

$$
\begin{aligned}
dr(t) &= -\alpha(r(t) - \tilde{\mu})dt + \tilde{\sigma}d\tilde{Z}(t) \\
\text{where } \tilde{\mu} &= \mu + \frac{1}{2}\sigma^2 \\
\tilde{\sigma} &= \alpha\sigma
\end{aligned}
$$

$$\begin{aligned} \text{and} \quad dP(t,T) &= P(t,T)\left[r(t)dt + S(t,T)d\tilde{Z}(t)\right] \\ \text{where} \quad S(t,T) &= -\frac{\tilde{\sigma}}{\alpha}\left(1 - e^{-\alpha(T-t)}\right) \end{aligned}$$

References

Flesaker, B., and Hughston, L.P. (1996) "Positive interest," Risk 9(1), 46-49.

Langetieg, T. (1980) "A multivariate model of the term structure," Journal of Finance 35, 71-97.

Rogers, L.C.G. (1997) "The potential approach to the term-structure of interest rates and foreign exchange rates," Mathematical Finance 7, 157-164.

Rutkowski, M. (1997) "A note on the Flesaker & Hughston model of the term structure of interest rates," Applied Mathematical Finance 4, 151-163.

Vasicek, O. (1977) "An equilibrium characterisation of the term structure," Journal of Financial Economics 5, 177-188.