A multifactor model for the term structure and inflation for long-term risk management with an extension to the equities market

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Abstract

We propose a structure for modelling simultaneously both nominal and index-linked bond prices and consumer price inflation with a view to application in the management of long-term interest-rate risk. The model exploits the framework developed by Flesaker & Hughston (1996) which provides a straightforward means of ensuring that nominal rates of interest remain positive.

Price inflation is driven by the difference between short nominal and index-linked interest rates.

Equities are modelled by considering first total returns. These are taken to equal the risk-free rate of interest plus a risk premium plus a volitility term. The risk premium, dividend yields and dividend growth are all modelled on a basis which is consistent with this starting point and with one another. The model takes into account various interactions between these variables and with other economic variables in the term-structure model.

The structure of the model allows us to include some factors which ensure realistic modelling of short-term dynamics while other factors have a much longer-term impact. Thus we are able to model much more effectively the long-term cycles in interest rates which we have observed, for example, in the UK over the last 100 years or so.

Keywords: multifactor; positive interest; Ornstein-Uhlenbeck; time-homogeneous; nominal rates; real rates; inflation.

Note on the current version of this technical note:

An earlier version of Sections 1 to 5 have appeared previously as Cairns (1999) Proceedings of the 9th AFIR Colloquium, Tokyo, Volume 3 pages 93-113.

Section 6 is presently in *draft* form and is currently unpublished. *This section of the paper may not be quoted without the explicit permission of the author.*

1 Introduction

This paper describes the development of a stochastic model for the combined term structure of:

- rates of interest on fixed-interest bonds;
- rates of interest on index-linked bonds;
- consumer price inflation.

The aim underlying the development of this model is to produce a structure which is arbitrage free and which produces both reasonable short and long-term dynamics. In particular, the model was developed for use as a tool in long-term risk management problems (for example, the model would be appropriate for the investigation of problems arising in life insurance or pensions) while allowing dynamic portfolio management over shorter time scales including the use of bond derivatives. With a few exceptions (for example, Tice & Webber, 1997) arbitrage-free term-structure models have tended to be driven by short-term risk management problems (for example, the pricing of bond derivatives).

Much of the detailed development in the paper (that is, in Section 2) concentrates on the fixed-interest model. This exploits the framework developed in recent years by Flesaker & Hughston (1996) (FH). (The FH approach was subsequently generalised by Rutkowski, 1997, and Rogers, 1997.) This approach is similar to that of Heath, Jarrow & Morton (1992) (HJM) in the sense that they describe a general framework for the development of arbitrage-free term-structure models. Both frameworks work with more readily observable quantities such as forward rates and bond prices rather than the risk-free rate of interest. The FH framework, in contrast to HJM, provides a relatively simple means of ensuring that nominal interest rates always remain positive. Here we use the FH framework to develop a time-homogeneous model which is driven by a multifactor Ornstein-Uhlenbeck (OU) process. By using appropriate parameter values, one or more of the OU factors can be set up to create long-term cycles in the term structure mimicking observed behaviour in the UK and elsewhere over the last 100 years. Without this feature it becomes very difficult to model, in an arbitrage-free framework, the larger fluctuations we have observed in, for example, long-term par yields.

In Section 3 we describe a two-factor model for real rates of interest as might be inferred from index-linked bond prices. This model is a generalisation of the Vasicek (1977) model (that is, a special case of the model proposed by Langetieg, 1980). A model for consumer price inflation is developed in which the rate of price inflation is equal to the difference between nominal and real short-term rates of interest adjusted for an inflation risk premium (to reflect a market preference for index-linked assets) and then subject to a zero mean error.

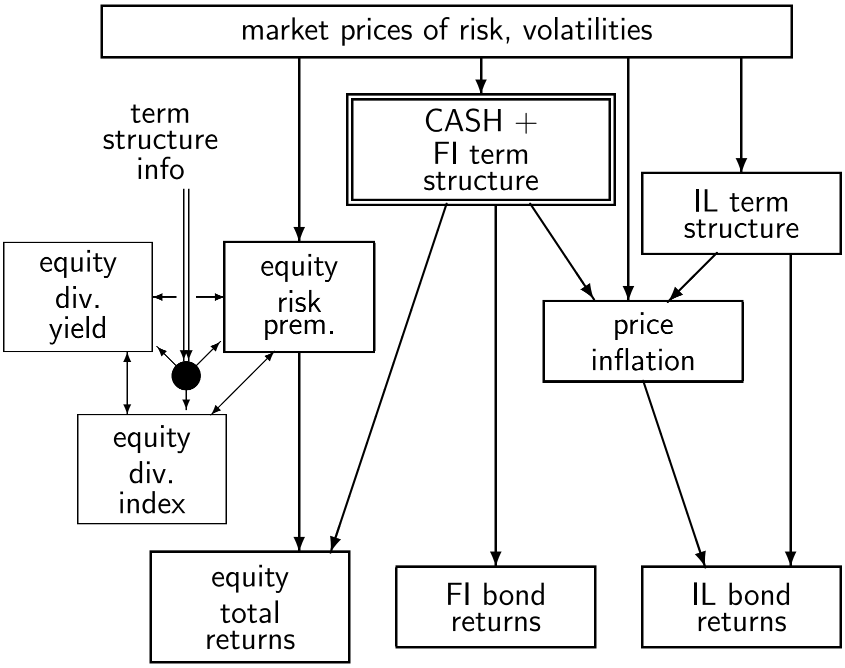
In Section 4 we discuss specific characteristics of the model and, in particular, justify why this model might be preferred to other multi-asset-class models used for long-term risk management.

The paper finishes with a discussion of the calibration of the fixed-interest part of the model.

1.1 A note on notation

For $i = 1, 2, 3$ let $Z_i(t)$ be n_i dimensional Brownian motion under the real-world measure P. Corresponding processes under the equivalent risk-neutral measure (Q) and terminal measure (P_∞) (where this is required) are denoted by $\tilde{Z}_i(t)$ and $\hat{Z}_i(t)$ respectively. The filtration generated by $Z_i(t)$ is denoted by $\mathcal{F}_t^i$. The filtration generated by $\{Z_1(t), \ldots, Z_i(t)\}$ is denoted by $\mathcal{F}_t^{(i)}$. $\tilde{Z}_i(t)$ and $Z_i(t)$ are linked by the market price of risk $\lambda_i(t)$:

$$d\tilde{Z}_i(t) = dZ_i(t) + \lambda_i(t)dt$$


market prices of risk, volatilities
term structure info
CASH + FI term structure
IL term structure
equity div. yield
equity risk prem.
price inflation
equity div. index
equity total returns
FI bond returns
IL bond returns

2 A model for fixed-interest bond prices

2.1 Background

In this section we propose a model which makes use of the framework developed by Flesaker & Hughston (1996), Rutkowski (1997) and Rogers (1997). Thus zero-coupon bond prices $P(t, T)$ are driven by the following family of stochastic processes.

Let $M(t, s)$ for $0 \leq t \leq s < \infty$ be a family of strictly positive stochastic processes over the index s which are martingales with respect to t under some probability measure P_∞. That is, given s, for $t < u < s$, $E_{P_\infty}[M(u, s)|\mathcal{F}_t] = M(t, s)$. Furthermore, we define $M(0, s) = 1$ for all s and assume that for each s, $M(t, s)$ is a diffusion process adapted to a finite (say n_1) dimensional Brownian motion, $\hat{Z}_1(t)$ (under P_∞).

Zero-coupon bond prices are defined as

$$P(t, T) = \frac{\int_T^\infty M(t, s)\phi(s)ds}{\int_t^\infty M(t, s)\phi(s)ds} \qquad (1)$$

A more general form of this framework was proposed by Rutkowski (1997) and Rogers (1997). Rutkowski (1997) defines

$$P(t, T) = \frac{E_{P_\infty}[A_T \mid \mathcal{F}_t]}{A_t}$$

where A_t is a strictly-positive supermartingale under the measure P_∞. The Flesaker & Hughston (1996) form which we will use in this paper is a special case of this framework, where $A_t = \int_t^\infty M(t, s)\phi(s)ds$, $M(t, s)$ is a positive martingale under P_∞, and $\phi(s) > 0$ for all s. Rutkowski (1997) and Rogers (1997) demonstrate that models of this general type are arbitrage free.

In this section we will make use of the FH formulation (Equation 1).

Since $M(0, s) = 1$ for all s we may infer that

$$\phi(s) = \frac{\partial}{\partial s} P(0, s)$$

up to a constant, non-zero scaling factor.

Instantaneous forward rates are then

$$\begin{aligned} f(t, T) &= -\frac{\partial}{\partial T} \log P(t, T) = \frac{M(t, T)\phi(T)}{\int_T^\infty M(t, s)\phi(s)ds} \\ \Rightarrow r(t) &= f(t, t) = \frac{M(t, t)\phi(t)}{\int_t^\infty M(t, s)\phi(s)ds} \end{aligned}$$

Although we can write down an expression for the short rate, $r(t)$, in this way it is not possible, in general, to express the dynamics of $r(t)$ in any simple fashion (for example, like we can with the Vasicek, 1977, model).

We can also write down expressions for bond volatilities which enables us to link the model into the framework of Heath, Jarrow & Morton (1992). Since $M(t,T)$ is a martingale under P_∞ for each T, we can write $dM(t,T) = M(t,T)\sigma_1^T(t,T)d\hat{Z}_1(t)$. We now define the vector

$$V(t,T) = \frac{\int_T^\infty M(t,s)\phi(s)\sigma_1(t,s)ds}{\int_T^\infty M(t,s)\phi(s)ds}$$

The dynamics of the zero-coupon bond prices can then be expressed in the form

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} &= r(t)dt + S_P(t,T)^T \left(d\hat{Z}_1(t) - V(t,t)dt\right) \\ \text{where } S_P(t,T) &= V(t,T) - V(t,t) \end{aligned}$$

It follows that the vector $S_P(t,T)$ is the price volatility function with each of its n_1 components defining the volatility of the price of a particular bond with respect to each of the n_1 sources of uncertainty.

Since we have expressed the price dynamics in the way given above we can immediately see that if

$$\tilde{Z}_1(t) = \hat{Z}_1(t) - \int_0^t V(s,s)ds$$

then $dP(t,T) = P(t,T)\left(r(t)dt + S_P(t,T)^T d\tilde{Z}_1(t)\right)$.

Suppose that the function $\sigma_1(t,T)$ has been defined in such a way that

$$E_{P_\infty}\left[\exp\left(\frac{1}{2}\int_0^t V_i(s,s)^2 ds\right)\right] < \infty \quad \text{for each } i$$

(for example, see Baxter & Rennie, 1996). Then, by the Cameron-Martin-Girsanov (CMG) Theorem, there exists a measure Q equivalent to P_∞ under which $\tilde{Z}_1(t)$ is an n_1-dimensional Brownian motion. Given the form of $dP(t,T)$ we can see that Q is the usual risk-neutral measure. Provided each $\sigma_{1i}(t,T)$ for all t, $T > t$, is bounded, then $V_i(s,s)$ must be bounded, so the CMG condition is satisfied.

We are then free to make a further change of measure from Q to the real-world measure P.

2.2 A specific multifactor equilibrium model

We have already expressed $M(t,T)$ in the following way

$$\begin{aligned} M(0,T) &= 1 \quad \text{for all } T \\ dM(t,T) &= M(t,T)\sigma_1(t,T)^T d\hat{Z}_1(t) \\ &= M(t,T)\sum_{i=1}^{n_1} \sigma_{1i}(t,T) d\hat{Z}_{1i}(t) \end{aligned}$$

where $\hat{Z}_{11}(t), \ldots, \hat{Z}_{1n_1}(t)$ are n_1 independent Brownian motions under P_∞.

Suppose now that $\sigma_{1i}(t,T) = \sigma_{1i}\exp[-\alpha_{1i}(T-t)]$. We have

$$\begin{aligned} M(t,T) &= \exp\left[\sum_{i=1}^{n_1}\left\{\int_0^t \sigma_{1i}(u,T)d\hat{Z}_{1i}(u) - \frac{1}{2}\int_0^t \sigma_{1i}(u,T)^2 du\right\}\right] \\ &= \exp\left[\sum_{i=1}^{n_1}\left\{\sigma_{1i}\int_0^t e^{-\alpha_{1i}(T-u)}d\hat{Z}_{1i}(u) - \frac{\sigma_{1i}^2}{4\alpha_{1i}}e^{-2\alpha_{1i}(T-t)}\left(1-e^{-2\alpha_{1i}t}\right)\right\}\right] \\ &= \exp\left[\sum_{i=1}^{n_1}\left\{\sigma_{1i}e^{-\alpha_{1i}(T-t)}X_{1i}(t) - \frac{\sigma_{1i}^2}{4\alpha_{1i}}e^{-2\alpha_{1i}(T-t)}\left(1-e^{-2\alpha_{1i}t}\right)\right\}\right] \\ \text{where } X_{1i}(t) &= \int_0^t e^{-\alpha_{1i}(t-u)}d\hat{Z}_{1i}(u) \quad \text{for } i=1,\ldots,n_1 \end{aligned}$$

Now we recognise the $X_{1i}(t)$ as standard Ornstein-Uhlenbeck processes (for example, see Øksendal, 1998): that is, $X_{1i}(t)$ is the solution to the stochastic differential equation $X_{1i}(0)=0$, $dX_{1i}(t) = -\alpha_{1i}X_{1i}(t)dt + d\hat{Z}_{1i}(t)$.

Now suppose that

$$\phi(s) = \phi\exp\left[-\beta s + \sum_{i=1}^{n_1}\left\{\sigma_{1i}e^{-\alpha_{1i}s}\tilde{X}_{1i}(0) - \frac{\sigma_{1i}^2}{4\alpha_{1i}}e^{-2\alpha_{1i}s}\right\}\right]$$

for some ϕ, β and $\tilde{X}_{1i}(0)$. Then

$$\int_T^\infty \phi(s)M(t,s)ds = \phi e^{-\beta t}\int_{T-t}^\infty \exp\left[-\beta u + \sum_{i=1}^{n_1}\left\{\sigma_{1i}e^{-\alpha_{1i}u}\tilde{X}_{1i}(t) - \frac{\sigma_{1i}^2}{4\alpha_{1i}}e^{-2\alpha_{1i}u}\right\}\right]du$$

where $\tilde{X}_{1i}(t) = X_{1i}(t) + e^{-\alpha_{1i}t}\tilde{X}_{1i}(0)$ is also a standard Ornstein-Uhlenbeck process but starting away from zero. Then we have

$$
\begin{aligned}
P(t,T) &= \frac{\int_{T-t}^{\infty} H(u,\tilde{X}_1(t))du}{\int_0^{\infty} H(u,\tilde{X}_1(t))du} \\
\text{where } H(u,x) &= \exp\left[-\beta u + \sum_{i=1}^{n_1}\left\{\sigma_{1i}e^{-\alpha_{1i}u}x_i - \frac{\sigma_{1i}^2}{4\alpha_{1i}}e^{-2\alpha_{1i}u}\right\}\right]
\end{aligned}
$$

Other specific models using Ornstein-Uhlenbeck processes as drivers within this positive-interest framework have been proposed by Rogers (1997).

2.3 The risk-free rate and forward rates

Applying the general formula in Section 1.1 we see that the forward-rate curve is

$$
\begin{aligned}
f(t,T) &= \frac{H(T-t,\tilde{X}_1(t))}{\int_{T-t}^{\infty} H(u,\tilde{X}_1(t))du} \\
&= \left\{\int_{T-t}^{\infty}\exp\left[-\beta\{u-(T-t)\} + \sum_{i=1}^{n_1}\left(\sigma_{1i}\tilde{X}_{1i}(t)\left[e^{-\alpha_{1i}u} - e^{-\alpha_{1i}(T-t)}\right]\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. -\frac{\sigma_{1i}^2}{4\alpha_{1i}}\left[e^{-2\alpha_{1i}u} - e^{-2\alpha_{1i}(T-t)}\right]\right)\right]du\right\}^{-1} \\
&= \left\{\int_0^{\infty}\exp\left[-\beta v + \sum_{i=1}^{n_1}\left\{\sigma_{1i}\tilde{X}_{1i}(t)e^{-\alpha_{1i}(T-t)}\left(e^{-\alpha_{1i}v}-1\right)\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. -\frac{\sigma_{1i}^2}{4\alpha_{1i}}e^{-2\alpha_{1i}(T-t)}\left(e^{-2\alpha_{1i}v}-1\right)\right\}\right]dv\right\}^{-1} \\
\Rightarrow r(t) &= \left\{\int_0^{\infty}\exp\left[-\beta v + \sum_{i=1}^{n_1}\left\{\sigma_{1i}\tilde{X}_{1i}(t)\left(e^{-\alpha_{1i}v}-1\right) - \frac{\sigma_{1i}^2}{4\alpha_{1i}}\left(e^{-2\alpha_{1i}v}-1\right)\right\}\right]dv\right\}^{-1}
\end{aligned}
$$

If we look more closely at the formula for $f(t,T)$ we can see that as T tends to infinity, $f(t,T)$ tends to β: that is, β is the constant long-term forward rate. (Note that Dybvig, Ingersoll & Ross, 1994, established that under the assumption of no arbitrage a model for the term-structure of interest rates must have a non-decreasing long-term *spot* rate.)

The par yield on irredeemable bonds (assuming continuous payment of coupons) is

$$
\rho(t) = \left[\int_0^{\infty} P(t,t+s)ds\right]^{-1} = \frac{\int_0^{\infty} H\left(u,\tilde{X}_{1i}(t)\right)du}{\int_0^{\infty} uH\left(u,\tilde{X}_{1i}(t)\right)du}
$$

and this can be developed in the same way as $f(t,T)$ and $r(t)$ above.

2.4 Equivalence of P_∞ and Q

Recall that $\tilde{Z}_{1i}(t) = \hat{Z}_{1i}(t) - V_i(t,t)dt$ where the $\tilde{Z}_{1i}(t)$ and $\hat{Z}_{1i}(t)$ are Brownian Motions under the risk-neutral measure Q and the terminal measure P_∞ respectively. Here

$$V_i(t,t) = \frac{\int_0^\infty H(u, \tilde{X}_1(t))\sigma_{1i} e^{-\alpha_{1i} u} du}{\int_0^\infty H(u, \tilde{X}_1(t)) du}.$$

Since $H(u,x) > 0$ for all $u > 0$, $-\infty < x < \infty$, we have $V_i(t,t) < \sigma_{1i}$ for all t. For equivalence between P_∞ and Q we require $E_{P_\infty}\left[\exp\left(\frac{1}{2}\int_0^t V_i(s,s)^2 ds\right)\right] < \infty$ for each i (the CMG condition). Since $V_i(t,t)$ is bounded this condition is satisfied.

2.5 Time homogeneity

From the form of $H(u,x)$, and $\tilde{X}_1(t) = \left(\tilde{X}_{11}(t), \ldots, \tilde{X}_{1n_1}(t)\right)^T$ we can see that the $P(t,T)$ are Markov and time homogeneous. As such the model plus knowledge of $\tilde{X}_1(t)$ gives us a set of theoretical prices which may differ from those observed. Under the original no-arbitrage framework (Flesaker & Hughston, 1996) initial observed prices form part of the input (hence the earlier note that $\phi(s) = \partial P(0,s)/\partial s$) but this results in the loss of time homogeneity. Both approaches have their own merits. Here the intention is that the number of factors, n_1, should be large enough to ensure that once the $\tilde{X}_{1i}(t)$ have been estimated there is a close correspondence (but not exact) between theoretical and observed prices for all t. It can then be argued that frictions in the market such as transaction costs and buying and selling spreads prevent exploitation of the price errors.

2.6 Practical considerations

The structure of this model is such that only a limited number of random factors ($\tilde{X}_{1i}(t)$ for $i = 1, \ldots, n_1$) need to be recorded in order for us to be able to reconstruct the evolution of the term structure through time, calculate prices, returns on assets and so on. This is in contrast to some no-arbitrage models based upon the Heath-Jarrow-Morton (1992) framework which require a record of the entire forward rate curve at all times.

In this the first stage of the development of this model it was not considered necessary to allow for correlation between the $\tilde{X}_{1i}(t)$. This is because the correlations did not enrich, in any obvious way, the range of shapes of forward-rate curves *etc.* which could be generated. This means that it would be unlikely that the more complex model would fit historical data significantly better than the model without correlations.

The $\tilde{X}_{1i}(t)$, for $i = 1, \ldots, n_1$, follow a standard Ornstein-Uhlenbeck process under P_∞. The processes are therefore particularly simple to simulate accurately under this measure given that, for $s > t$, $\tilde{X}_{1i}(s)$ is normally distributed. The nature of the changes of measure for each of the $\tilde{X}_{1i}(t)$ means that the $\tilde{X}_{1i}(s)$ given $\tilde{X}_{1i}(t)$ are no longer normally distributed under Q. For simulation purposes, we are interested in the real-world measure P which has not really been discussed so far. If a constant market price of risk is employed relative to Q then the same problem exists (that $\tilde{X}_{1i}(s)$ is not normally distributed exists – although over a one-month period the normal approximation is reasonable). As an alternative we can employ a constant change of drift between $\hat{Z}_1(t)$ and $Z_1(t)$. This ensures that the $\tilde{X}_{1i}(t)$ still follow Ornstein-Uhlenbeck processes under P (now with non-zero means). A less desirable consequence of this, though, is that this does occasionally allow risk-premia to become negative from time to time. The frequency of this clearly depends upon the parametrisation of the model and the size of the change of measure with a low frequency being tolerable for the sake of ease of simulation of the $\tilde{X}_{1i}(t)$.

It is necessary to carry out numerical integration in order to compute bond prices and interest rates on a given date and given $\tilde{X}_1(t)$. However, this step can be done in a straightforward and accurate way, since it only involves one-dimensional integration. Furthermore, with only a little extra work we can use numerical integration to calculate the distribution of many quantities (how straightforward this is depends upon the relationship between P_∞ and P described above).

The calibration of this part of the model is discussed in Section 5.

3 Real rates of interest, index-linked bonds and inflation

3.1 Basic principles

We start here by developing some general ideas before proposing a specific model.

Let $C(t)$ be the consumer price index at time t for a particular currency. Let $Q(t,T)$ be the price at time t for a payment of $C(T)/C(t)$ at time T (that is, the $Q(t,T)$ represent index-linked, zero-coupon prices). We assume for simplicity that $C(t)$ is known at time t and that index-linked bonds index payments without a time lag. Real rates of interest can be derived from $Q(t,T)$ in the following simple ways:

$$\begin{aligned}
\text{Spot rates: } R_Q(t,T) &= -\frac{1}{T-t}\log Q(t,T) \\
\text{Forward rates: } f_Q(t,T) &= -\frac{\partial}{\partial T}\log Q(t,T) \\
\text{Real risk-free rate: } r_Q(t) &= R_Q(t,t) = f_Q(t,t)
\end{aligned}$$

In the index-linked bond market there is no real risk-free asset (that is, an asset which returns over the interval t to $t + dt$, for small dt, the increase in $C(t)$ plus the real risk-free rate $r_Q(t)$ with no volatility). In practice, the real forward and spot-rate curves are inferred from a limited number of index-linked coupon bonds. From either curve we can infer what $r_Q(t)$ would be if it existed.

Here we propose an equilibrium derivation of the $Q(t,T)$. We start with a diffusion model for $r_Q(t)$ under the risk-neutral measure Q (which we extend from coverage of the fixed-interest market to the index-linked market) and calculate prices according to the following formula:

$$Q(t,T) \quad = \quad E_Q\left[\exp\left(-\int_t^T r_Q(s)ds\right) \;\middle|\; \mathcal{F}_t^2\right]$$

Under this model we have

$$\begin{aligned} dQ(t,T) &= Q(t,T)\left(r_Q(t)dt + S_Q(t,T)^T d\tilde{Z}_2(t)\right) \\ &= Q(t,T)\left\{\left(r_Q(t) + \lambda_2(t)^T S_Q(t,T)\right)dt + S_Q(t,T)^T dZ_2(t)\right\} \end{aligned}$$

Here, $\tilde{Z}_2(t)$ is n_2-dimensional Brownian motion under Q, $\tilde{Z}_2(t)$ is independent of $\tilde{Z}_1(t)$ (this assumption can be relaxed easily) and $\mathcal{F}_t^2 = \sigma\left(\{\tilde{Z}_2(s) \mid 0 \le s \le t\}\right)$.

The consumer prices index is a positive process which is declared on a monthly basis. Here we will model it as a continuous-time diffusion process:

$$dC(t) = C(t)\left[\mu_C(t)dt + \sigma_3(t)dZ_3(t)\right]$$

where $Z_3(t)$ (which is $n_3 = 1$-dimensional Brownian motion under the real world probability measure P) is independent of $Z_1(t)$ and $Z_2(t)$.

We will assume that $\mu_C(t)$ is measurable with respect to the filtration, $\mathcal{F}^{(2)}$, generated by $Z_1(t)$ and $Z_2(t)$: that is, the drift of the price index is determined by past and present bond prices. (We will see below that under Q the drift is equal to $r(t) - r_Q(t)$ under Q.) Furthermore, we will assume that $\sigma_3(t)$ is small but non-zero. This reflects the assumption that $C(t)$ is a commodities index and that at least some of the individual commodities making up the index have prices which change in an unpredictable way. The requirement that $\sigma_3(t) > 0$ also enables inflation to have different drifts under the real-world and risk-neutral measures.

Now the $Q(t,T)$ do not give the prices of tradeable assets, since the definition of the payment at T is continually being changed. On the other hand, if we define $L(t,T) = C(t)Q(t,T)$, then the $L(t,T)$ do represent the prices of tradeable assets since the payment at T is always $C(T)$. Then we have

$$\begin{aligned}
dL(t,T) &= Q(t,T)dC(t) + C(t)dQ(t,T) + dC(t)dQ(t,T) \\
&= L(t,T)\left(\mu_C(t)dt + \sigma_3(t)dZ_3(t)\right) \\
&\quad + L(t,T)\left\{\left(r_Q(t) + \lambda_2(t)^T S_Q(t,T)\right)dt + S_Q(t,T)^T dZ_2(t)\right\} + 0 \\
&= L(t,T)\left((\mu_C(t) + r_Q(t) + \lambda_2(t)^T S_Q(t,T))dt + S_Q(t,T)^T dZ_2(t) + \sigma_3(t)dZ_3(t)\right)
\end{aligned}$$

(Independence of $Z_3(t)$ and $Z_2(t)$ means that $dC(t).dQ(t,T) = 0$.)

Since $L(t,T)$ is a tradeable asset we can also write its dynamics as

$$\begin{aligned}
dL(t,T) &= L(t,T)\left(r(t)dt + S_L(t,T)^T d\tilde{Z}'(t)\right) \\
\text{where } d\tilde{Z}'(t) &= \begin{pmatrix} d\tilde{Z}_2(t) \\ d\tilde{Z}_3(t) \end{pmatrix} \\
\text{and } S_L(t,T) &= \begin{pmatrix} S_Q(t,T) \\ \sigma_3(t) \end{pmatrix}
\end{aligned}$$

It follows, by recalling that $d\tilde{Z}_i(t) = dZ_i(t) + \lambda_i(t)dt$, that

$$\begin{aligned}
r(t) &= \mu_C(t) + r_Q(t) - \lambda_3(t)\sigma_3(t) \\
\Rightarrow \mu_C(t) &= r(t) - r_Q(t) + \lambda_3(t)\sigma_3(t)
\end{aligned}$$

from which we see that $dC(t) = C(t)\left[\left(r(t) - r_Q(t)\right)dt + \sigma_3(t)d\tilde{Z}_3(t)\right]$.

It is reasonable to discuss the choice of $\lambda_3(t)$ at this point. Investors are generally more interested in real returns rather than nominal returns, especially where inflation is a relatively uncertain process. We can conjecture, therefore, that if a real risk-free asset existed then it would have a lower expected return than a nominal risk-free asset. As a consequence $\mu_C(t)$ should be less than $r(t) - r_Q(t)$. This means that the market price of inflation risk, $\lambda_3(t)$, should be less than zero (if $\sigma_3(t) > 0$). An appropriate choice of $\lambda_2(t)$ will ensure that longer-dated, risky, index-linked bonds give the right level of risk premium over nominal cash and relative to fixed-interest bonds.

3.2 A specific model for index-linked bond prices

We consider here a two-factor equilibrium model for index-linked prices which is a generalisation of the Vasicek (1977) model. A similar approach to fixed-interest bond prices has been taken, for example, by Langetieg (1980). The present model is also a special case of that developed by Tice & Webber (1997) for the nominal risk-free rate (by taking their third factor $p(t)$ to be identically equal to zero).

Suppose that

$$\begin{aligned} dr_Q(t) &= -\alpha_{21}\left(r_Q(t) - m_Q(t)\right)dt + \sigma_{21} d\tilde{Z}_{21}(t) \\ \text{where } dm_Q(t) &= -\alpha_{22}\left(m_Q(t) - \mu_Q\right)dt + \sigma_{22} d\tilde{Z}_{22}(t) \\ \text{or } dy_Q(t) &= -A_Q\left(y_Q(t) - \tilde{\mu}_Q\right) + S_Q d\tilde{Z}_2(t) \\ \text{where } y_Q(t) &= \left(r_Q(t), m_Q(t)\right)^T \\ \tilde{\mu}_Q &= \left(\mu_Q, \mu_Q\right)^T \\ A_Q &= \begin{pmatrix} \alpha_{21} & -\alpha_{21} \\ 0 & \alpha_{22} \end{pmatrix} \\ S_Q &= \begin{pmatrix} \sigma_{21} & 0 \\ 0 & \sigma_{22} \end{pmatrix} \end{aligned}$$

The process $m_Q(t)$ is a standard Ornstein-Uhlenbeck process with mean-reversion level μ_Q. This process should itself be interpreted as the local mean reversion level of the main process $r_Q(t)$. Superficially this model looks like the Hull & White (1990) model, but in their development $m_Q(t)$ is a deterministic function which is determined by the initial set of prices $Q(0,T)$.

Under this model we find that

$$\begin{aligned} Q(t,T) &= \exp\left[A_Q(T-t) - B_{Q1}(T-t)r_Q(t) - B_{Q2}(T-t)m_Q(t)\right] \\ \text{where } B_{Q1}(s) &= \frac{1-e^{-\alpha_{21}s}}{\alpha_{21}} \\ B_{Q2}(s) &= \frac{\alpha_{21}}{\alpha_{21}-\alpha_{22}}\left[\frac{1-e^{-\alpha_{22}s}}{\alpha_{22}} - \frac{1-e^{-\alpha_{21}s}}{\alpha_{21}}\right] \end{aligned}$$

$$\begin{aligned} A_Q(s) = {} & (B_{Q1}(s) - s)\left(\mu_Q - \frac{\sigma_{21}^2}{2\alpha_{21}^2}\right) + B_{Q2}(s)\mu_Q - \frac{\sigma_{21}^2 B_{Q1}(s)^2}{4\alpha_{21}} \\ & + \frac{\sigma_{22}^2}{2}\left[\frac{s}{\alpha_{22}^2} - 2\frac{(B_{Q2}(s)+B_{Q1}(s))}{\alpha_{22}^2} + \frac{1}{(\alpha_{21}-\alpha_{22})^2}\frac{(1-e^{-2\alpha_{21}s})}{2\alpha_{21}}\right. \\ & - \frac{2\alpha_{21}}{\alpha_{22}(\alpha_{21}-\alpha_{22})^2}\frac{(1-e^{-(\alpha_{21}+\alpha_{22})s})}{(\alpha_{21}+\alpha_{22})} \\ & \left. + \frac{\alpha_{21}^2}{\alpha_{22}^2(\alpha_{21}-\alpha_{22})^2}\frac{(1-e^{-2\alpha_{22}s})}{2\alpha_{22}}\right] \end{aligned}$$

When we choose the market prices of risk, $\lambda_2(t)$, these should be consistent with $\lambda_1(t)$ and $\lambda_3(t)$ and our views on the relationship between fixed-interest and index-linked bonds. For example, we have already noted that short-dated index-linked bonds should return less than short-dated fixed-interest bonds through the market-price of inflation risk, $\lambda_3(t)$. This will automatically affect returns on longer-dated bonds in the same way, but longer-term inflation risks may mean that we wish to have a larger or a smaller difference between long-dated index-linked and fixed-interest bonds.

4 Structure of the model

In describing this model we can note the following points.

First, the model is Markov. Other approaches (for example, Vector ARMA models) include non-Markov elements. For example, some variables in the model might depend upon price inflation over the last two years rather than just the current rate of price inflation. However, such models can be given a state-space representation, which records at time t all necessary variables to allow simulation of the observations at time $t+1$. Such extended models are Markov in structure even though some elements are not directly observable in the natural sense. It is a reasonable assumption (although this is not essential) that current prices take account of all available information: past as well as present. For example, if the future risk-free rate of interest depends upon rates over the last two years then current bond prices should reflect this directly meaning that the relevant history is incorporated into the current set of prices.

Second, the model defines nominal and real interest rates and then inflation is driven by the difference between the short rates. A more natural construction might start with inflation and build a term-structure model on top of that (for example, Wilkie, 1995). The present model is Markov and so inflation could be described at the top level and bond prices below that. However, the resulting formulation of the model would look much less compact than it is in its present form.

A third point which can be mentioned relates to other models used for long-term risk management. In some cases such models may provide good forecasts of the medium and long term while supporting characteristics which give rise to unrealistic short-term predictions: that is, compared to predictions derived from current market and economic data (for example, government policy). Under such circumstances, it is common practice to modify short-term dynamics of the model while keeping the longer term version of the model unchanged. As an example, it is often the case with the Wilkie (1995) model that inflation is modelled in the short term using lower volatility and with inflation expectations modified to reflect current market information. With the current model we feel that such alterations should not be necessary since the inclusion of a full term-structure (of interest rates and inflation) means that the model immediately reflects fully an accurate view of future expectations of interest rates and inflation. Appropriate choice of volatilities should then ensure appropriate levels of uncertainty in both short-term as well as long-term dynamics.

Fourthly the model allows all nominal rates of interest to take values arbitrarily close to zero. In contrast, some other models with positive interest (for example, the multi-factor Cox-Ingersoll-Ross model described in Duffie, 1996) impose non-zero lower bounds on all nominal rates of interest other than the risk-free rate. Such constraints, other than the lower bound of zero, are rather arbitrary and unjustifiable.

5 Calibration of the fixed-interest model: discussion

The parameters for this model can be approached in a number of different ways. We will describe two here.

First recall a result of Dybvig, Ingersoll & Ross (1994). They proved that for an interest rate model to be arbitrage free it is necessary that the infinite-maturity spot rate must be non-decreasing. In light of this result it becomes a challenge to produce a sufficient level of volatility on yields-to-redemption on long-dated coupon bonds (as observed, for example, in the UK) without making short-term interest rates too volatile. Consistent levels of volatility in both short and long rates can be achieved under the present model by choosing appropriate values for the α_{1i} and σ_{1i}.

5.1 Historical data

Historical data may be comprised of a mixture of coupon-bond prices, treasury bill rates and, more recently, prices of zero-coupon bonds, and bond futures and options.

It should be borne in mind that the model was devised with the intention that one or more of the α_{1i} (say α_{1n_1}) should be relatively low. This allows for long-term fluctuations in the general level of interest rates (particularly those on long-dated coupon bonds). A consequence of this is that estimated values of the process $\tilde{X}_{1n_1}(t)$ will be insufficient to get a reliable estimate of α_{1n_1}. On the other hand, the shapes of the fitted yield curves on individual dates do depend upon the α_{1i}, so that additional data about α_{1n_1} can be gained from the range of bond price data.

Full statistical analysis includes modelling how the prices of individual bonds vary relative to their theoretical prices over time. Here we propose a different approach which should (although this is not tested) produce almost as effective answers.

Let $\theta = (\beta, \alpha_1, \sigma_1)$ be the fixed parameter set and π_t be the set of prices on a given date. The error structure of the π_t given θ and $\tilde{X}_1(t)$ is the subject of studies by Cairns (1998, 1999) (although those papers use a simpler descriptive model for the forward-rate curve). Thus, for each t, given θ and π_t we can estimate $\tilde{X}_1(t)$ independent of all other dates (so-called *descriptive* modelling). The full likelihood function can be expressed in the following way:

$$L(\theta; x_1; \pi) = L_1(\theta; x_1) \prod_{t \in \mathcal{T}} L_2(\theta; x_1(t); \pi_t) = L_1(\theta; x_1) L_2(\theta; x_1; \pi)$$

where $\mathcal{T}$ is the reference index, $L_2(\theta; x_1(t); \pi_t)$ is the likelihood function for time t only for estimation of θ and $x_1(t)$, and $L_1(\theta; x_1) = L_1(\theta; \{x_1(t) : t \in \mathcal{T}\})$ is the likelihood function for the time series $x_1(t)$. (This full likelihood assumes that actual-minus-theoretical price errors for individual stocks on different dates are independent.)

We conjecture that the following estimation procedure gives a good approximation to the result of full maximisation of $L(\theta; x_1; \pi)$ over θ and x_1 with substantially less computing effort:

Stage 1:
For a given θ and $t \in \mathcal{T}$, let $\hat{x}_1(t)(\theta)$ maximise $L_2(\theta; x_1(t); \pi_t)$ over $x_1(t)$.

Stage 2:
Maximise $L(\theta; \hat{x}_1(\theta); \pi)$ over θ where $\hat{x}_1(\theta) = \{\hat{x}_1(t)(\theta) : t \in \mathcal{T}\}$.

The curve fitting required in Stage 1 in the context of interest-rate data is known as *descriptive modelling*. This is described in more detail elsewhere (Cairns, 1998, 1999).

This conjecture is based upon the following argument:

We know that the size of the dataset $\pi = \{\pi_t : t \in \mathcal{T}\}$ will be much larger than the size of $\{x_1(t) : t \in \mathcal{T}\}$. A consequence of the relatively large quantity of price data is that any deviation in $x_1(t)$ from $\hat{x}_1(t)(\theta)$ would result in a much more severe penalty in $L_2(\theta; x_1(t); \pi_t$ (unless the model is over-parametrised) compared to the potential gain in $L_1(\theta; x_1)$. As a result we conjecture that the result, $(\bar{\theta}, \bar{x}_1)$, of full maximisation of $L(\theta; x_1; \pi)$ would give $\bar{x}_1 \approx \hat{x}_1(\bar{\theta})$.

We should also investigate the adequacy of the model. If the choice of model and, in particular, the number of factors n_1 is acceptable then we should find the following:

- The quality of fit on any particular date, t, given $\hat{\theta}$ and $\hat{x}_1(t)(\hat{\theta})$ should not vary substantially over time. For example, the model should be able to mimic the full range of yield curves which we observe in the market. Furthermore, the differences between estimated and observed prices should be sufficiently small to avoid (because of transactions costs) exploitation of apparent arbitrage opportunities.
- The estimated $\hat{x}_1(t)(\hat{\theta})$ should evolve in a way which is consistent with the underlying model. Should it be found that the processes $\hat{x}_{1i}(t)(\hat{\theta})$ for $i = 1, \ldots, n_1$ are correlated then the mode should be adjusted accordingly, although this will result in rather more parameters than we might prefer.

5.2 Qualitative methods

Alternative methods to calibration involve making prior statements about *target* quantities such as:

- the mean value of specific interest rates (for example, the 3-month treasury-bill rate, and 5 and 25-year par yields);
- the level of short-term variability of specified rates;
- the level of long-term variability of specified rates;
- the degree of influence of the various driving factors on different interest rates in the short and long term.

Parameter values can then be found by analytical methods or by simulation to match as closely as possible the targets.

The prior statements are typically based upon historical experience but take subjective views on the relative importance of certain periods of data. Subjective judgements are especially important for the treatment of factors which are subject to long cycles (that is, for low α_{1i}) where there is insufficient data.

Consider, for example, par yields on long-dated bonds and irredeemable bonds in the UK (for example, see Wilkie, 1995, Figure 6.1). Over a period of 100 years or so, these yields have ranged from about 2.5% up to 15%. The target for 25-year par yields should, therefore, cover this range with reasonable probabilities of attaining both 2.5% and 15% over, say, a one-hundred-year period. It is specifically this requirement which dictates the need for one of the α_{1i} to be relatively low, although the existence of relatively long cycles suggests this as well.

The non-linear dependence of prices and interest rates upon the $\tilde{X}_{1i}(t)$ complicates matters somewhat. However, we can make some *crude* approximations based upon the Taylor expansion which turn out to be quite effective. Thus,

$$\begin{aligned}
f(t,T) &= f(t,T)(\tilde{X}_1(t)) \\
&\approx f(t,T)(0)\left(1+\sum_{i=1}^{n_1}\sqrt{2\alpha_{1i}}d_i(T-t)\tilde{X}_{1i}(t)\right) \\
&\approx \beta\left(1+\sum_{i=1}^{n_1}\sqrt{2\alpha_{1i}}d_i(T-t)\tilde{X}_{1i}(t)\right) \\
\text{where}\quad d_i(u) &= \frac{\sigma_{1i}e^{-\alpha_{1i}u}}{(\beta+\alpha_{1i})}\sqrt{\frac{\alpha_{1i}}{2}}
\end{aligned}$$

Note that $Var\left(\sqrt{2\alpha_{1i}}\tilde{X}_{1i}(t)\right) \to 1$ as $t \to \infty$. Thus the long-term effect of $\tilde{X}_{1i}(t)$ on $f(t, t+s)$ is $\beta d_i(s)$ in contrast to the short-term effect of local volatility in $\tilde{X}_{1i}(t)$ which is $\beta\sqrt{2\alpha_{1i}}d_i(s)$.

The par yield on irredeemable bonds is

$$\begin{aligned}
\rho(t) = \rho(t)\left(\tilde{X}_{1i}(t)\right) &= \left[\int_0^\infty P(t, t+s)ds\right]^{-1} \\
&= \frac{\int_0^\infty H\left(u, \tilde{X}_{1i}(t)\right) du}{\int_0^\infty uH\left(u, \tilde{X}_{1i}(t)\right) du} \\
&\approx \beta\left[1 + \sum_{i=1}^{n_1} \frac{\sigma_{1i}\beta}{(\beta+\alpha_{1i})^2}\sqrt{\frac{\alpha_{1i}}{2}}\tilde{X}_{1i}(t)\right]
\end{aligned}$$

Actual variances turn out to be a bit higher than those predicted by these formulae because of the non-linear dependence of $f(t, T)$ on $\tilde{X}_1(t)$. However, we are provided with a useful starting point. Furthermore, we can get a useful guide to the relative importance of each of the $\tilde{X}_{1i}(t)$ on various interest rates.

5.3 Example

For illustration, the following parameter values (with $n_1 = 3$) were chosen partly based on trial and error and partly using the linearisation.

$\beta = 0.05$	(this needs to be sufficiently low to get some 25-year par yields as low as 3% with reasonable frequency)
$\alpha_1 = (0.4, 0.2, 0.05)$	
$\sigma_1 = (0.7, 0.3, 0.4)$	

The dependencies quoted in Table 1 can be seen graphically in Figures 1, 2 and 3.

In Figure 1 we have plotted the 25-year par yield over a typical 100-year period: that is, we can see periods of stable low rates interspersed with bursts of high rates. Figure 1 also plots $\tilde{X}_{1i}(t)$ for $i = 1, 2, 3$ over the same period. This allows us to see that certain features in the dynamics of the 25-year par yield can be explained by variations in each of the three driving factors. For example, the broad level of the yield is driven by $\tilde{X}_{13}(t)$ while more local peaks and dips are caused by local peaks and dips in $\tilde{X}_{11}(t)$ and $\tilde{X}_{13}(t)$. These short-term dependencies concur (in as far as we can assess this visually) with the final row of Table 1.

In Figure 2 we plot a longer simulation run and give scatter plots which allow us to get a better picture of the dependency of 25-year par yields on the $\tilde{X}_{1i}(t)$. In particular, we can see the strong dependency noted in Table 1 (and that this is non-linear) on $\tilde{X}_{13}(t)$ and the weak long-term dependency on $\tilde{X}_{11}(t)$ and $\tilde{X}_{12}(t)$.

Long-term Standard deviations (%)	Individual S.D. due to X_{11}	 X_{12}	 X_{13}	Total S.D. predicted	 sample
$f(t, t + \frac{1}{4})$	3.15	1.90	3.16	5.07	5.05
$f(t, t + 5)$	0.47	0.70	2.46	2.60	3.10
$f(t, t + 25)$	0.00	0.01	0.91	0.91	1.11
$\rho(t)$	0.39	0.38	1.58	1.67	*2.70*(*)
Short-term Standard deviations (%)					
$f(t, t + \frac{1}{4})$	2.82	1.14	0.99	3.19	4.04
$f(t, t + 5)$	0.42	0.44	0.78	0.99	1.11
$f(t, t + 25)$	0.00	0.01	0.29	0.29	0.33
$\rho(t)$	0.35	0.24	0.50	0.65	*1.25*(*)

Table 1: Standard deviations driven by individual factors and overall standard deviations for selected rates of interest. Predicted standard deviations estimated using the linearisation. (*) The sample standard deviation for $\rho(t)$ is, in fact, the sample standard deviation of the 25-year par yield.

Figure 3 gives equivalent plots for 3-month spot rates. As expected there is a greater degree of short-term volatility and of long-term variation compared to the 25-year par yields. We can also see how long-term variability depends in more equal terms on $\tilde{X}_{11}(t)$ and $\tilde{X}_{13}(t)$, and (to a lesser extent) $\tilde{X}_{12}(t)$. A closer analysis of simulated 3-month rates shows that short-term rates of interest sometimes experience long periods of low stable rates and other periods of considerable volatility. Such behaviour is entirely consistent with that observable in many developed countries. For example, in Germany rates have been low and stable for many years having gone through a period of considerable turmoil earlier in the century. Overall 3-month rates could be seen to range from 0.1% (as we have seen in Japan) up to 45% (consistent with, for example, some East European countries).

An alternative approach to the modelling of these cycles was proposed by Tice & Webber (1994). Their three-factor equilibrium model incorporates a particular form of non-linearity which gives rise to an underlying chaotic behaviour with high and low interest-rate cycles as a means of explaining historical behaviour of interest rates, for example, in the UK. Here we have demonstrated experimentally that the present model (with linear underlying processes) apparently can mimic equally well this behaviour.

6 Equities

[Note: this Section is presently in *draft* form and is currently unpublished. *This section of the paper may not be quoted without the explicit permission of the author.*]

Let us now turn to equities. We define:

$$\begin{aligned} P_0(t) &= \text{total return equity index} \\ P_1(t) &= \text{price index without reinvestment of dividends} \\ y(t) &= \text{gross dividend yield payable contunuously} \\ D(t) &= \text{dividend index} \\ &= P_1(t)y(t) \end{aligned}$$

First consider the total return and price indices. It's dynamics can be written using the following stochastic differential equation:

$$\begin{aligned} \frac{dP_0(t)}{P_0(t)} &= (r(t) + \rho_p(t))\, dt + \sum_{j=1}^{n_1} \sigma_{41j}(t) dZ_{1j}(t) \\ &\quad + \sum_{j=1}^{2} \sigma_{42j}(t) dZ_{2j}(t) + \sigma_{431}(t) dZ_3(t) + \sigma_{441}(t) dZ_4(t) \\ \text{and } \frac{dP_1(t)}{P_1(t)} &= (r(t) + \rho_p(t) - y(t))\, dt + \sum_{j=1}^{n_1} \sigma_{41j}(t) dZ_{1j}(t) \\ &\quad + \sum_{j=1}^{2} \sigma_{42j}(t) dZ_{2j}(t) + \sigma_{431}(t) dZ_3(t) + \sigma_{441}(t) dZ_4(t) \end{aligned}$$

where $Z_4(t)$ is a standard Brownian motion under P independent of $Z_1(t)$, $Z_2(t)$ and $Z_3(t)$. We anticipate that the $\sigma_{41j}(t)$ will be negative ensuring that instantaneous equity-price and bond-price changes are positively correlated. Thus we regard the equity price as being the discounted value of future dividend payments with a fall in this value if interest rates rise unexpectedly.

Suppose that

$$\begin{aligned} \frac{dD(t)}{D(t)} &= \mu_D(t)dt + \sum_{j=1}^{n_1} \sigma_{51j}(t) dZ_{1j}(t) + \sum_{j=1}^{2} \sigma_{52j}(t) dZ_{2j}(t) \\ &\quad + \sigma_{531}(t) dZ_3(t) + \sigma_{541}(t) dZ_4(t) + \sigma_{551}(t) dZ_5(t) \end{aligned}$$

where $Z_5(t)$ is a standard Brownian motion under P independent of $Z_1(t)$, $Z_2(t)$, $Z_3(t)$ and $Z_4(t)$.

We can also write

$$\begin{aligned}\frac{dy(t)}{y(t)} &= \mu_y(t)dt + \sum_{j=1}^{n_1}\sigma_{61j}(t)dZ_{1j}(t) + \sum_{j=1}^{2}\sigma_{62j}(t)dZ_{2j}(t) \\ &\quad +\sigma_{631}(t)dZ_3(t) + \sigma_{641}(t)dZ_4(t) + \sigma_{651}(t)dZ_5(t)\end{aligned}$$

Since $y(t) = D(t)/P_1(t)$ we have

$$\begin{aligned}dy(t) &= dD(t)\left(\frac{1}{P_1(t)}\right) + D(t)d\left(\frac{1}{P_1(t)}\right) + dD(t)d\left(\frac{1}{P_1(t)}\right) \\ \Rightarrow \mu_y(t) &= \mu_D(t) - r(t) - \rho_p(t) + y(t) \\ &\quad + \sum_{j=1}^{n_1}\sigma_{41j}(t)^2 + \sum_{j=1}^{2}\sigma_{42j}(t)^2 + \sigma_{431}(t)^2 + \sigma_{441}(t)^2 - \sum_{j=1}^{n_1}\sigma_{41j}(t)\sigma_{51j}(t) \\ &\quad - \sum_{j=1}^{2}\sigma_{42j}(t)\sigma_{52j}(t) - \sigma_{431}(t)\sigma_{531}(t) - \sigma_{441}(t)\sigma_{541}(t) \qquad (2) \\ \sigma_{61j}(t) &= \sigma_{51j}(t) - \sigma_{41j}(t) \\ \sigma_{62j}(t) &= \sigma_{52j}(t) - \sigma_{42j}(t) \\ \sigma_{631}(t) &= \sigma_{531}(t) - \sigma_{431}(t) \\ \sigma_{641}(t) &= \sigma_{541}(t) - \sigma_{441}(t) \\ \sigma_{651}(t) &= \sigma_{551}(t)\end{aligned}$$

We will assume that dividends grow in a predictable fashion rather than reacting immediately to market volatility: that is, $\sigma_{5ij}(t) = 0$ for all i and j.

We will also assume that $\sigma_{4ij}(t) = 0$ for $i = 2, 3$ and all j. This means that volatility in equity prices arises from two sources: the fixed-interest bond market; and the fourth source, $Z_4(t)$, which is independent of the bond markets and price inflation. Hence:

$$\begin{aligned}\sigma_{61j}(t) &= -\sigma_{41j}(t) \text{ for } j = 1, \ldots, n_1 \\ \sigma_{641}(t) &= -\sigma_{441}(t) \\ \text{and } \sigma_{6ij} &= 0 \text{ otherwise}\end{aligned}$$

In the following paragraphs we will also make use of the following processes $R(t)$ and $R_Q(t)$ which satisfy the stochastic differential equations:

$$\begin{aligned}dR(t) &= -\alpha_R R(t)dt + dr(t) \\ dR_Q(t) &= -\alpha_{RQ} R_Q(t)dt + dr_Q(t)\end{aligned}$$

We propose the following simple models for $\mu_D(t)$ and $\rho_p(t)$:

$$
\begin{aligned}
\mu_D(t) &= \theta_0 + \theta_1 y(t) + \theta_2 r(t) + \theta_3 \left(r(t) - r_Q(t)\right) + \theta_4 R(t) + \theta_5 \left(R(t) - R_Q(t)\right) \\
\rho_p(t) &= \rho_0 + \rho_1 y(t) + \rho_2 r(t) + \rho_3 r_Q(t) + \rho_4 R(t) + \rho_5 R_Q(t) \\
&\quad + \rho_6 \left(\sum_{j=1}^{n_1} \sigma_{41j}(t)^2 + \sigma_{44}(t)^2 \right)
\end{aligned}
$$

We note some points to explain the choice of structure in this model:

- It is often felt that if dividend yields, $y(t)$ are too low (or too high) then effect of mean reversion in $y(t)$ will mean a fall (or a rise) in equity prices. However, it can equally well reflect expectations that dividends will rise at a faster rate than normal. Thus dividend yields will indeed revert to more normal levels without a fall in prices.

  This means that θ_1 will be negative.

- The term $\theta_3 \left(r(t) - r_Q(t)\right)$ determines the extent to which dividend increases reflect changes in consumer prices. θ_3 will typically lie between 0 and 1. If θ_3 equals 1 then dividends fully reflect changes in prices (everything else being equal). More likely, though, $\theta_3 < 1$ meaning that dividends do not keep up with prices in times of high inflation.

- Changes in equity prices already reflect changes in interest rates to the extent that the $\sigma_{41j}(t)$ are less than 0. That is, if interest rates rise then dividends are discounted at higher rates of interest and equity prices fall. However, there is a longer-term effect on dividends. Most companies have some form of short-term debt which is subject to variable short-term rates of interest. Thus, if the risk-free rate of interest, $r(t)$, rises, interest payments on these loans will rise. This will reduce the amount of money which will be available for distribution in the form of future dividends.

  Any immediate impact is modelled in the volatility terms $\sigma_{51j}(t)$ (that is, the immediate change in prices as a result of a change in discount factors). The effect on dividends will be longer term, however, and this is reflected in the term $\theta_2 r(t)$.

  We require, therefore, that $\theta_2 < 0$. The magnitude of θ_2 depends upon the extent to which underlying companies rely on short-term debt.

- We have included $R(t)$ and $R_Q(t)$ to reflect the possibility that there is a longer delay in the effect of changes in $r(t)$ and $r_Q(t)$. In particular, we will assume that $\theta_4 = -\theta_2$. Thus, unanticipated changes in $r(t)$ will not be reflected immediately in $\mu_D(t)$. Instead they will gradually emerge as $R(t)$ decays to zero.

Similarly, changes in price inflation $r(t) - r_Q(t)$ may only work their way through to the rate of dividend growth gradually. We allow for some immediate effect on the growth rate so that $-\theta_3 < \theta_5 < 0$.

- Generally we associate high dividend yields with higher risk premia and low dividend yields with greater stability and lower risk premia. In particular, if equity prices fall sharply ($y(t)$ rises) investors often equate the mean reversion effect in $y(t)$ with higher price rises: that is, the market will correct some of the fall in prices. This can be modelled in $\rho_p(t)$ by taking $\rho_1 > 0$.
- When interest rates are high we expect to see greater volatility and also anticipate increased risk premia. High risk premia are also associated with recent price falls, here as a result of an unanticipated rise in interest rates. This means that we should take $\rho_2 > 0$.
- We will consider the effect of $r_Q(t)$ on the risk premium through the factor ρ_3 later as its impact is not so clear.
- It is not obvious that changes in $r(t)$ or $r_Q(t)$ should have any delayed impact. We therefore assume that ρ_4 and ρ_5 will be equal to zero.
- We expect the risk premium to be larger when prices are more volatile. This is modelled here by including the term $\rho_6 \left(\sum_{j=1}^{n_1} \sigma_{41j}(t)^2 + \sigma_{44}(t)^2\right)$ with $\rho_6 > 0$. We note though that the level of volatility in equity prices might be linked to the level of volatility in bond prices which in turn is closely correlated with $r(t)$. To some extent, then, the risk-return link could be modelled through ρ_2 only with ρ_6.

We now write:

$$\begin{aligned} \mu_y(t) &= -\xi_1 \left(y(t) - \bar{y}(t)\right) \\ \text{where } \bar{y}(t) &= \xi_0 + \xi_2 r(t) + \xi_3 r_Q(t) + \xi_4 R(t) + \xi_5 R_Q(t) + \xi_6 \left(\sum_{j=1}^{n_1} \sigma_{41j}(t)^2 + \sigma_{44}(t)^2\right) \end{aligned}$$

$\bar{y}(t)$ represents the natural or target dividend yield for the current state of the economy. From equation (2) and the subsequent discussion we have

$$\xi_1 = -\theta_1 + \rho_1 - 1$$

We require that $\xi_1 > 0$ to ensure that $y(t)$ is mean reverting. We have already specified that $\theta_1 < 0$ and $\rho_1 > 0$. The new requirement thus means that $\rho_1 > \max\{1 + \theta_1, 0\}$. Given ξ_1 we then have

$$\begin{aligned}
\xi_0 &= (\theta_0 - \rho_0)/\xi_1 \\
\xi_2 &= (\theta_2 + \theta_3 - \rho_2 - 1)/\xi_1 \\
\xi_3 &= -(\theta_3 + \rho_3)/\xi_1 \\
\xi_4 &= (\theta_4 + \theta_5 - \rho_4)/\xi_1 = (-\theta_2 + \theta_5)/\xi_1 \text{ assuming } \theta_4 = -\theta_2 \text{ and } \rho_4 = 0 \\
\xi_5 &= -(\theta_5 + \rho_5)/\xi_1 = -\theta_5/\xi_1 \text{ assuming } \rho_5 = 0 \\
\xi_6 &= (1 - \rho_6)/\xi_1
\end{aligned}$$

We have not yet considered the value of ρ_3. Let us look first at the likely effect of $r_Q(t)$ on $\bar{y}(t)$. We associate low real yields on index-linked bonds with low dividend yields on equities. This implies that we should have $\xi_3 > 0$. But $\xi_3 = -(\theta_3 + \rho_3)/\xi_1$ and $\theta_3 > 0$, $\xi_1 > 0$, which implies that $\rho_3 = -\xi_1\xi_3 - \theta_3 < 0$: that is, low real yields should be associated with high risk premia. This is not an obvious relationship. However, it is a consequence of the likely effects of $r_Q(t)$ on dividends and dividend yields. With some thought we can see, in fact, that a possible consequence of falling yields is rising prices. This can be a result of an increased risk premium: that is, we should indeed have $\rho_3 < 0$.

6.1 Stationarity

Recall that we can write

$$\begin{aligned}
\frac{dy(t)}{y(t)} &= -\xi_1 \left(y(t) - \bar{y}(t)\right) dt + \sum_{i,j} \sigma_{6ij}(t) dZ_{ij}(t) \\
\Rightarrow y(t) &= y(0) \exp\left[\int_0^t \mu_y(s) ds - \frac{1}{2}\int_0^t \sigma_y(s)^2 ds + \sum_{i,j} \sigma_{6ij}(t) dZ_{ij}(t)\right] \qquad (3) \\
\text{where } \sigma_y(t)^2 &= \sum_{i,j} \sigma_{6ij}(t)^2
\end{aligned}$$

As $y(t) \to 0$, $\mu_y(t) \to \xi_1 \left[\xi_0 + \xi_2 r(t) + \xi_3 r_Q(t) + \xi_4 R(t) + \xi_5 R_Q(t) + \xi_6 \sigma_y(t)^2\right]$. It is not sufficient, that this limit is positive on average. Instead, we see from equation (3) that we must take into account the Ito correction $-\sigma_y(t)^2/2$. As a consequence, it is *necessary* that

$$\xi_1 \left[\xi_0 + \xi_2 r(t) + \xi_3 r_Q(t) + \xi_6 \sigma_y(t)^2\right] - \frac{1}{2}\sigma_y(t)^2$$

is positive on average. (Note that $R(t)$ and $R_Q(t)$ have been omitted from this equation as they have zero mean.) This condition would also be sufficient if $r(t)$, $r_Q(t)$ and the $\sigma_{6ij}(t)$ are all constant.

The condition helps to ensure that $y(t)$ does not tend to 0 as t tends to infinity. This has been confirmed by simulations where it was found that a small positive drift $\mu_y(t)$ when $y(t)$ was close to 0 was not sufficient to prevent $y(t)$ from getting smaller still.

If that the parameterisation ensures that $y(t)$ does not tend to zero and that $\mu_y(t)$ is negative for large $y(t)$, it normally follows that $y(t)$ is stationary. However, we can also note that with an appropriate choice of parameters $y(t)$, like the fixed-interest model, can move in long cycles. This means that the model can exhibit what appears to be non-stationary behaviour over quite long periods of time.

6.2 Further comments

Initial numerical tests suggest that inclusion of the $R(t)$ and $R_Q(t)$ factors do not affect substantially the results of a simulation. The most significant effect is that the inclusion of $\theta_5 R_Q(t)$ in the expression for expected dividend growth increases the level of volatility in that quantity. The impact on dividend yields and on equity returns is rather smaller.

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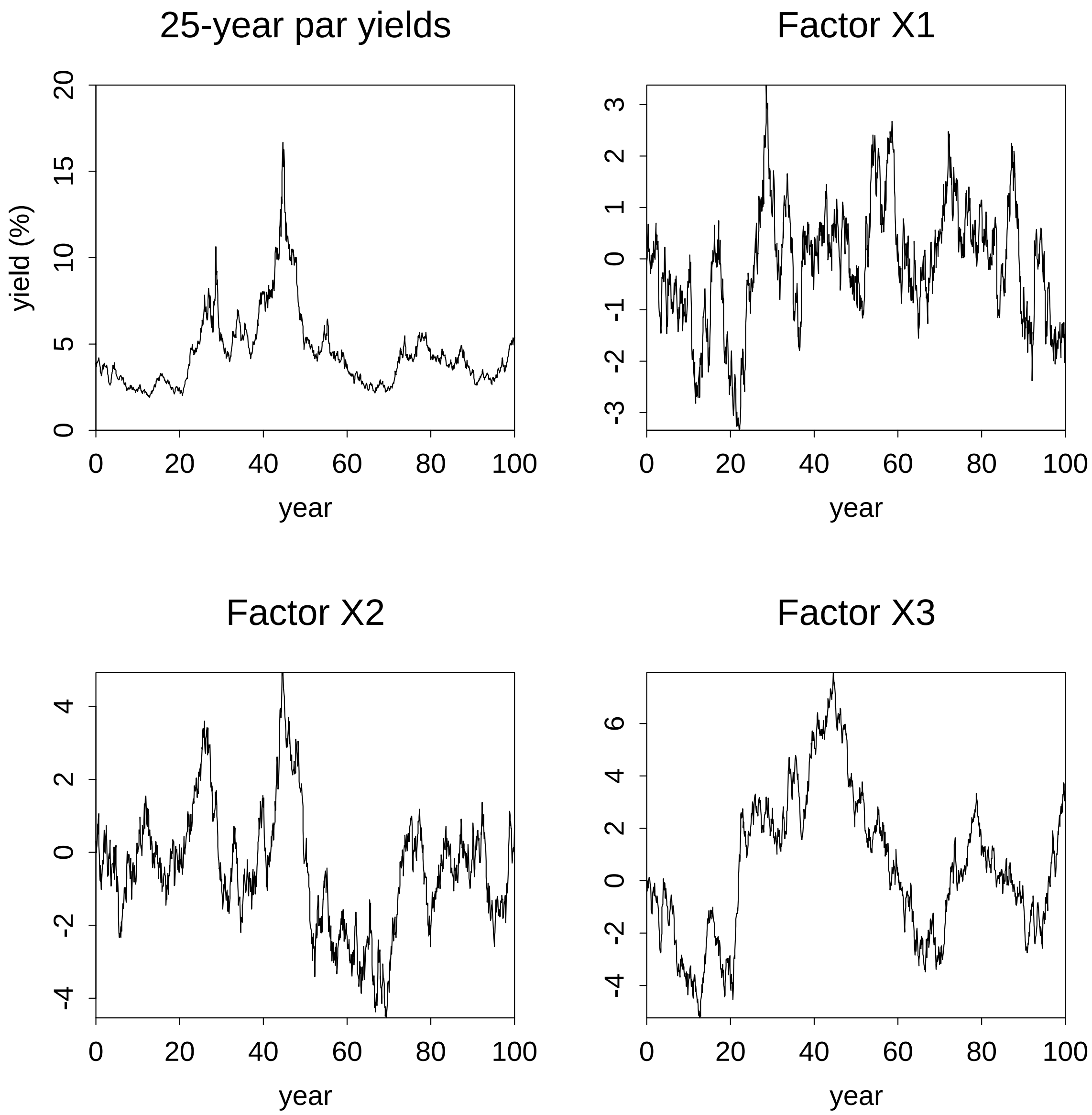

Figure 1: 25-year par yields. Top left: variation over a 100-year period. Top right and bottom left/right: variation of $\tilde{X}_{1i}(t)$ for $i = 1, 2, 3$.

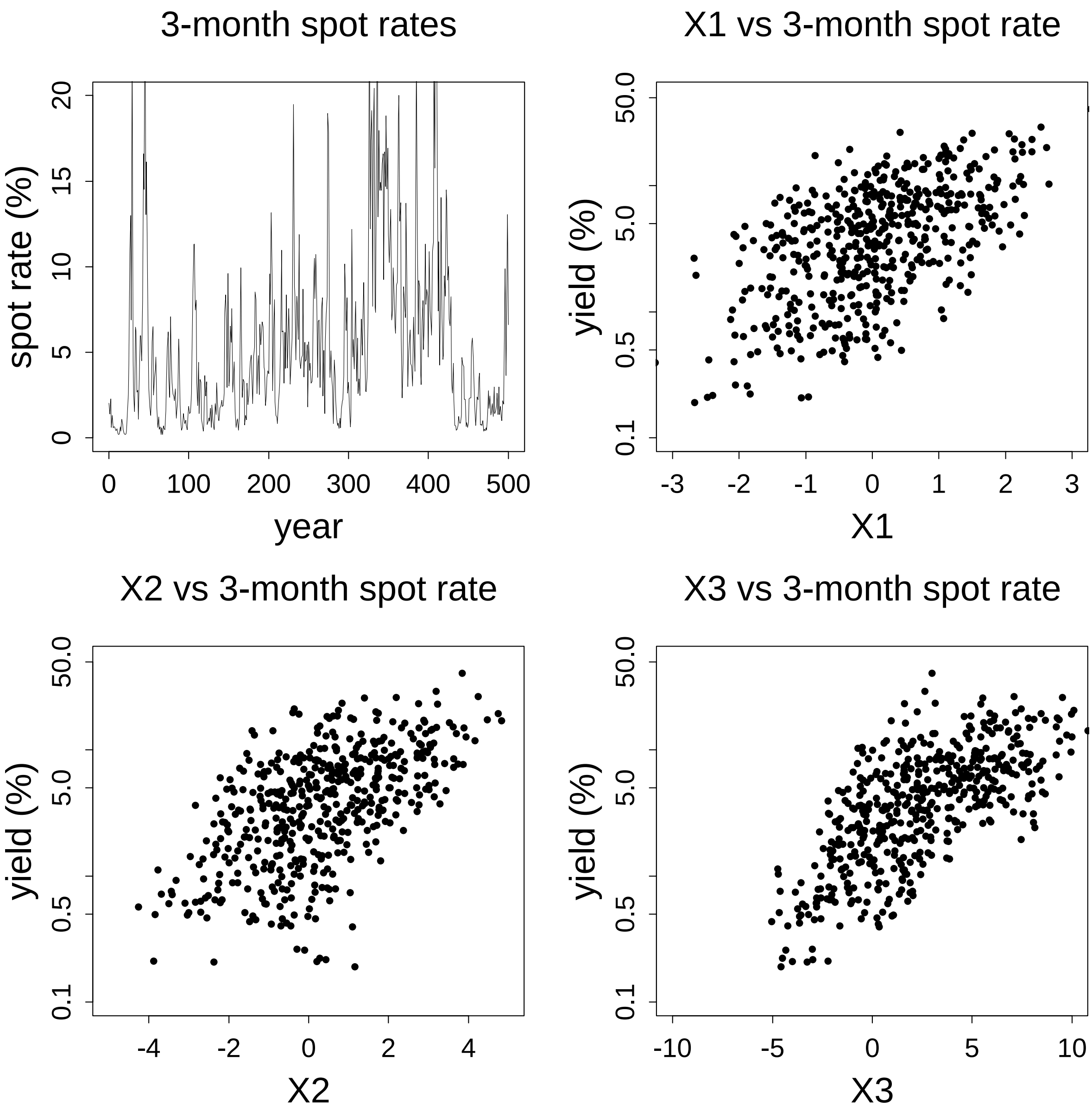

Figure 2: 25-year par yields. Top left: variation over a 500-year period. Top right and bottom left/right: Scatter plots of 25-year par yields against $\tilde{X}_{1i}(t)$ for $i = 1, 2, 3$.

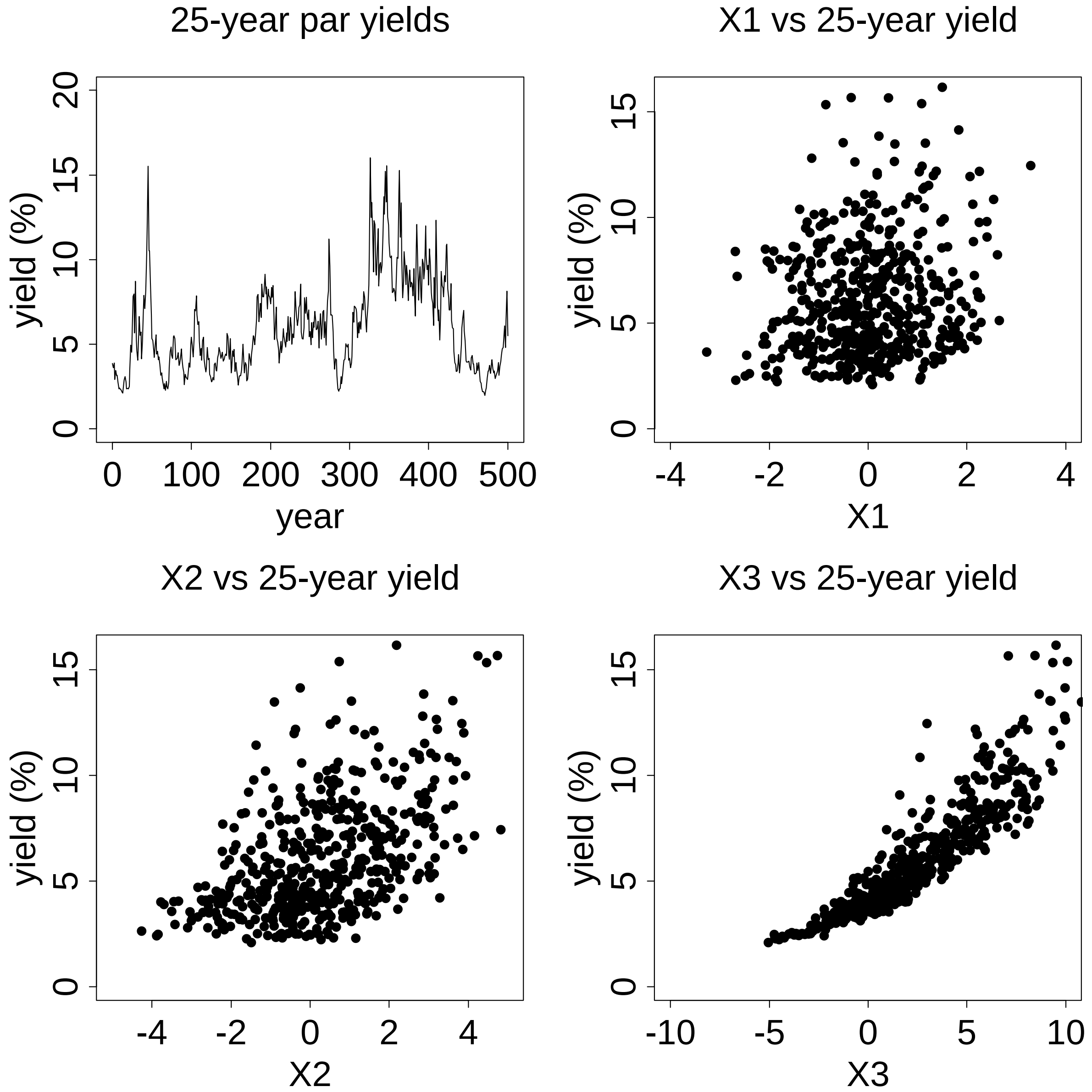

Figure 3: 3-month spot rates. Top left: variation over a 500-year period. (Some values ranging from 20% up to 45% have been cut off to allow us to make out more of the main detail.) Top right and bottom left/right: Scatter plots of 25-year par yields against $\tilde{X}_{1i}(t)$ for $i = 1, 2, 3$.

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Appendix A: The Ornstein-Uhlenbeck Process

Suppose $dX(t) = -\alpha X(t)dt + \sigma dZ(t)$.

Let $Y(t) = \exp(\alpha t)X(t)$. Then, by Ito's formula:

$$\begin{aligned} dY(t) &= \alpha e^{\alpha t}X(t)dt + e^{\alpha t}dX(t) \\ &= \sigma e^{\alpha t}dZ(t) \\ \Rightarrow Y(t) &= Y(0) + \sigma \int_0^t e^{\alpha u}dZ(u) \\ \Rightarrow X(t) &= e^{-\alpha t}X(0) + \sigma \int_0^t e^{-\alpha(t-u)}dZ(u) \end{aligned}$$

It follows that, for $t < s$, $X(s)$ given $\mathcal{F}_t$ is normally distributed with

$$\begin{aligned} E\left[X(s) \mid \mathcal{F}_t\right] &= e^{-\alpha(s-t)}X(t) \\ Var\left[X(s) \mid \mathcal{F}_t\right] &= \sigma^2 \frac{\left(1 - e^{-2\alpha(s-t)}\right)}{2\alpha} \end{aligned}$$