Valuation and Hedging of LPI Liabilities

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Abstract

This paper considers the market or economic valuation and the hedging of Limited Price Indexed (LPI) liabilities. This involves finding optimal static and dynamic hedging strategies which minimize the riskiness of the investment portfolio relative to the liability.

In this paper we do not aim to find the perfect hedge in a perfect world. Instead it is assumed that optimisation is restricted to three commonly-used asset classes in pension funds: cash; long-term (or irredeemable) fixed-interest bonds; and long-dated index-linked bonds. The paper then develops a workable but approximate method for actuaries suitable for hedging and valuation which is based upon mean-variance hedging and linear regression. This reduces the problem that theoretically optimal solutions are difficult to establish. The approach is illustrated with various numerical examples and we compare the results of the approximately-optimal hedging strategy with static strategies.

Keywords: LPI liability; static hedging; Regular rebalancing; dynamic hedging.

1 Introduction

In this paper, we investigate some hedging methods for particular kinds of pension liability. Unlike short-term hedging problems (for example, derivative pricing, hedging and reserving), a pension fund normally requires a long-term view because of the long-term nature of the liabilities. The levels of contribution income and benefit outgo are comparatively stable and predictable. Thus, pension fund investment strategies are less constrained by short-term considerations allowing the actuary and fund managers to focus on long-term investment decisions. Pension fund income comes from the investment returns, and the employer's and the employees' contributions. A central feature of the decision making process is the choice of assets. These must strike the right balance between risk and return: for example, by maximising the expected return subject to an acceptable level of risk.

For a pension fund to meet a particular liability, it is important to choose the most appropriate investment strategy. We can investigate the effects of adopting a variety of investment strategies, and, in particular, establish the likelihood of

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meeting the fund's objectives. Nowadays, simulation can be used to determine the possible effects of various investment strategies. According to the results, actuaries can make well informed suggestions for future contribution rates and the expected surpluses from pension funds and so choose a suitable investment strategy whch provides the best balance between risk and return for a particular scheme. (See, for example, Huang, 2000, Cairns & Parker, 1997, Cairns, 2000, Dufresne, 1988, 1989, 1990, Haberman & Sung 1994.)

Now consider the valuation of liabilities and assets. Until recently in the UK, values of both assets and liabilities were determined by discounting cash flows at an assumed rate which is normally taken to represent the future long-term rate of return on investments. In recent years, ideas arising in financial economics have been applied to the valuation of insurance related liability when taken along side the market value of the assets. This is similar to estimating market prices for liabilities (the so-called *fair value*). This is discussed further by Head *et al.* (2000). They proposed various methods that take assets into the balance sheet at market value. The principle of fair value is discussed further by Cairns (2001).

In a discrete-time model, since the number of outcomes after each time-step is infinite, the market is incomplete: that is, few of a pension fund's liabilities can be precisely matched or *replicated*[3]. As a *benchmark*[4] it is desirable to find the hedging strategy which minimises the level of risk associated with a specific liability. Where perfect hedging (matching) is possible (for example, Black & Scholes, 1973) the measure of risk is irrelevant. Under more realistic models, perfect hedging is not possible and the measure of risk is relevant. Here we use the concept of mean-variance hedging well known to actuaries (see, for example, Wise, 1984a,b, 1987a,b, 1989, Wilkie, 1985, and Keel & Müller, 1995). However, mean-variance hedging is even more firmly established within the field of financial mathematics (see, for example, Musiela & Rutkowski, 1997, Chapter 4, and references therein). Alternative measures of risk include Value-at-Risk (in the present context this means *quantile hedging*; see Föllmer & Leukert, 1999, 2000) and semi-variance (Clarkson, 1995). Ingeneral these different risk measures give rise to different hedging strategies and estimates of value. However, in many situations these differences are small and not worth the considerable argument which rages around them. Cairns (2001) argues why, out of these, mean-variance hedging is, perhaps, the most appropriate choice.

[3]If a liability or financial derivative can be *replicated* we mean the following. The appropriate initial investment in combination with a suitable dynamic hedging strategy using standard traded assets we are able to reproduce *exactly* (that is, with certainty) the liability cashflow or the derivative payoff without the need for further injections of cash (positive or negative).

[4]We stress the word *benchmark*. This is intended as an objective point of reference. The objectives of the pension fund may mean that a different investment mix from the benchmark is appropriate.

2 Basic principles

In order to develop our ideas further we introduce some notation.

- $V(t)$ represents the liability value at time t.
- $F(t^-)$ represents the fund size just *before* any net injections of cash at time t.
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- $S_1(t)$ is the value of a cash account at time t. In particular, $S_1(t+1)/S_1(t)$ equals the return over each year on the one-year, zero-coupon bond. It follows that $S_1(t+1)$ is known at time t. This gives rise to the commonly used notion that $S_1(t)$ represents the risk-free investment: that is, risk-free over each one-year time horizon.
- $S_k(t)$ for $k = 2, \ldots, m$ represents the value (with reinvestment of dividend or coupon income) of a unit investment at time 0 in risky asset k. (In this paper we will use $m = 3$.)
- We will assume that $S_i(0) = 1$ for all $i = 1, \ldots, m$.
- The vector $p(t) = (p_1(t), \ldots, p_m(t))'$ represents the proportions at time t invested in the various assets $i = 1, \cdots m$ with $\sum_{i=1}^m p_i(t) = 1$ for all $t = 0, \cdots T$. Besides its dependence upon t, the $p_i(t)$ can depend upon the fund size at time t, current market conditions or on the history of the process up to and including time t.

Using this notation we have:

$$F(t^-) \quad = \quad F(t-1) \sum_{i=1}^{m} p_i(t-1) \frac{S_i(t)}{S_i(t-1)}.$$

An asset-allocation strategy is *self-financing* if $F(t^-) = F(t)$ for $t = 1, 2, \ldots, T-1$.

Two kinds of mean-variance hedging are classified as follows (see for example, Musiela & Rutkowski, 1997):

- Variance-minimizing hedging
  This type of hedging assumes that an investment strategy is *self-financing* (that is, there are no external injections or removals of cash except at the outset) and concentrates on minimizing the *tracking error* at the terminal date only. In other words, we aim to minimise:

  $$Var\left(\frac{V(T) - F(T)}{S_1(T)}\right)$$

by choosing an appropriate initial fund size $\hat{F}(0)$, and asset strategies $\hat{p}(t)$.

When we are following a variance-minimising strategy $\hat{F}(0)$ is regarded as the appropriate price or value of the liability being hedged. Between times 0 and T the fund size is not likely to be equal to the price at that time.

- Risk-minimizing hedging (see, also, Schweizer and Föllmer, 1988)

  This type of hedging is more flexible since it is not necessarily self-financing. In this case, an optimality criterion is required at each date before the terminal date:
  The values of the liabilities are determined backwards-recursively. Let $V(t)$ be the value of the liability at time t. $F(t-1)$ represents the funds available at time $t-1$, invested in the proportions $p_i(t-1)$. We define $K(t)$ to be the shortfall (or tracking error) at time t: that is,

$$K(t) = V(t) - F(t^-) = V(t) - F(t-1)\sum_{i=1}^{m} p_i(t-1)\frac{S_i(t)}{S_i(t-1)}.$$

  The local risk-minimisation criterion requires that we minimise $Var[K(t)|\mathcal{F}_{t-1}]$ over $F(t-1)$ and the $p_i(t-1)$ subject to $E[K(t)|\mathcal{F}_{t-1}] = 0$ (where $\mathcal{F}_{t-1}$ represents the available information up to and including time $t-1$). (This is equivalent to the unconstrained minimisation $E[K(t)^2|\mathcal{F}_{t-1}]$.) The optimal $\hat{F}(t-1)$ (given $\mathcal{F}_{t-1}$) then provides us with our liability value at time $t-1$.[5]

  At each time t additional finance of $K(t)$ is provided to ensure that the fund size is at all times equal to the value of the liability: that is, $F(t) = F(t^-) + K(t) = V(t)$. Thus, in contrast to variance-minimising hedging, this strategy is not self financing.

In Sections 4 and 5 we discuss the cases of static hedging and regular rebalancing hedging to predetermined proportions. In Sections 6, 7 and 8 we investigate the possibility of dynamic hedging for certain liabilities and have comparisons of the hedging efficiency among static hedging, regular rebalancing hedging, and dynamic hedging. In this paper we focus the investigation of the hedging strategy on Limited Price Indexation (LPI) liabilities.

3 The definition of LPI

The valuation of the accrued liability is an important part of a pension scheme funding valuation. In recent years, Limited Price Indexation (LPI) has, by law,

[5] In a complete market there will always exist suitable processes $V(u)$ and $p(u)$ such that the optimised $K(t)$ are all equal to zero with certainty. Under such circumstances the optimal strategy will be self financing and replicating.

become a necessary feature of UK pension schemes. There are several forms of LPI and we describe two of them:

- Type 1 is the limited indexation of pensions in deferment introduced in the 1986 Pensions Act. Deferred pensions of early leavers from final salary pension schemes, are increased at the lesser of some cap rate (UK 5% per annum) compounded over the whole period, and the actual increase in the Retail Prices Index in the UK (consumer prices index in a more international context) again measured over the full period. For an early leaver at time t retiring at a later time T the pension payable at a time T will be:

$$pen(T) = \min\left\{\frac{RPI(T)}{RPI(t)}, 1.05^{T-t}\right\} \times pen(t)$$

  where $pen(t)$ is the deferred pension calculated at the date of exit before statutory revaluation and $RPI(t)$ is the value of the Retail Prices Index at time t.

- Type 2 is the limited indexation of pensions once in payment, introduced under the 1990 Social Security Act. Under this type of indexation, a comparison is made year-on-year between the cap rate (UK 5% pa) and the annual increase in the Retail Prices Index and the lesser of the two increases is awarded. The pension at time t, $pen(t)$, is then:

$$pen(t) = pen(t-1) \times min\left\{\frac{RPI(t)}{RPI(t-1)}, 1.05\right\}$$

For some cases, we can additionally apply a floor (such as 0% increases per annum) to Types 1 and 2 as well as a ceiling. We will not consider this possibility because breaches of typical floors are not common in practice. In contrast to liabilities in respect of active employees, LPI liabilities are well defined (the former are subject to argument over the extent of salary risk and over the division between past and future service liabilities).

In this paper we will consider three types of hedging in order to establish a value for an LPI liability of type 2: static; rebalancing; and dynamic hedging. Optimal strategies in each case will be derived for three types of liability: fixed pension increases; fully index-linked (RPI) pensions; and LPI pensions.

4 Hedging strategies considered

4.1 Static hedging

In this section we investigate the use of static hedging to establish a first approximation to the value of an LPI liability. The aim is to achieve this using the standard asset classes used by pension funds rather than achieve perfect matching of assets and liabilities using a more complex and detailed range of bonds. Here static hedging means we hold a fixed quantity of each asset over the full term, n (that is, buy and hold). In investigating the LPI liability, we consider long-dated index-linked bonds, consols (that is, UK, irredeemable, fixed-interest bonds) and cash in the portfolio. Compared to index-linked bonds, consols may be more volatile relative to inflation. However, since LPI includes a fixed cap, consols may be found to provide a useful asset for hedging outcomes where the cap locks in (see, for example, Cairns, 1999).

Because an LPI liability falls between two other liabilities, to investigate the hedging for LPI liability, it is useful for us to consider these two liabilities which are: Fixed Percentage (FP) liability; and Retail Price Indexation (RPI) liability.

For an FP liability, the pension in payment is subject to fixed percentage increases, r, (for example, 5%) each year. That is,

$$pen(t) = (1 + r)^t \times pen(0)$$

For an RPI liability, the pension in payment increases in line with the retail prices index: that is,

$$pen(t) = \frac{RPI(t)}{RPI(0)} \times pen(0)$$

For an LPI liability, the pension in payment, $pen(t)$, is governed by the equation:

$$pen(t) = pen(t-1) \times min\left\{\frac{RPI(t)}{RPI(t-1)}, 1 + r\right\}$$

We will assume here that $r = 0.05$ or 5% in line with UK regulations.

With an FP liability, we consider three cases of investment strategies in the portfolio:

- Holding cash only in the portfolio.
- Holding consols only in the portfolio.
- Holding cash, consols and index-linked bonds in the portfolio.

With an RPI liability, we consider two investment strategies in the portfolio:

- Holding index-linked bonds only in the portfolio.
- Holding cash, consols and index-linked bonds in the portfolio.

With an LPI liability, we consider only one case in the portfolio:

- Holding cash, consols and index-linked bonds in the portfolio.

Let $S_1(t), S_2(t), S_3(t)$ be the prices at time t of one unit of cash, irredeemable fixed-interest bonds (consols) and irredeemable index-linked bonds, with $S_i(0) = 1$ for $i = 1, 2, 3$. We now define: $R_i(t) = S_i(t)/S_i(t-1)$ = total return on asset i from $t-1$ to t
with $i = 1$ (cash), $i = 2$ (consols) and $i = 3$ (I-L bonds).

We also define $M^\alpha(t)$ = liability index of type α, where $\alpha = F, R, L$ represents the type of liability (fixed, RPI and LPI respectively). Thus, starting with $M^\alpha(0) = 1$, we have, for $t = 1, 2, \ldots$:

$$M^\alpha(t) = \begin{cases} (1+r)M^\alpha(t-1) = (1+r)^t & \text{for } \alpha = F \\ \frac{RPI(t)}{RPI(t-1)} M^\alpha(t-1) = \frac{RPI(t)}{RPI(0)} & \text{for } \alpha = R \\ \min\left(\frac{RPI(t)}{RPI(t-1)}\ ,\ 1+r\right) M^\alpha(t-1) & \text{for } \alpha = L \end{cases}$$

We now define the objective function for a liability of $M^\alpha(T)$ due at time T:

$$SB = Var\left(\frac{\sum_{i=1}^3 x_i^\alpha S_i(T) - M^\alpha(T)}{S_1(T)}\right) + \theta\left(E\left[\frac{\sum_{i=1}^3 x_i^\alpha S_i(T) - M^\alpha(T)}{S_1(T)}\right]\right)^2 \quad (4.1)$$

We minimise SB over the x_i^α (the amounts invested at time 0 in each asset) to obtain the optimal asset-allocation strategy. This approach is similar to that of Wise (1984a,b, 1987a,b, 1989) and Wilkie (1985).

We can immediately note that x_1^α has no impact on the variance $Var\left([\sum_{i=1}^3 x_i^\alpha S_i(T) - M^\alpha(T)]/S_1(T)\right)$. Thus if $\theta > 0$ then the optimal solution is to minimise the Variance over x_2^α and x_3^α first, before choosing x_1^α in a way which ensures that the expectation $E\left([\sum_{i=1}^3 x_i^\alpha S_i(T) - M^\alpha(T)]/S_1(T)\right) = 0$. There is no unique solution when $\theta = 0$.

It is convenient to consider the case $\theta = 1$ when the objective SB is to minimize the expected value of the square of the tracking error: that is,

$$E\left[\left(\frac{\sum_{i=1}^3 x_i^\alpha S_i(T) - M^\alpha(T)}{S_1(T)}\right)^2\right]$$

For notational convenience, we define:

$$\begin{aligned}
\psi_i &= E\left[\frac{S_i(T)}{S_1(T)}\right] \\
\omega_{i,j} &= Cov\left(\frac{S_i(T)}{S_1(T)}, \frac{S_j(T)}{S_1(T)}\right) \\
\gamma_i^\alpha &= Cov\left(\frac{S_i(T)}{S_1(T)}, \frac{M^\alpha(T)}{S_1(T)}\right) \\
\psi_0^\alpha &= E\left[\frac{M^\alpha(T)}{S_1(T)}\right] \\
\omega_0^\alpha &= Var\left[\frac{M^\alpha(T)}{S_1(T)}\right]
\end{aligned}$$

where $i, j = 1, 2, 3$ and $\alpha = F, R, L$.

For the purpose of developing the optimal asset allocation for objectives SB, we now define:

$$\begin{aligned}
X^\alpha &= (x_1^\alpha, x_2^\alpha, x_3^\alpha)' \\
\Omega &= (\omega_{i,j})_{i,j=1}^3 \\
\Gamma^\alpha &= (\gamma_1^\alpha, \gamma_2^\alpha, \gamma_3^\alpha)' \\
\Psi &= (\psi_1, \psi_2, \psi_3)'
\end{aligned}$$

Assume that $\theta = 1$. SB is quadratic in x_1, x_2 and x_3: that is,

$$SB = X^{\alpha\prime}\Omega X^\alpha - 2X^{\alpha\prime}\Gamma^\alpha + \omega_0^\alpha + \left(X^{\alpha\prime}\Psi - \psi_0^\alpha\right)^2$$

Since the $\omega_{ij} = 0$ and $\gamma_i^\alpha = 0$ whenever $i = 0$ or $j = 0$. The solution to this is:

$$\begin{pmatrix} x_2^\alpha \\ x_3^\alpha \end{pmatrix} = \hat{\Omega}^{-1}\hat{\Gamma}^\alpha \quad \text{where } \hat{\Omega} = \begin{pmatrix} \omega_{22} & \omega_{23} \\ \omega_{32} & \omega_{33} \end{pmatrix} \text{ and } \hat{\Gamma}^\alpha = \begin{pmatrix} \gamma_2^\alpha \\ \gamma_3^\alpha \end{pmatrix}.$$

$$\text{Then } x_1^\alpha = \psi_1^{-1}\left(\psi_0^\alpha - \sum_{i=2}^{3} x_i^\alpha \psi_i\right).$$

4.2 Dynamic hedging

4.2.1 Introduction

We generally find that the optimal static hedge has relative proportions in each asset which vary with time to payment T and also with the prevailing economic conditions at time 0. This indicates that hedging strategies which vary over both time and with changing economic conditions will outperform static hedges.

The particular economic conditions we will make use of are as follows:
$y_1(t)$ = log inflation rate from time $t-1$ up to time t;
$y_2(t)$ = log dividend yield at time t;
$y_3(t)$ = log consols yield at time t;
$y_4(t)$ = the 1-year risk-free rate of interest from t to $t+1$ *plus 1*
$y_5(t)$ = log index-linked yield at time t;
$y(t) = (y_1(t), \ldots, y_5(t))'$.
(This choice of economic factors reflects the later use in this paper of the Wilkie (1995) model. Other selections could be used where appropriate.)

A simulation approach for finding the optimal asset allocation for the one-year LPI liability is discussed by Dai (1998). Here we develop an approach which extends his work to the multi-period case. In an attempt to avoid excessive use of simulation, we aim to find simple formulae which give good (and, we hope, practical) approximations to the optimal asset allocation. To find these formulae for an LPI liability, we first investigate fixed and RPI liabilities in the knowledge that the LPI liability is strongly related to both cases. For convenience, we disregard transaction costs associated with rebalancing in the model.

Before constructing formulae for approximating the optimal asset allocation for LPI liabilities, we need to build some fundamental models. Section 5.1 provides a multivariate regression model to estimate the Wilkie Model. In Section 5.2 we build a model for approximating the value of the liability which is a linear function of the inflation rate, log(dividend yield), log(consols yield), one-year zero-coupon rate and log(index-linked yield). These two models provide sufficient information to construct the formulae for asset allocations for dynamic hedging.

In Section 6 we set up the steps for finding the optimal asset allocation for the linearized liability. Section 7 displays some numerical results for both FP and RPI liabilities. In Section 8, we determine a methodology for obtaining an approximation to the optimal asset allocation for an LPI liability. In Section 8.1 we propose a model to connect both FP and RPI liabilities with an LPI liability. Section 8.2 describes the method used to test the efficiency of dynamic hedging. Section 8.3 discusses the optimal asset allocation for FP, RPI and LPI liabilities. In Section 8.4 we assess if the theoretical method is sufficiently accurate for estimating the LPI liability.

In Section 9 we list some numerical results and make comparisons among static hedging, regular rebalancing and dynamic hedging strategies for various LPI liabilities.

4.2.2 True optimization for dynamic hedging

For an LPI liability, we consider cash, consols and irredeemable index-linked bonds as the available assets for hedging. Index-linked bonds and consols are risky assets in the portfolio: that is, unlike the cash account (one-year zero-coupon bonds), the values of these investments are not known one year in advance. We suppose that the economic model governing future liabilities and asset returns is Markov and time homogeneous. We then let the vector $y(t)$ represent relevant market conditions at time t: that is, knowledge of $y(t)$ is sufficient for probability forecasts of the future.

Recall that $R_i(t)$ is the total return on asset i from $t-1$ to t. Define the vector $R(t) = (R_1(t), R_2(t), R_3(t))'$.

Let $\tilde{x}_j^\alpha(t, y(t), T-t)$ be the optimal amount of asset j for a true optimization at time t for an α-liability due in $T-t$ years given market conditions $y(t)$ at time t.

Let $\bar{x}_j^\alpha(t, y(t), T-t)$ be the approximately optimal amount in asset j based on a linearised optimisation procedure (still to be described) at time t given $y(t)$ and $T-t$ years to payment.

Let:

- $\tilde{V}^\alpha(t, y(t), T-t)$ be the true optimal value at t for the liability $M^\alpha(T)$ payable at time T given $y(t)$;
- $\bar{V}^\alpha(t, y(t), T-t)$ be the approximate optimal value at t given $y(t)$ for this liability based upon a series of linear approximations (still to be described) between t and T.

Now the economic model $y(t)$ is Markov and time homogeneous. It follows that $\tilde{V}^\alpha(t, y(t), T-t) = M^\alpha(t)\tilde{V}^\alpha(0, y(t), T-t)$ meaning that it is sufficient for us to establish the form of $\tilde{V}^\alpha(0, y, T-t)$ only. Similarly, $\bar{V}^\alpha(t, y(t), T-t) = M^\alpha(t)\bar{V}^\alpha(0, y(t), T-t)$, $\tilde{x}^\alpha(t, y(t), T-t) = M^\alpha(t)\tilde{x}^\alpha(0, y(t), T-t)$ and $\bar{x}^\alpha(t, y(t), T-t) = M^\alpha(t)\bar{x}^\alpha(0, y(t), T-t)$. We also note that given $y(t) = y$, $M^\alpha(t) = M^\alpha$ we can write $M^\alpha(t+1) \stackrel{\mathcal{D}}{=} M^\alpha \times M^\alpha(1)$ given $y(0) = y$.

The $\tilde{x}_i^\alpha(t, y(t), T-t)$ and $\tilde{V}^\alpha(t, y(t), T-t)$ are established by means of a backwards recursion starting at $t = T-1$ and stepping backwards a year at a time to $t = 0$. Thus, for a general t, we aim to choose $\tilde{x}_i^\alpha(t, y(t), T-t)$ so that

$$E\left[\left(\sum_{i=1}^{3} \tilde{x}_i^\alpha(t, y(t), T-t)R_i(t+1) - \tilde{V}^\alpha(t+1, y(t+1), T-t-1)\right)^2 \,\middle|\, y(t) = y\right]$$

is minimized. Using the Markov property with $M^\alpha(t) = M^\alpha$ this is equivalent to minimisation of:

$$E\left[\left(\sum_{i=1}^{3} M^\alpha \times \tilde{x}_i^\alpha(0, y, T-t)R_i(1) - M^\alpha \times M^\alpha(1)\tilde{V}^\alpha(0, y(1), T-t-1)\right)^2 \,\middle|\, y(0) = y\right]$$

With the optimal asset allocation at time t, $\tilde{x}_i^\alpha(t, y(t), T-t)$, we then define the economic value of the liability at time t as:

$$\tilde{V}_i^{\alpha}(t, y(t), T-t) = \sum_{j=1}^{3} \tilde{x}_j^{\alpha}(t, y(t), T-t)$$

Then, we reduce t by 1 and find the optimal asset allocation at time $t-1$.

This approach is similar to the Risk-Minimisation approach proposed by Schweizer and Föllmer (1988).

To develop formulae for the optimal asset allocation for the LPI liability, we need to set up some economic models.

5 Model structure and assumptions for hedging

For the purpose of finding the optimal asset allocation for LPI liabilities, we need some underlying models for the additional mathematical calculations. In this section we use multivariate regression to build a vector autoregressive model for the market variables and a linear estimated liability model.

5.1 The vector autoregressive model

Let us suppose that there is some underlying stochastic economic model such as the Wilkie (1995) model or the TY model (Yakoubov, Teeger & Duval, 1999). Often such models are sufficiently complex to render optimisation infeasible. Here we propose the use of a simple vector autoregressive model (VAR(1)) as an approximation to these more complex models. The VAR model is fitted by the use of multivariate regression on simulated data generated by the more complex model. For the three assets under consideration the Wilkie (1995) model requires the following 5 drivers:

$$\begin{aligned}
y_1(t) &= \text{(annualised) inflation rate from } t-1 \text{ to } t \\
y_2(t) &= \text{(historical) log dividend yield at } t \\
y_3(t) &= \text{log consols yield at } t \\
y_4(t) &= 1/(\text{price at } t \text{ for zero-coupon bond maturing at } t+1) \\
&= 1 + \text{risk-free interest rate from } t \text{ to } t+1 \\
y_5(t) &= \text{log real yield on IL bonds at } t.
\end{aligned}$$

For other models this set may be larger or smaller than 5 and may contain different elements.

Consider a long simulation run using the underlying stochastic economic model running from time 0 to time N. This gives us values for the market indicators $y_i(t)$ for $i = 1, \ldots, 5$ and $t = 0, 1, \ldots, N$ and for the total returns $R_i(t)$ for $i = 1, 2, 3$ and $t = 1, \ldots, N$. Let:

$$\begin{aligned}
X(t) &= (X_1(t), \ldots, X_5(t))' \\
Y(t) &= (Y_1(t), \ldots, Y_8(t))' \\
\text{where } X_i(t) &= y_i(t-1) \quad \text{for } i = 1, \ldots, 5 \\
\text{and } Y_i(t) &= \begin{cases} y_i(t) & \text{for } i = 1, \ldots, 5 \\ R_{i-5}(t) & \text{for } i = 6, 7, 8 \end{cases}
\end{aligned}$$

Let μ_{Xi} be the unconditional mean of $X_i(t)$ and μ_{Yi} be the unconditional mean of $Y_i(t)$. Then for $i = 1, \ldots, 5$ we have $\mu_{Xi} = \mu_{Yi}$.

This leads us to the following mutivariate regression model:

$$Y(t) - \mu_Y = A(X(t) - \mu_X) + \epsilon(t) \quad \text{for } t = 1, \ldots, N$$

where $\epsilon(t) = (\epsilon_1(t), \ldots, \epsilon_8(t))'$ and $\epsilon(1), \ldots, \epsilon(N)$ are i.i.d. $\sim MVN(\underline{0}, C_\epsilon)$, $A = (A_{ij})$ is an 8×5 matrix and $C_\epsilon = (C_{ij})$ is an 8×8 covariance matrix.
For convenience later we will write:

$$A = \begin{pmatrix} A_y \\ A_R \end{pmatrix}, \qquad \mu_Y = \begin{pmatrix} \mu_y \\ \mu_R \end{pmatrix}, \qquad \text{and} \quad \epsilon(t) = \begin{pmatrix} \epsilon^{(y)}(t) \\ \epsilon^{(R)}(t) \end{pmatrix}$$

where A_y is the first 5 rows of A, A_R is the last 3 rows of A, $\mu_y = \mu_X$ is the first 5 elements of μ_Y, μ_R is the last 3 elements of μ_Y, $\epsilon^{(y)}(t)$ is the first 5 elements of $\epsilon(t)$ and $\epsilon^{(R)}(t)$ the last 3 elements of $\epsilon(t)$. We can then write:

$$\begin{aligned} y(t) &= \mu_y + A_y[y(t-1) - \mu_y] + \epsilon^{(y)}(t) \\ \text{and} \quad R(t) &= \mu_R + A_R[y(t-1) - \mu_y] + \epsilon^{(R)}(t) \end{aligned}$$

Now let $U = (U_{ij})$ be the 5×5 simulation covariance matrix for $X(t)$ and $W = (W_{ij})$ be the 8×5 covariance matrix for $Y(t)$ and $X(t)$: that is,

$$\begin{aligned} U_{ij} &= \frac{1}{N}\sum_{k=1}^{N}(X_i(k) - \hat{\mu}_{Xi})(X_j(k) - \hat{\mu}_{Xj}) \quad \text{for } i,j = 1,\ldots,5 \\ W_{ij} &= \frac{1}{N}\sum_{k=1}^{N}(Y_i(k) - \hat{\mu}_{Yi})(X_j(k) - \hat{\mu}_{Xj}) \quad \text{for } i = 1,\ldots,8 \text{ and } j = 1,\ldots,5 \\ \text{where} \quad \hat{\mu}_{Xi} &= \frac{1}{N}\sum_{k=1}^{N} X_i(k) \quad \text{for } i = 1,\ldots,5 \\ \text{and} \quad \hat{\mu}_{Yi} &= \frac{1}{N}\sum_{k=1}^{N} Y_i(k) \quad \text{for } i = 1,\ldots,8 \end{aligned}$$

Then (for example, see Srivasta & Carter, 1983):

$$\hat{A} = WU^{-1}$$

Now let $\hat{\epsilon}(k) = Y(k) - \hat{\mu}_Y - \hat{A}(X(k) - \hat{\mu}_X)$. Then:

$$\begin{aligned} \hat{C}_\epsilon &= (\hat{C}_{ij}) = \frac{1}{N}\sum_{k=1}^{N} \hat{\epsilon}(k)\hat{\epsilon}(k)' \\ \text{or} \quad \hat{C}_{ij} &= \frac{1}{N}\sum_{k=1}^{N} \hat{\epsilon}_i(k)\hat{\epsilon}_j(k) \quad \text{for } i,j = 1,\ldots,8 \end{aligned}$$

5.2 The Linear Estimated Liabilities Model

In this section we set up a model for FP, RPI and LPI liabilities for developing suitable formulae for asset allocation. In the methodology of dynamic hedging we use the backward recursion method to derive the asset allocation. For convenience, we choose to approximate the liabilities by linear functions of the market indicators $y_1(t), \ldots, y_5(t)$. Using such a linear approximation in combination with the VAR model, we can easily obtain formulae for the optimal asset allocation year-by-year from the last year of the term by using the backward recursion method.

Using Taylor's expansion, we have:

$$\begin{aligned}\tilde{V}^\alpha(0, y(t), T-t) &= \tilde{V}^\alpha(0, \mu_y, T-t) + \left(\left. \frac{\partial \tilde{V}^\alpha(0, y, T-t)}{\partial y} \right|_{y=\mu_y} \right)' (y(t) - \mu_y) \\ &\quad + o(|y(t) - \mu_y|)\end{aligned}$$

If the error term $o(|y(t) - \mu_y|)$ is small, we can disregard it and consider the liability as a linear function of $y(t)$: that is,

$$\begin{aligned}\tilde{V}^\alpha(0, y, T-t) &\approx a^\alpha_{T-t}(y - \mu_y) + b^\alpha_{T-t} \\ \text{where} \quad a^\alpha_{T-t} &= (a^\alpha_{1,T-t}, a^\alpha_{2,T-t}, a^\alpha_{3,T-t}, a^\alpha_{4,T-t}, a^\alpha_{5,T-t}) \\ &= \left. \frac{\partial \tilde{V}^\alpha(0, y, T-t)}{\partial y} \right|_{y=\mu_y} \\ b^\alpha_{T-t} &= \tilde{V}^\alpha(0, \mu_y, T-t).\end{aligned}$$

We define $\bar{V}^\alpha(0, y, T-t) = a^\alpha_{T-t}(y - \mu) + b^\alpha_{T-t}$ to be the linear approximation.

6 Finding the optimal asset allocation

In this section we consider the fixed and RPI liabilities ($\alpha = F$ and $\alpha = R$ only) and set up the steps for finding the approximate optimal asset allocation for the linear approximation liabilities $\bar{V}^\alpha(0, y, T-t)$ and $\bar{V}^\alpha(t, y(t), T-t) = M^\alpha(t)\bar{V}^\alpha(0, y(t), T-t)$:

Step 1: Find $\bar{x}_i^\alpha(t-1, y(t-1), T-t+1)$.
For a general t, we first aim to find the $\bar{x}_i^\alpha(t-1, y(t-1), T-t+1)$ for $i = 1, 2, 3$ which minimizes:

$$E\left[\left(\sum_{i=1}^{3} \bar{x}_i^\alpha(t-1, y(t-1), T-t+1)R_i(t) - \bar{V}_i^\alpha(t, y(t), T-t)\right)^2 \middle| y(t-1)\right]$$

Since $\bar{x}_i^\alpha(t-1, y(t-1), T-t+1) = M^\alpha(t-1)\bar{x}_i^\alpha(0, y(t-1), T-t+1)$ and $\bar{V}^\alpha(t, y(t), T-t) = M^\alpha(t)\bar{V}^\alpha(0, y(t), T-t)$ this is equivalent to minimisation of:

$$E\left[\left(\sum_{i=1}^{3} \bar{x}_i^\alpha(0, y, T-t+1)R_i(1) - M^\alpha(1)\bar{V}^\alpha(0, y(1), T-t)\right)^2 \middle| y(0) = y\right].$$

For the purpose of constructing formulae for the optimal asset allocation, we need to linearize $M^\alpha(1)\bar{V}^\alpha(0, y(1), T-t)$: that is,

$$M^\alpha(1)\bar{V}^\alpha(0, y(1), T-t) \approx \hat{a}_{T-t}^\alpha(y(1) - \mu_y) + \hat{b}_{T-t}^\alpha$$

where $\hat{a}_{T-t}^\alpha$ is a row vector where:

$$\begin{aligned} \hat{a}_{j,T-t}^\alpha &= \frac{\partial}{\partial y_j(1)}\Big(M^\alpha(1)\bar{V}^\alpha(0, y(1), T-t)\Big)\Big|_{y(1)=\mu_y} \\ \hat{b}_{T-t}^\alpha &= M^\alpha(1)\bar{V}^\alpha(0, y(1), T-t)\big|_{y(1)=\mu_y}. \end{aligned}$$

Note that the $\hat{a}_{j,s}^\alpha$ and $\hat{b}_s^\alpha$ are different from the $a_{j,s}^\alpha$ and b_s^α.
Since $R_1(1)$ (the return on cash) is known at time 0, for any $\bar{x}_2^\alpha(0, y(0), T-t+1)$ and $\bar{x}_3^\alpha(0, y(0), T-t+1)$ we can find an $\bar{x}_1^\alpha(0, y(0), T-t+1)$ for which:

$$E\left[\sum_{i=1}^{3} \bar{x}_i^\alpha(t-1, y(t-1), T-t+1)R_i(t) \right.$$
$$\left. -M^\alpha(t-1)\left\{\hat{a}_{T-t}^\alpha(y(t) - \mu_y) - \hat{b}_{T-t}^\alpha\right\} \middle| y(t-1), M^\alpha(t-1)\right] = 0.$$

Thus, step 1 is equivalent to minimization of:

$$Var\left[\sum_{i=2}^{3} \bar{x}_i^\alpha(t-1, y(t-1), T-t+1)R_i(t) \right.$$
$$\left. -M^\alpha(t-1)\left\{\hat{a}_{T-t}^\alpha(y(t)-\mu_y)-\hat{b}_{T-t}^\alpha\right\}\middle| y(t-1), M^\alpha(t-1)\right]$$

with $\bar{x}_1^\alpha(t-1, y(t-1), T-t+1)$ chosen subsequently to satisfy:

$$E\left[\sum_{i=1}^{3} \bar{x}_i^\alpha(t-1, y(t-1), T-t+1)R_i(t) \right.$$
$$\left. -M^\alpha(t-1)\left\{\hat{a}_{T-t}^\alpha(y(t)-\mu_y)-\hat{b}_{T-t}^\alpha\right\}\middle| y(t-1), M^\alpha(t-1)\right] = 0.$$

Recall (Section 5.1) that the model of Multivariate Regression is given by:

$$\begin{aligned} y(t) &= \mu_y + A_y[y(t-1)-\mu_y] + \epsilon^{(y)}(t) \\ \text{and} \quad R(t) &= \mu_R + A_R[y(t-1)-\mu_y] + \epsilon^{(R)}(t) \\ \text{where} \quad \epsilon(t) &= \begin{pmatrix} \epsilon^{(y)}(t) \\ \epsilon^{(R)}(t) \end{pmatrix} \sim MVN(0, C_\epsilon). \end{aligned}$$

Thus, we have:

$$\begin{aligned} M^\alpha(1)\bar{V}_i^\alpha(0, y(1), T-t) &\approx \hat{a}_{T-t}^\alpha(y(1)-\mu_y) + \hat{b}_{T-t}^\alpha \\ &= \hat{a}_{T-t}^\alpha\left(A_y(y(0)-\mu_y)+\epsilon^{(y)}(1)\right) + \hat{b}_{T-t}^\alpha. \end{aligned}$$

According to the normal approximation, the minimization of the objective function:

$$Var\left(\sum_{i=2}^{3} \bar{x}_i^\alpha(0, y, T-t+1)R_i(1) - \hat{a}_{T-t}^\alpha(y(1)-\mu_y) - \hat{b}_{T-t}^\alpha \middle| y(0)=y\right)$$

can be rewritten as (abbreviating $\bar{x}_k^\alpha(0, y, T-t+1)$ by z_k):

$$\begin{aligned} & Var\left(z_2\epsilon_7(1) + z_3\epsilon_8(1) - \hat{a}_{T-t}^\alpha\epsilon^{(y)}(1)\right) \\ = \; & (-1, z_2, z_3)\begin{pmatrix} \tilde{C}_{11}(T-t+1) & \tilde{C}_{17}(T-t+1) & \tilde{C}_{18}(T-t+1) \\ \tilde{C}_{17}(T-t+1) & C_{77} & C_{78} \\ \tilde{C}_{18}(T-t+1) & C_{78} & C_{88} \end{pmatrix}\begin{pmatrix} -1 \\ z_2 \\ z_3 \end{pmatrix} \end{aligned}$$

where the optimal values for z_2 and z_3 give us $\bar{x}_i^\alpha(0, y, T-t+1)$ for $i = 2, 3$.

For convenience we write $Z = (z_2, z_3)'$,

$$\tilde{C}_{11}(T-t+1) = \hat{a}^{\alpha}_{T-t} \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{pmatrix} \hat{a}^{\alpha}_{T-t}{}'$$

$$\tilde{C}_{17}(T-t+1) = \hat{a}^{\alpha}_{T-t} \begin{pmatrix} C_{17} \\ C_{27} \\ C_{37} \\ C_{47} \\ C_{57} \end{pmatrix} \quad \text{and} \quad \tilde{C}_{18}(T-t+1) = \hat{a}^{\alpha}_{T-t} \begin{pmatrix} C_{18} \\ C_{28} \\ C_{38} \\ C_{48} \\ C_{58} \end{pmatrix}.$$

In order to develop the formula for the asset allocation, $\bar{x}^{\alpha}_i(0, y, T-t+1)$, we define:

$$C_Z = \begin{pmatrix} C_{77} & C_{78} \\ C_{78} & C_{88} \end{pmatrix} \quad \text{and} \quad h_Z(T-t+1) = -\begin{pmatrix} \tilde{C}_{17}(T-t+1) \\ \tilde{C}_{18}(T-t+1) \end{pmatrix}.$$

Then the objective function becomes:

$$\begin{aligned} f(Z) &= Var\left(\sum_{i=2}^{3} z_i R_i(1) - \hat{a}^{\alpha}_{T-t}(y(1) - \mu_y) - \hat{b}^{\alpha}_{T-t} \,\middle|\, y(0) = y \right) \\ &= Z' C_Z Z + 2 h_Z(T-t+1)' Z + \tilde{C}_{11}(T-t+1). \end{aligned}$$

To establish the optimal values of $\bar{x}^{\alpha}_2(0, y, T-t+1)$ and $\bar{x}^{\alpha}_3(0, y, T-t+1)$, we minimize the objective function $f(Z)$ over z_2 and z_3: that is,

$$\frac{\partial f(Z)}{\partial Z} = 2 C_Z Z + 2 h_Z(T-t+1) = 0.$$

We then obtain the optimal asset allocation of consols and index-linked bonds at time 0 as follows:

$$\hat{Z} = \begin{pmatrix} \bar{x}^{\alpha}_2(0, y, T-t+1) \\ \bar{x}^{\alpha}_3(0, y, T-t+1) \end{pmatrix} = -C_Z^{-1} h_Z(T-t+1)$$

We note that C_Z is constant and that $h_Z(T-t+1)$ depends upon $T-t+1$ and μ_y but not on y. It follows that $\bar{x}^{\alpha}_2(0, y, T-t+1)$ and $\bar{x}^{\alpha}_3(0, y, T-t+1)$ depend upon $T-t+1$ and μ_y only and not on y.

With the values of $\bar{x}^{\alpha}_2(0, y, T-t+1)$ and $\bar{x}^{\alpha}_3(0, y, T-t+1)$ established, we can find $\bar{x}^{\alpha}_1(0, y, T-t+1)$ to ensure that:

$$E\left[\sum_{i=1}^{3} \bar{x}^{\alpha}_i(0, y, T-t+1) R_i(1) - M^{\alpha}(1) \bar{V}^{\alpha}_i(0, y(1), T-t) \,\middle|\, y(0) = y \right] = 0.$$

Hence, the optimal asset allocation of cash at time $t-1$ is:

$$\begin{aligned}
&\bar{x}_1^\alpha(t-1, y, T-t+1) = M^\alpha(t-1)\bar{x}_1^\alpha(0, y, T-t+1) \\
= \quad & \frac{M^\alpha(t-1)}{E[R_1(1)|y(0)=y]}\Bigg[\left(\hat{a}_{T-t}^\alpha E[y(1)-\mu_y|y(0)=y] + \hat{b}_{T-t}^\alpha\right) \\
&\qquad - (\bar{x}_2^\alpha(0, y, T-t+1)E[R_2(1)|y(0)=y]) \\
&\qquad - (\bar{x}_3^\alpha(0, y, T-t+1)E[R_3(1)|y(0)=y])\Bigg]
\end{aligned}$$

where

$$\begin{aligned}
E[R_1(1)|y(0)=y] &= y_4 \\
E[R_2(1)|y(0)=y] &= \mu_7 + A_R[2,](y-\mu_y) \\
E[R_3(1)|y(0)=y] &= \mu_8 + A_R[3,](y-\mu_y) \\
E[y(1)-\mu_y|y(0)=y] &= A_y(y-\mu_y).
\end{aligned}$$

In contrast to $\bar{x}_2$ and $\bar{x}_3$, it is clear that $\bar{x}_1^\alpha(t-1, y(t-1), T-t+1)$ does depend on $y(t-1)$.

Step 2: Set up the estimated economic value of the liability.

Once we have determined the optimal asset allocation at a given time, the estimated economic value of the liability is then defined as the sum of the values of the holdings in the three assets. Thus:

$$\begin{aligned}
\sum_{i=1}^{3}\bar{x}_i^\alpha(t-1, y(t-1), T-t+1) &= \hat{V}^\alpha(t-1, y(t-1), T-t+1) \\
\text{or} \quad \sum_{i=1}^{3}\bar{x}_i^\alpha(0, y, T-t+1) &= \hat{V}^\alpha(0, y, T-t+1).
\end{aligned}$$

This will be non-linear in y. It represents the best estimate of the liability at time $t-1$ given the linear approximations at times $t, t+1, \ldots, T$. Thus, the estimated economic value of the liability is:

$$\begin{aligned}
&\hat{V}^\alpha(t-1, y(t-1), T-t+1) \\
= \quad & \bar{x}_2^\alpha(t-1, y(t-1), T-t+1) + \bar{x}_3^\alpha(t-1, y(t-1), T-t+1) \\
& + \frac{1}{y_4(t-1)}\Bigg\{M^\alpha(t-1)\left(\hat{a}_{T-t}^\alpha A_y(y(t-1)-\mu_y) + \hat{b}_{T-t}^\alpha\right) \\
&\qquad - \bar{x}_2^\alpha(t-1, y(t-1), T-t+1)\left(\mu_7 + A_R[2,](y(t-1)-\mu_y)\right) \\
&\qquad - \bar{x}_3^\alpha(t-1, y(t-1), T-t+1)\left(\mu_8 + A_R[3,](y(t-1)-\mu_y)\right)\Bigg\}.
\end{aligned}$$

Step 3: Linearize the estimated economic value of the liability.
Let:

$$\bar{V}^{\alpha}(t-1, y(t-1), T-t+1) = M^{\alpha}(t-1)\left[a^{\alpha}_{T-t+1}(y(t-1)-\mu) + b^{\alpha}_{T-t+1}\right]$$

where a^{α}_{T-t+1} is a row vector and:

$$\begin{aligned} a^{\alpha}_{j,T-t+1} &= \left.\frac{\partial \hat{V}^{\alpha}(0, y, T-t+1)}{\partial y_j}\right|_{y=\mu_y} \\ b^{\alpha}_{T-t+1} &= \hat{V}^{\alpha}(0, \mu_y, T-t+1). \end{aligned}$$

The parameters of the linear liability are as follows.
For $j = 1, 2, 3, 5$:

$$\begin{aligned} & a^{\alpha}_{j,T-t+1} \\ = & \frac{1}{\mu_4}\left\{\hat{a}^{\alpha}_{T-t}A_y[,j] - \bar{x}^{\alpha}_2(0, \mu_y, T-t+1)A_R[2,j] - \bar{x}^{\alpha}_3(0, \mu_y, T-t+1)A_R[3,j]\right\} \end{aligned}$$

where $A_y[,j]$ represents the jth column of A_y.
For $j = 4$:

$$\begin{aligned} & a^{\alpha}_{4,T-t+1} \\ = & \frac{1}{\mu_4}\left\{\hat{a}^{\alpha}_{T-t}A_y[,4] - \bar{x}^{\alpha}_2(0, \mu_y, T-t+1)A_R[2,4] - \bar{x}^{\alpha}_3(0, \mu_y, T-t+1)A_R[3,4]\right\} \\ & -\frac{1}{\mu_4^2}\left(\hat{b}^{\alpha}_{T-t} - \bar{x}^{\alpha}_2(0, \mu_y, T-t+1)\mu_7 - \bar{x}^{\alpha}_3(0, \mu_y, T-t+1)\mu_8\right) \end{aligned}$$

and the average liability is:

$$\begin{aligned} b^{\alpha}_{T-t+1} = & \hat{b}^{\alpha}_{T-t}\mu_4^{-1} + \bar{x}^{\alpha}_2(0, \mu_y, T-t+1)(1-\mu_7\mu_4^{-1}) \\ & +\bar{x}^{\alpha}_3(0, \mu_y, T-t+1)(1-\mu_8\mu_4^{-1})). \end{aligned}$$

Now, for the purpose of constructing approximate formulae for the optimal asset allocation, we aim to linearize:

$$M^{\alpha}(1)\bar{V}^{\alpha}(0, y(1), T-t+1) \approx \hat{a}^{\alpha}_{T-t+1}(y(1)-\mu_y) + \hat{b}^{\alpha}_{T-t+1}$$

For the FP liability, the parameters of the linear liability with pension increase, f, are:

$$\begin{aligned} \hat{a}^{F}_{j,T-t+1} &= (1+f)a^{F}_{j,T-t+1} \\ \hat{b}^{F}_{T-t+1} &= (1+f)b^{F}_{T-t+1} \end{aligned}$$

and for the RPI liability, the parameters of the linear liability with pension increase are:

$$
\begin{aligned}
\hat{a}^R_{1,T-t+1} &= b^R_{T-t+1} + (1+\mu_1)a^R_{1,T-t+1} \\
\hat{a}^R_{j,T-t+1} &= (1+\mu_1)a^R_{j,T-t+1} \quad j = 2,3,4,5 \\
\text{and } \hat{b}^R_{T-t+1} &= (1+\mu_1)b^R_{T-t+1}
\end{aligned}
$$

Step 4: Reduce t by 1 and go to step 1.

Using the backward method, we can derive the asset allocation step-by-step from the last year back to the first year.

7 Some numerical results

In Sections 5 and 6 we constructed formulae for approximately optimal asset allocations for FP liability and RPI liability for dynamic hedging. In this section we provide some numerical results obtained using those formulae.

First, by making a single 10000-year simulation of the Wilkie Model, we obtained values of A and C_ϵ as follows:

$$\hat{A} = \begin{pmatrix} 0.58202 & 0.00167 & 0.00037 & -0.00558 & -0.00086 \\ 0.07273 & 0.56505 & 0.00310 & -0.11103 & -0.00805 \\ 0.35627 & -0.00500 & 0.93178 & 0.00303 & -0.05921 \\ 0.02054 & 0.00077 & 0.01200 & 0.73368 & -0.00252 \\ -0.00262 & 0.00058 & -0.00489 & 0.04145 & 0.54888 \\ 0 & 0 & 0 & 1 & 0 \\ -0.37395 & 0.00689 & 0.14036 & 0.04523 & 0.07199 \\ 0.60820 & 0.00131 & 0.00546 & -0.04873 & 0.51492 \end{pmatrix}$$

(Recall that $y_1(t)$ = inflation rate from time $t-1$ to t, $y_2(t)$ = log(dividend yield) at t, $y_3(t)$ = log(consols yield) at t, $y_4(t)$ = 1+risk-free interest rate from t to $t+1$, $y_5(t)$ = log(real yield on index-linked bonds) at t.)

$$\hat{C}_\epsilon = \begin{pmatrix} 0.00199 & 0.00332 & 0.00112 & 0.00006 & 0.00000 & 0 & -0.00113 & 0.00207 \\ 0.00332 & 0.02961 & 0.00525 & 0.00039 & -0.00010 & 0 & -0.00528 & 0.00356 \\ 0.00112 & 0.00525 & 0.00782 & 0.00059 & 0.00308 & 0 & -0.00789 & -0.00207 \\ 0.00006 & 0.00039 & 0.00059 & 0.00019 & 0.0002 & 0 & -0.00050 & -0.00014 \\ 0.00000 & -0.00010 & 0.00308 & 0.00020 & 0.00407 & 0 & -0.00309 & -0.00428 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.00113 & -0.00528 & -0.00789 & -0.00050 & -0.00309 & 0 & 0.00804 & 0.00207 \\ 0.00207 & 0.00356 & -0.00207 & -0.00014 & -0.00428 & 0 & 0.00207 & 0.00669 \end{pmatrix}$$

Consols and index-linked bonds are risky assets in the portfolio whereas cash is the riskless asset. The matrix of $\hat{C}_\epsilon$ shows that the return on consols has the largest conditional variance of 0.00804 or 0.090^2.

The unconditional means of $(y_1(t), y_2(t), y_3(t), y_4(t), y_5(t), R_1(t), R_2(t), R_3(t))'$ are (to 4 significant figures):

$$\hat{\mu}' = (0.04827, -3.197, -2.570, 1.065, -3.218, 1.065, 1.084, 1.093)$$

From the values of $\hat{\mu}$, we see that index-linked bonds offer the highest average return 1.093. Cash provides the lowest average return 1.065.

The Wilkie model is used here for illustration only. In principle, the autoregressive approximation could be applied directly to any stochastic asset model or estimated directly from historical data.

t	$T-t$	$\bar{x}_1^F(0,\mu_y,T-t)$	$\bar{x}_2^F(0,y(0),T-t)$	$\bar{x}_3^F(0,y(0),T-t)$	Liability multiplier
0	10	0.2999	0.4972	0.0221	1
1	9	0.3722	0.4479	0.0204	1.05
2	8	0.4458	0.3969	0.0185	1.05^2
3	7	0.5206	0.3443	0.0162	1.05^3
4	6	0.5966	0.2902	0.0137	1.05^4
5	5	0.6736	0.2345	0.0110	1.05^5
6	4	0.7515	0.1773	0.0081	1.05^6
7	3	0.8300	0.1189	0.0052	1.05^7
8	2	0.9083	0.0595	0.0024	1.05^8
9	1	0.9855	0	0	1.05^9

Table 7.1: Asset allocation for for FP liabilities (5% increase) with terms up to 10 years. The $\bar{x}_1^F(0,y(0),T-t)$ are those for $y(0)=\mu_y$. The $\bar{x}_i^F(0,y(0),T-t)$ for $i=2,3$ do not depend upon $y(0)$. For a 10-year liability the $\bar{x}_i^F(0,y(0),T-t)$ should be multiplied by the *Liability multiplier*.

Now we consider a ten-year term liability for both FP and RPI pensions.

We first list the optimal asset allocation of cash, consols and index-linked bonds for FP liability from time 0 to 9 as in Table 7.1.

For example, consider the row $T-t=8$ years. The table tells us that, on average, we require 0.4003 in cash, 0.4220 in consols and 0.0283 in index-linked to hedge from time 0 to time 1 a payment of 1.05^8 at time 8. For a 10 year dynamic hedge, we multiply row s by 1.05^{10-s}.

For FP liability, we see from Table 7.1 that we should invest most funds (about 61% of the assets) in consols at the beginning of the term. We also notice that, for FP liability, we should hold very few assets in index-linked bonds in the portfolio as we might expect. This is because, for FP liability, pension liability increases by a fixed percentage (here 5%) every year and so we know the exactly liability at the end of the term. Thus, consols are the better choice for this type of liability. We also see that the optimal investments shift gradually from consols early on into cash (100% in the final year).

As noted earlier the optimal values of $\bar{x}_2^F(t,y(t),T-t)$ and $\bar{x}_3^F(t,y(t),T-t)$ do not depend upon the value of $y(t)$. We are able to find the optimal values of $\bar{x}_1^F(t,y(t),T-t)$ making:

$$E\left[\sum_{i=1}^{3}\bar{x}_i^F(t,y(t),T-t)R_i(t+1)-\bar{V}_i^F(t+1,y(t+1),T-t-1)\mid y(t)\right]=0.$$

The values shown in Table 7.1 are for $y(t)=\mu_y$. From Table 7.1, the asset allocation of consols and index-linked bonds are decreasing functions of time t,

t	$T-t$	b^F_{T-t}	$a^F_{1,T-t}$	$a^F_{2,T-t}$	$a^F_{3,T-t}$	$a^F_{4,T-t}$	$a^F_{5,T-t}$
0	10	0.8192	0.0002	-0.0064	-0.4169	-1.5682	-0.0374
1	9	0.8405	0.0010	-0.0064	-0.3601	-1.7493	-0.0342
2	8	0.8611	0.0017	-0.0063	-0.3034	-1.9091	-0.0305
3	7	0.8812	0.0022	-0.0060	-0.2472	-2.0378	-0.0265
4	6	0.9005	0.0026	-0.0055	-0.1925	-2.1221	-0.0220
5	5	0.9191	0.0026	-0.0047	-0.1403	-2.1434	-0.0172
6	4	0.9370	0.0024	-0.0037	-0.0925	-2.0763	-0.0122
7	3	0.9540	0.0017	-0.0025	-0.0509	-1.8857	-0.0072
8	2	0.9702	0.0008	-0.0011	-0.0188	-1.5239	-0.0029
9	1	0.9855	0	0	0	-0.9250	0

Table 7.2: Liability valuation for 5% fixed pension liabilities. The b^F_{T-t} column gives the liability value for a $T-t$-year liability starting from $y(0) = \mu_y$. The $a^F_{i,T-t}$ show how sensitive the values are to deviations in $y_i(0)$ from μ_i.

and the asset allocation of cash is an increasing function of time t. This reflects the decreasing duration of the liability in relation to the two dominant assets which have high and low durations.

Recall that the linear approximation to the liability is:

$$\begin{aligned} \bar{V}^F(t, y(t), T-t) &= (1+f)^t \left(a^F_t (y - \mu_y) + b^F_t \right) \\ \text{or} \quad \bar{V}^F(0, y, T-t) &= a^F_t (y - \mu_y) + b^F_t \end{aligned}$$

Values for the b^F_{T-t} and $a^F_{i,T-t}$ are given in Table 7.2.

We see from Table 7.2 that, for the FP liability, the linearized liability is strongly related to the risk-free return on one-year zero-coupon bonds (that is, the column headed $a^F_{4,T-t}$). Also, the liability is related to the consol real yield, especially in the earlier years of the term.

The b^F_{T-t} column gives the average liability. We can note, for example, that for $T-t = 10$ the average liability implies an average discount rate of 7.1% over the 10 years. This reflects the heavier investments in cash in the later years.

Consider next the RPI liability.

In a similar way to the calculation of the FP liability, we can investigate the case of an RPI liability over a ten-year term. Table 7.3 shows that the optimal asset allocations of cash, consols and index-linked bonds for RPI liability from $t = 0$ to 9. From Table 7.3, we see that we should hold a very high proportion of assets in index-linked bonds, especially in the early years of the term. This makes sense since, for an RPI liability, IL bonds provide a reasonable match in the long run for the RPI-linked pension increases. Like the FP liability, with the RPI liability we

t	$T-t$	$\bar{x}_1^R(0,\mu_y,T-t)$	$\bar{x}_2^R(0,y(0),T-t)$	$\bar{x}_3^R(0,y(0),T-t)$	Liability multiplier
0	10	-0.1047	0.0420	0.7532	1
1	9	-0.0803	0.0305	0.7722	$M^R(1)$
2	8	-0.0456	0.0143	0.7866	$M^R(2)$
3	7	0.0024	-0.0081	0.7943	$M^R(3)$
4	6	0.0679	-0.0383	0.7928	$M^R(4)$
5	5	0.1556	-0.0778	0.7783	$M^R(5)$
6	4	0.2714	-0.1275	0.7454	$M^R(6)$
7	3	0.4216	-0.1849	0.6847	$M^R(7)$
8	2	0.6104	-0.2369	0.5779	$M^R(8)$
9	1	0.8337	-0.2393	0.3836	$M^R(9)$

Table 7.3: Asset allocation for for RPI liabilities with terms up to 10 years. The $\bar{x}_1^R(0,y(0),T-t)$ are for $y(0)=\mu_y$. The $\bar{x}_i^R(0,y(0),T-t)$ for $i=2,3$ do not depend upon $y(0)$. For a 10-year liability the $\bar{x}_i^R(0,y(0),T-t)$ should be multiplied by the *Liability multiplier*. In the present context the liability multiplier $M^R(t)$ equals $RPI(t)/RPI(0)$.

t	$T-t$	b_{T-t}^R	$a_{1,T-t}^R$	$a_{2,T-t}^R$	$a_{3,T-t}^R$	$a_{4,T-t}^R$	$a_{5,T-t}^R$
0	10	0.6905	-0.0302	-0.0015	-0.0028	-0.0296	-0.8073
1	9	0.7225	-0.0173	-0.0014	0.0049	-0.1714	-0.8149
2	8	0.7552	0.0004	-0.0012	0.0138	-0.3379	-0.8128
3	7	0.7886	0.0243	-0.0008	0.0238	-0.5256	-0.7980
4	6	0.8224	0.0559	-0.0001	0.0342	-0.7263	-0.7661
5	5	0.8561	0.0969	0.0008	0.0441	-0.9233	-0.7119
6	4	0.8894	0.1481	0.0018	0.0515	-1.0874	-0.6292
7	3	0.9214	0.2065	0.0029	0.0540	-1.1708	-0.5118
8	2	0.9513	0.2563	0.0034	0.0480	-1.0987	-0.3560
9	1	0.9780	0.2434	0.0026	0.0299	-0.7601	-0.1700

Table 7.4: Liability valuation for RPI pension liabilities. The b_{T-t}^R column gives the liability value for a $T-t$-year liability starting from $y(0)=\mu_y$. The $a_{i,T-t}^R$ show how sensitive the values are to deviations in $y_i(0)$ from μ_i.

shift gradually towards cash in the later years. For example, in RPI liability we should hold 0.753 in index-linked bonds, 0.042 in consols and short by an average of 0.105 in cash to minimise risk in the first year of a term. In the last year of the term we require 0.384 in index-linked bonds, -0.239 in consols and an average holding of 0.834 in cash. This presumably means that cash is a better hedge for an RPI over one year than index-linked bonds. It reflects the relative certainty of the liability one year ahead, the certainty (in nominal terms) of cash versus the relative riskiness of index-linked bonds (because of the variable real yield) in the short term.

Values for the b^R_{T-t} and $a^R_{i,T-t}$ are given in Table 7.4.

From Table 7.4, we notice that, for the RPI liability, the linearized liability is affected by the price of one-year zero-coupon bonds (especially for the later years of the term), by the real yield on index-linked bonds (especially for the earlier years of the term), and by the inflation rate (especially for the later years of the term). The fact that, for example, $a^R_{1,9} = -0.030 < 0$ indicates that even if the rate of inflation is currently high (suggesting a higher liability) returns on the matching assets must be correspondingly higher in the long run (that is, the liability is actually lowered).

8 Optimal asset allocation for LPI liability

For an LPI liability, the pension increase rate is the lower of the fixed rate (5%) and RPI: that is,

$$M^L(t) = M^L(t-1) \quad \min\left\{\frac{RPI(t)}{RPI(t-1)}, 1.05\right\}$$

It follows that an LPI liability has strong links with both the FP and RPI liabilities (for example, we can immediately note that the liability will be lower than the corresponding fixed and RPI liabilities[6]). We will use the RPI and FP liabilities developed earlier in this chapter to propose approximate formulae for an LPI liability by choosing an appropriate function which satisfactorily distributes the proportions of these two types of liability.

8.1 Model setting for the LPI liability

In this section we aim to build a suitable model to connect the FP and RPI liabilities with the LPI liability. Since the main factor affecting the relationship among these three liabilities is the inflation rate, we assume for simplicity that this connection is a function only of the inflation rate.

We now assume that there exists two functions of inflation rate, $p(T-t, y_1(t))$ and $q(T-t, y_1(t))$, which make the approximation of the asset allocation for the LPI liability as follows:

$$\hat{x}_j^L(t, y(t), T-t) = p(T-t, y_1(t))\bar{x}_j^R(t, y(t), T-t) + q(T-t, y_1(t))\bar{x}_j^F(t, y(t), T-t)$$

where $j = 1, 2, 3$.

Note that it will not be possible in general to find a p and q such that the linear combination of $\bar{x}_j^R$ and $\bar{x}_j^F$ is precisely equal to the true $\bar{x}_j^L$. Instead we aim to find a p and q which make $\hat{x}_j^L = p\bar{x}_j^R + q\bar{x}_j^F$ the best approximation (in some sense) to $\bar{x}_j^L$. Note also that $\bar{x}_1^R$ and $\bar{x}_1^F$ depend upon all of the $y_i(t)$. To this extent $\hat{x}_1^L$ will also depend upon $y(t)$. $\hat{x}_2^L$ and $\hat{x}_3^L$ will depend on $y_1(t)$ and μ_y only since $p(T-t, y_1(t))$ and $q(T-t, y_1(t))$ are functions only of $y_1(t)$ and $\bar{x}_2^R$, $\bar{x}_2^F$, $\bar{x}_3^R$ and $\bar{x}_3^F$ are dependent on μ_y.

Then the estimated LPI liability is taken to be:

$$\begin{aligned} & \hat{V}^L(t, y(t), T-t) \\ = \; & p(T-t, y_1(t))\bar{V}^R(t, y(t), T-t) + q(T-t, y_1(t))\bar{V}^F(t, y(t), T-t) \end{aligned}$$

As indicated earlier, $p(T-t, y_1(t))$ and $q(T-t, y_1(t))$ are functions only of $T-t$ and $y_1(t)$, the rate of inflation from $t-1$ to t. If current inflation rates are low

[6] Note that if X and Y are random variables then $E[\min\{X, Y\}] \leq \min\{E[X], E[Y]\}$ with strict inequality if $Pr[X > Y] > 0$ and $Pr[X < Y] > 0$.

(that is, much lower than 5%), then we anticipate that the LPI liability will be closer to the RPI liability than if current inflation rates are high, especially when $T-t$ is small. In this case $p(T-t, y_1(t))$ should be near to one and $q(T-t, y_1(t))$ should be near to zero, and vice versa if recent inflation has been high (much higher than 5%). Following this observation, we propose these two functions to be:

$$p(T-t, y_1(t)) = \frac{\gamma^p_{T-t} \exp(-\alpha_{T-t}(y_1(t) - \beta^p_{T-t}))}{1 + \exp(-\alpha_{T-t}(y_1(t) - \beta^p_{T-t}))} \tag{8.1}$$

$$\text{and} \quad q(T-t, y_1(t)) = \frac{\gamma^q_{T-t} \exp(\alpha_{T-t}(y_1(t) - \beta^q_{T-t}))}{1 + \exp(\alpha_{T-t}(y_1(t) - \beta^q_{T-t}))}. \tag{8.2}$$

To establish $p(T-t, y_1(t))$ and $q(T-t, y_1(t))$ we need to minimize the SS function below over $p = p(T-t, y_1(t))$ and $q = q(T-t, y_1(t))$ where:

$$\begin{aligned} SS = E\Bigg[\Big\{ & \left(p\bar{x}^R(t, y(t), T-t) + q\bar{x}^F(t, y(t), T-t)\right)^T R(t+1) \\ & - \tilde{V}^L(t+1, y(t+1), T-t-1) \Big\}^2 \,\Bigg|\, y(t), LPI(t) \Bigg] \end{aligned}$$

This function is minimised for each $y_1(t)$ giving different values of p and q for each $y_1(t)$. (This step is implemented before we parametrize p and q according to Equations (8.1) and (8.2).)

To obtain estimates for $p(T-t, y_1(t))$ and $q(T-t, y_1(t))$ we differentiate SS with respect to p and q and equate to zero in combination with 40000 simulations as an approximation to exact expectation. This is implemented for 21 values of $y_1(t)$ ($y_1^{(i)}(t) = \mu_y + 0.02(i-11)$ for $i = 1, \ldots, 21$). The estimated p and q values for $T-t = 1$ are plotted in Figure 8.1.

With the estimated optimal values of $\hat{p}(T-t, y_1(t))$ and $\hat{q}(T-t, y_1(t))$, the next step is to estimate the values of α_{T-t}, β^p_{T-t}, β^q_{T-t}, γ^p_{T-t} and γ^q_{T-t}. These are determined by minimizing:

$$\begin{aligned} S(\alpha, \beta^p, \beta^q, \gamma^p, \gamma^q) = & \sum_{i=1}^{21} \left[p(T-t, y_1^{(i)}(t))(\alpha, \beta^p, \gamma^p) - \hat{p}(T-t, y_1^{(i)}(t))\right]^2 \\ & + \sum_{i=1}^{21} \left[q(T-t, y_1^{(i)}(t))(\alpha, \beta^q, \gamma^q) - \hat{q}(T-t, y_1^{(i)}(t))\right]^2 \end{aligned} \tag{8.3}$$

We can see from Figure 8.1 that the quality of fit for $\hat{p}(1, y_1(t))$ and $\hat{q}(1, y_1(t))$ is good. We then can obtain graphs simlar to Figure 8.1 for $t = 0, \cdots, T-2$. In

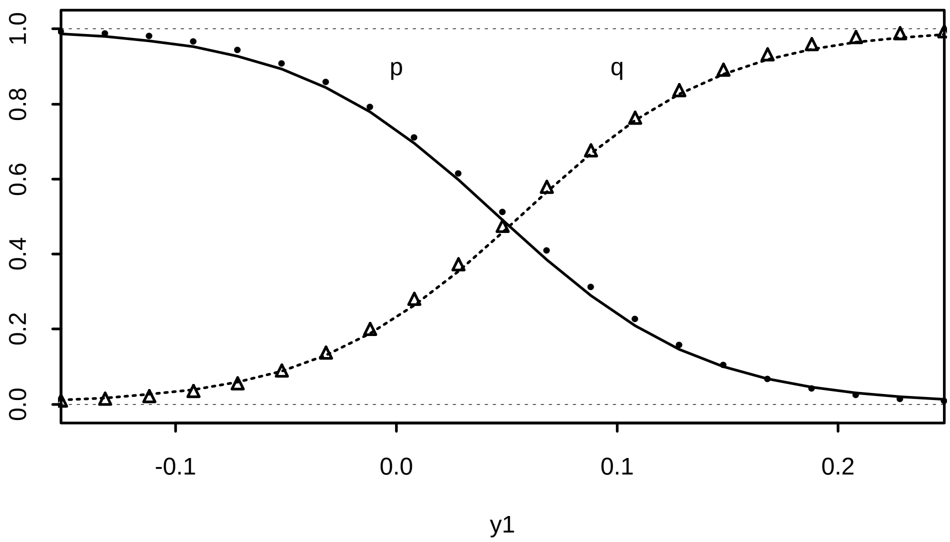

Figure 8.1: Estimated $\hat{p}$ (dots) and $\hat{q}$ (triangles) against different inflation rates for $T-t=1$. Fitted curves $p(1,y_1)$ (solid curve) and $q(1,y_1)$ (dotted curve).

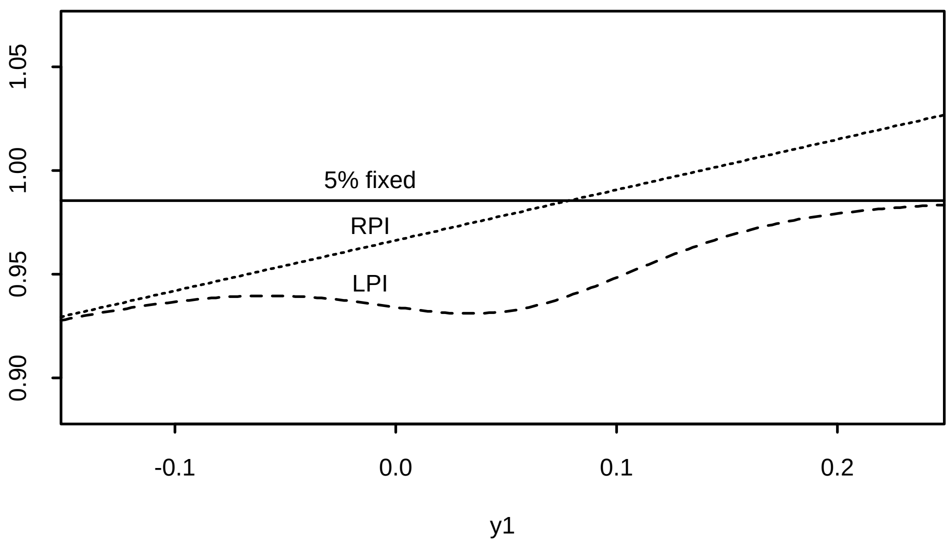

Figure 8.2: Estimated liability values for different values of y_1. The liability is due for payment in $T-t$ =1 year.

t	$T-t$	α_{T-t}	β^p_{T-t}	β^q_{T-t}	γ^p_{T-t}	γ^q_{T-t}
0	10	8.910	0.06997	0.08294	0.8726	0.7617
1	9	8.917	0.07028	0.08317	0.8824	0.7867
2	8	8.960	0.07032	0.08313	0.8925	0.8125
3	7	9.062	0.07014	0.08286	0.9028	0.8391
4	6	9.257	0.06898	0.08157	0.9132	0.8669
5	5	9.583	0.06538	0.07782	0.9242	0.8950
6	4	10.100	0.06032	0.07252	0.9380	0.9211
7	3	9.824	0.06213	0.07422	0.9531	0.9437
8	2	7.237	0.05264	0.07998	0.9775	0.9998
9	1	21.607	0.04649	0.05591	1.0000	1.0000

Table 8.1: Values of α_{T-t}, β^p_{T-t}, β^q_{T-t}, γ^p_{T-t} and γ^q_{T-t} for $T = 10$ and $t = 0, ..., 9$.

Figure 8.2 we plot the liability values for $T - t = 1$. As expected we find that the LPI value is close to the RPI liability when y_1 is very low and close to the 5% fixed liability when y_1 is very high. We can also see that the LPI liability is very much lower than both the RPI and fixed liabilities when y_1 is close to the LPI threshold of 5% indicating that the effect of the stochastic minimum ($\min\{1.05, RPI(1)/RPI(0)\}$) is significant.

The backward method is used to calculate in sequence the values for α_{T-t}, β^p_{T-t}, β^q_{T-t}, γ^p_{T-t} and γ^q_{T-t} for $T - t = 1, \ldots, T$. At each time $T - t + 1$ we calculate $\hat{p}(T - t + 1, y_1)$ and $\hat{q}(T - t + 1, y_1)$ on the assumption that:

$$\begin{aligned} & \bar{V}^L(t, y(t), T-t) \\ = \; & p(T-t, y_1(t))\bar{V}^R(t, y(t), T-t) + q(T-t, y_1(t))\bar{V}^F(t, y(t), T-t) \end{aligned}$$

where the parametric forms for p and q (Equations 8.1 and 8.2) are used with the already estimated values for the function parameters.

The estimated values of α_{T-t}, β^p_{T-t}, β^q_{T-t}, γ^p_{T-t} and γ^q_{T-t} for $t = T - 1$ to 0 are presented in Table 8.1). These estimates for $t = 0, ..., T - 1$, give us a means of connecting the FP and RPI liabilities with the LPI liability. From this, we are able to deduce the approximately optimal asset allocations for the LPI liability.

In Figures 8.3 and 8.4 we plot the functions p and q and the liability estimates for $T - t = 10$. In contrast to Figure 8.1 the p and q functions are flatter and have upper limits which are below 1 (that is, $g^p_{10} = 0.87$ and $g^q_{10} = 0.76$, see Table 8.1). Even more striking is the comparison with Figure 8.2. The three liability curves are now almost independent of y_1. The difference between LPI and RPI amounts to a difference in the assumed rate of increase in the pension of 1.5% per annum. The size of this difference reflects the magnitude of the volatility in price inflation. It is also appropriate here to compare RPI with 5% fixed increases. Average price inflation is just below 5% whereas the liability values suggest something rather

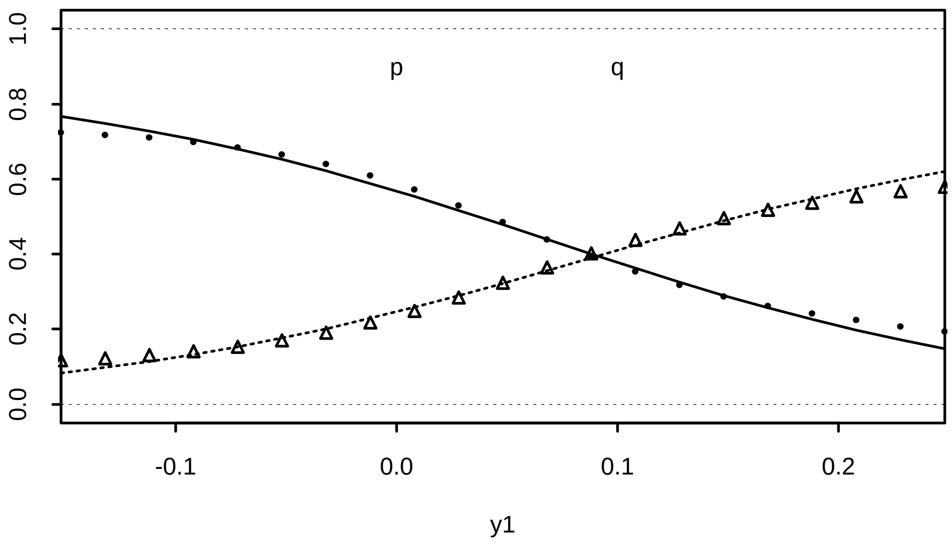

Figure 8.3: Estimated $\hat{p}$ (dots) and $\hat{q}$ (triangles) against different inflation rates for $T - t = 10$. Fitted curves $p(10, y_1)$ (solid curve) and $q(10, y_1)$ (dotted curve).

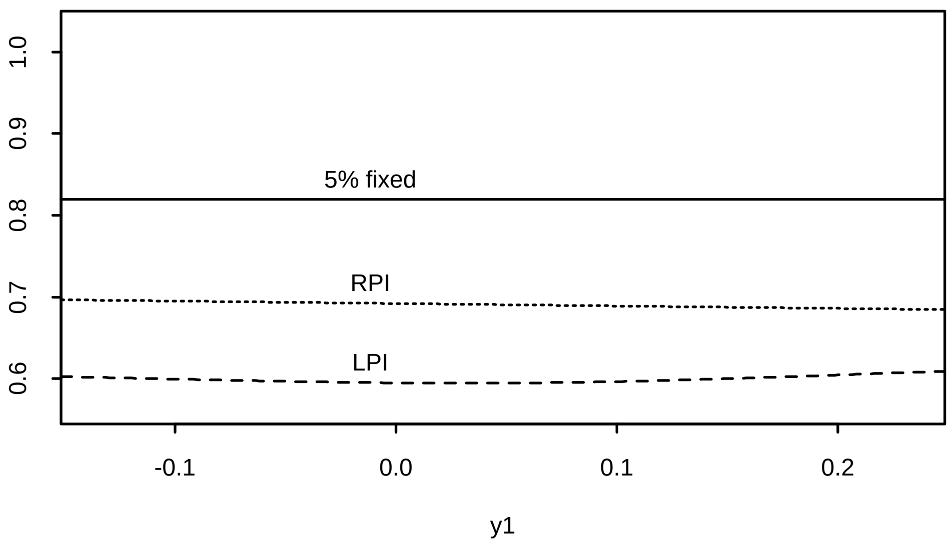

Figure 8.4: Estimated liability values for different values of y_1. The liability is due for payment in $T - t$ =10 years.

larger. The bigger difference is a result of the hedging portfolio for each liability. In the case of RPI the liability is hedged with a much larger proportion invested in index-linked bonds which have a higher expected rate of return resulting in a lower liability.

Figure 8.5 displays illustrations of asset allocations in the three different types of liabilities when $y(t) = \mu_y$. We can see more clearly how the various allocations change over time. For example, in all cases cash becomes less important as term to payment, $T-t$, increases. Also we can see that longer-term LPI liabilities make use of a mixture of consols and IL bonds as we might have expected.

8.2 The efficiency of dynamic hedging

From the preceeding sections we have derived the formula for the optimal asset allocation, $\hat{x}_j^\alpha(t, y(t), n)$ where $\alpha = F, R, L.$ and $j = 1, 2, 3$. In this section we can now use these formulae to examine the effectiveness of the proposed (approximate) dynamic hedging strategy.

In dynamic strategies, the extra cash required at time t is:

$$C_t^\alpha = \sum_{j=1}^{3} \bar{x}_j^\alpha(t-1, y(t-1), T-t+1)R_j(t) - \bar{V}^\alpha(t, y(t), T-t)$$

Thus, C_t^α is the difference between the new liability at t, $\bar{V}^\alpha(t, y(t), T-t)$, and the value at t of the available assets held from $t-1$ to t.

Then present value at time 0 of the total extra cash required up to time T is:

$$TC^\alpha = \sum_{t=1}^{n} C_t^\alpha \frac{S_1(0)}{S_1(t)}$$

where $S_1(u)$ is the the unit value of a cash account at time u (that is, $S_1(0) = 1$ and $S_1(u+1) = S_1(u)R_1(u+1) = S_1(u)y_4(u)$). This measure is consistent with those commonly used in financial mathematics (see, for example, Musiela & Rutkowski, 1997, Chapter 4). It is, in particular, consistent with the approach taken in earlier sections of minimising variances over each time step.

To test the efficiency of dynamic hedging we need to calculate the values of $E[TC^\alpha]$ and $Var(TC^\alpha)$ by making numerous simulations. Numerical results of comparisons with other hedging strategies are shown in Section 9.

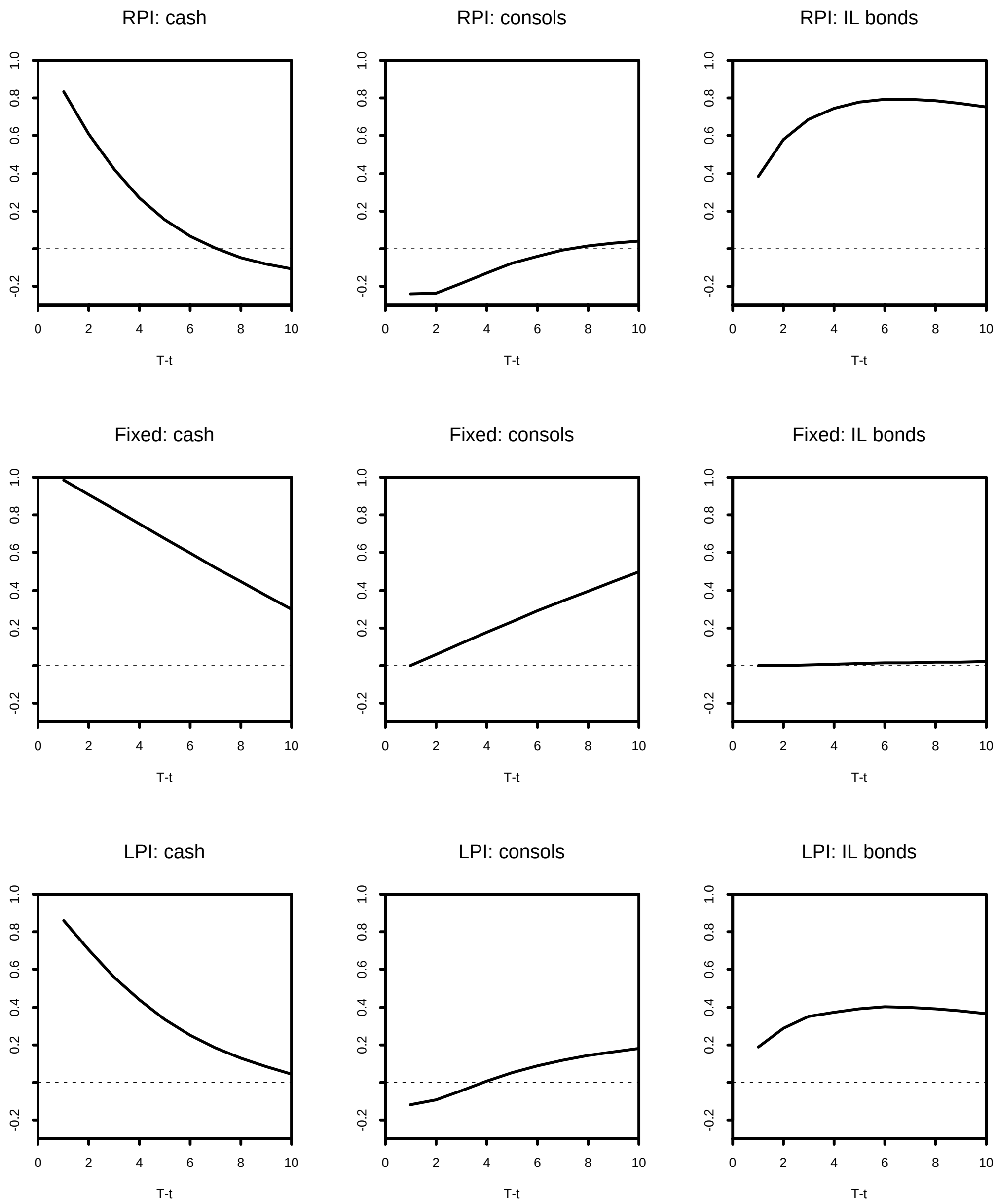

Figure 8.5: Asset allocations for a 10-year liability for the cases of RPI, FP and LPI liabilities. Values plotted are representative values for $y(t) = \mu_y$.

	$E[TC^\alpha]$	$Var[TC^\alpha]$	$E[TC^{\alpha 2}]$
5% Fixed			
Static (cash only)	-0.000873	0.012321	0.012322
Static (consols only)	-0.024918	0.020034	0.020655
Static (three assets)	0.000278	0.005150	0.005150
Dynamic	0.007496	0.004585	0.004641
RPI			
Static (IL bonds only)	-0.006211	0.004409	0.004448
Static (three assets)	-0.000728	0.003825	0.003825
Dynamic	0.005707	0.003527	0.003559
LPI			
Static (cash only)	0.001417	0.015153	0.015154
Static (consols only)	-0.029089	0.036549	0.037394
Static (IL bonds only)	-0.0304	0.014991	0.015915
Static (three assets)	0.000656	0.004866	0.004866
Dynamic	0.039129	0.002924	0.004455

Table 9.1: Comparison of static and dynamic hedging strategies for a 10-year liability. Statistics are based on 40,000 simulations.

9 Comparison of hedging strategies

In this section we will consider numerical results to allow comparison of the static and dynamic hedging strategies.

To assess the effect of static strategies, as in dynamic hedging, we denote TC^α as the present value at time 0 of total extra cash for the static strategies at time T for pensions $\alpha = F, R, L$. We have:

$$TC^\alpha = \frac{1}{S_1(T)}\left[M^\alpha(T) - \sum_{i=1}^{3} \hat{x}_i^\alpha \frac{S_i(T)}{S_i(0)}\right]$$

where $\alpha = F, \quad R$ or L. Or objective with static hedging was to minimise the function SB (Equation (4.1)) which is:

$$SB = E[TC^{\alpha 2}].$$

We will now consider how much of an improvement in SB is provided by the switch to dynamic hedging (based on the linear approximations for 5%-fixed and RPI liabilities and the non-linear approximation for LPI liabilities). We will assume that we are starting from neutral conditions at time 0 (that is, $y(0) = \mu_y$) and $T = 10$.

Results for a 10-year liability are presented in Table 9.1. The value of TC^α was calculated for each of 40,000 simulations.

First, we note that (with the exception of the RPI liability) static hedging proves to give a substantial improvement in performance relative to investment in a single asset class (for example, consols only for the fixed pension liability).

Second, we can observe from this that the proposed form of dynamic hedging does reduce the primary objective function $E[TC^{\alpha 2}]$. However, the improvement is not substantial suggesting that dynamic hedging does not help greatly over static hedging.

An important observation to note is that all of the dynamic hedging strategies have $E[TC^{\alpha}]$ significantly different from 0 (LPI especially so). This is not the case (explicitly by construction) for static hedging. For the 5%-fixed and RPI liabilities this will be the result of the linear approximation, but, in any event, the error is relatively insignificant.

The larger bias in the LPI liability is a significant factor contributing to the size of the objective function $E[TC^{\alpha 2}]$. In contrast we can see that of all the liability types we can see that dynamic hedging works best for LPI when we consider its effect on the variance $Var[TC^{\alpha}]$.

Further investigation suggests that much of the bias arises close to the liability payment date. This indicates that we should focus our attention in the future on improving both the dynamic hedging and our assessment of the liability value for shorter term liabilities. Conversely longer dated liabilities (for example, 20 years) benefit more from dynamic hedging relative to static hedging.

An alternative line of investigation is to replace consols and long-dated index-linked bonds with zero-coupon fixed-interest and IL bonds maturing on the same date as the pension liability.

10 Conclusions

In this paper, we use the methods of static and regular rebalancing to hedge LPI liabilities. For static hedging, we find that investing solely in cash, index-linked bonds or long-dated bonds creates higher errors than when holding a suitable mixture of the three assets in the portfolio.

With dynamic hedging, we develop mathematical formulae for finding an approximation to the optimal asset allocation for hedging FP, RPI and LPI liabilities. For the FP liability, it is shown that in the portfolio most funds should be invested in consols at the begining of the term with very few assets in index-linked bonds. We also find that larger proportions of risky asset should be held in the early years of the term in the portfolio with fewer risky assets in later years. In other words, the allocations in consols and index-linked bonds are decreasing functions of time t (for a fixed payment date T), and cash is an increasing function of time t. This reflects the decreasing duration of the liability.

For an RPI liability, a high proportion of index-linked bonds should be held especially in the early years of the term. Like the FP liability, more risky assets should be held in the early years of a term and more cash in later years. When making comparisons between FP, RPI and LPI liabilities, all the lines of asset allocation curves for the three types of liabilities are similar and especially for both RPI and LPI liabilities. When the current inflation rate is very high (significantly above 5%), then the optimal asset allocation of the LPI liability is closer to those of the FP liability. Also, if the inflation rate is very low (always lower than 5%), then the optimal asset allocation of the LPI liability will be closer to that of the RPI liability.

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