

Continuous-Time Pension-Fund Modelling

Andrew J.G. Cairns^{1 2}

Department of Actuarial Mathematics and Statistics,
Heriot-Watt University,
Riccarton, Edinburgh, EH14 4AS,
United Kingdom

Abstract

This paper considers stochastic pension fund models which evolve in continuous time and with continuous adjustments to the contribution rate and to the asset mix. A generalization of constant proportion portfolio insurance is considered and an analytical solution is derived for the stationary distribution of the funding level. In the case where a risk-free asset exists this is a translated-inverse-gamma distribution.

Numerical examples show that the continuous-time model gives a very good approximation to more widely used discrete time models, with, say, annual contribution rate reviews, and using a variety of models for stochastic investment returns.

Keywords: continuous time; stochastic differential equation; risk-free rate; continuous proportion portfolio insurance.

¹E-mail: A.Cairns@ma.hw.ac.uk

²WWW: <http://www.ma.hw.ac.uk/~andrewc/>

1 Introduction

In this paper we consider continuous time stochastic models for pension fund dynamics. The general form of this simple model is:

$$dX_t = X_t d\delta(t, X_t) + (N + D(X_t) - B)dt$$

where X_t = funding level at time t
= Assets/Liabilities at time t
 $d\delta(t, X_t)$ = real return between t and $t + dt$
over salary growth
 N = normal contribution rate
 $D(t, X_t)$ = adjustment to the contribution rate
for surplus or deficit
and B = rate of benefit outgo (as a proportion
of the actuarial liability)

(Note that the description of the model given here allows for the distribution of investment returns to depend upon the funding level.)

Here, it is assumed that the level of benefit outgo is constant through time relative to the actuarial liability.

Related to the funding level is the target funding level, L , which will normally be equal to 1 but this need not be the case. This reserve is related to the normal contribution rate, the level of benefit outgo and the valuation rate of interest in excess of salary growth, δ_v , in the following way:

$$\frac{dL}{dt} = \delta_v L + N - B = 0$$

That is, if the experience of the fund is precisely as expected then interest on the fund plus the normal contribution rate will be precisely sufficient to pay the benefits. Thus $B - N = \delta_v L$.

Similar continuous time models have been considered by Dufresne (1990). A discrete time version of the model has been considered in more detail and in various forms by Cairns and Parker (1996), Dufresne (1988, 1989, 1990) and Haberman (1992, 1994).

This paper will discuss various special cases of the model. The first case is where $d\delta(t, X_t)$ does not depend upon X_t and, in effect, reflects a static investment pol-

icity with independent and identically distributed returns. This case has previously been considered by Dufresne (1990) who showed that the stationary distribution of the fund size was Inverse Gaussian and here we verify his result using different techniques.

The second case will consider Continuous Proportion Portfolio Insurance. This is a special type of investment strategy which holds a greater proportion of its assets in low risk stocks when the funding level X_t is low. Several sub-cases are investigated including one in which a risk-free asset exists and one in which it does not. The latter indicates that selling a particular asset class short could be a problem and as a consequence certain constraints are put in place. These constraints prevent the fund from going short on the higher risk assets when the funding level is low and place an upper limit on the amount by which the fund can go short on low risk assets when the funding level is high. In all cases, a closed form solution can be found for the limiting (stationary) density function of X_t . When there exists a risk-free asset, this distribution is Translated-Inverse-Gaussian (TIG).

Much of the analysis relies on the following result:

Theorem 1.1

Let the continuous-time stochastic process X_t satisfy the stochastic differential equation

$$dX_t = (\alpha + \beta X_t + \gamma X_t^2)^{1/2} dZ_t + \mu dt - \nu X_t dt$$

subject to the constraints on the parameters $\alpha > 0$, $\gamma > 0$, $\beta^2 - 4\alpha\gamma \leq 0$, $\mu > 0$ and $\nu > 0$.

(a) If $\beta^2 - 4\alpha\gamma < 0$, the stationary density function of X_t is

$$f_X(x) = k' \exp \left[2a \tan^{-1} \frac{x+b}{c} \right] (\alpha + \beta x + \gamma x^2)^{-1-\nu/\gamma}$$

for $-\infty < x < \infty$

$$\text{where } a = \frac{1}{\sqrt{4\alpha\gamma - \beta^2}} \left(\frac{\nu\beta}{\gamma} + 2\mu \right)$$

$$b = \frac{\beta}{2\gamma}$$

$$c = \frac{\sqrt{4\alpha\gamma - \beta^2}}{2\gamma}$$

where k' is a normalizing constant.

(b) If $\beta^2 - 4\alpha\gamma = 0$ and $X_0 > b$, the stationary density function of X_t is

$$f_X(x) = k(\theta, \phi)(x+b)^{-\theta} \exp[-\phi/(x+b)] \text{ for } b < x$$

$$\text{where } b = \frac{\beta}{2\gamma}$$

$$\theta = 2 \left(1 + \frac{\nu}{\gamma} \right)$$

$$\phi = \frac{\nu\beta + 2\mu\gamma}{\gamma^2}$$

that is, the Translated-Inverse-Gamma distribution with parameters $-b$, $\theta - 1 > 0$ and $\phi > 0$ ($TIG(-b, \theta - 1, \phi)$). (If $X \sim TIG(k, \alpha, \beta)$ then $(X - k)^{-1} \sim \text{Gamma}(\alpha, \beta)$.)

Proof See Cairns (1996).

2 Model 1: Static investment strategy

This model takes the simplest case possible. In the absence of other cashflows the value of the assets will follow Geometric Brownian motion. Thus

$$d\delta(t, X_t) = d\delta_t = \delta dt + \sigma dZ_t$$

where Z_t is standard Brownian Motion.

In particular investment returns are uncorrelated and do not depend upon the funding level at any point in time. Such a model is appropriate if the trustees of the fund operate a static asset allocation strategy: that is, the proportion of the fund invested in each asset class remains fixed.

The deficit at time t is $L - X_t$ and the adjustment for this deficit to the contribution rate is

$$D(X_t) = k.(L - X_t).$$

$k = 1/\bar{a}_{\overline{m}|}$ is the spread factor, and m is the term of amortization.

This method is sometimes referred to as the spread method of amortization (for example, see Dufresne, 1988).

In continuous time this model has been considered by Dufresne (1990).

The stochastic differential equation for the fund size is

$$dX_t = (\delta dt + \sigma dZ_t)X_t + (N - B + k(L - X_t))dt = -vX_t dt + \sigma X_t dZ_t + \mu dt$$

where $v = k - \delta$ and $\mu = (k - \delta_v)L$.

2.1 Properties of X

Let X be a random variable with the stationary distribution of X_t . (Cairns and Parker, 1996, show that such processes are stationary and ergodic.)

Now X_t falls into the collection of stochastic processes covered in Theorem 1.1 Thus by Theorem 1.1(b) X has an Inverse Gamma distribution with parameters $\theta - 1$ and ϕ where $\theta = 2 \left(1 + \frac{v}{\sigma^2}\right)$ and $\phi = \frac{v\beta + 2\mu\sigma^2}{\sigma^4}$ (that is, $X^{-1} \sim \text{Gamma}(\theta - 1, \phi)$). For this to be a proper distribution (that is, one which has a density which integrates to 1) we require that $\theta > 1$. This therefore imposes the further condition that $k > \delta - \frac{1}{2}\sigma^2$. Stronger conditions on k are required to ensure that X has finite moments.

The stationary distribution of X_t was found by Dufresne (1990), Proposition 4.4.4, but here we have derived it in a different way by making use of Theorem 1.1.

Let $M_j = E(X^j)$ where j is a non-negative integer. Then it is easy to show that for $j < \theta - 1$

$$M_j = \frac{\phi^j}{(\theta - 2)(\theta - 3)\dots(\theta - (j + 1))}$$

For $j \geq \theta - 1$, M_j is infinite.

Using these equations we see that

$$\begin{aligned} E(X) &= \frac{k - \delta_v}{k - \delta} L \\ E(X^2) &= \frac{(k - \delta_v)^2}{(k - \delta)(k - \delta - \frac{1}{2}\sigma^2)} L^2 \\ \Rightarrow \text{Var}(X) &= \frac{(k - \delta_v)^2 \frac{1}{2}\sigma^2}{(k - \delta^2)(k - \delta - \frac{1}{2}\sigma^2)} L^2 \end{aligned}$$

Note that it is possible for the process to be stationary but to have an infinite mean.

Using this information we can calculate, for example, $Pr(X > x_0)$ where x_0 is the government statutory limit of 105% of the actuarial liability calculated on the UK statutory valuation basis. This figure gives a guide to the frequency in the long run of breaches of this upper limit.

2.2 Hitting Times

The problems described below are included as open problems.

Suppose $T = \inf\{t : X_t = x\}$. Since X_t is stationary it cannot be true that $E[s(X_T)] = E[s(X_0)]$. If, on the other hand, $0 < x < X_0 < y$ and $T = \inf\{t : X_t \leq x \text{ or } \geq y\}$ then $E[s(X_T)] = s(x) \cdot \text{Pr}(X_T = x) + s(y) \cdot \text{Pr}(X_T = y) = E[s(X_0)]$. Since no closed form for $s(x)$ exists this problem must be solved numerically.

The problem can be generalized to allow us to gain further information about a stopping time T . Suppose we are interested in the first time, T , that the process X_t reaches some level x or hits an upper or a lower bound (y or x). We can at least in principle obtain the moment generating function for T by generalizing the approach described in Section 2.1.

Let $Y_t = f(t, X_t) = F(t)G(X_t)$, which we wish to be a martingale.

Then by Ito's formula we have

$$\begin{aligned} dY &= \dot{F}Gdt + FG'dX + \frac{1}{2}FG''(dX)^2 \\ &= FG'\sigma X dZ + \left[\dot{F}G + \frac{1}{2}\sigma^2 X^2 FG'' - vXFG' + \mu FG' \right] dt \end{aligned}$$

For Y_t to be a martingale we therefore require the $[.]dt$ term to be equal to zero. That is

$$\begin{aligned} -\frac{\dot{F}(t)}{F(t)} &= \frac{1}{G(x)} \left[\frac{\sigma^2 x^2}{2} G''(x) + (-vx + \mu)G'(x) \right] = \lambda \\ \Rightarrow F(t) &= F_0 \exp(-\lambda t) \end{aligned}$$

and $G(x)$ satisfies:

$$x^2 G''(x) + (-\alpha x + \beta)G'(x) - \theta G(x) = 0$$

where $\alpha = 2v/\sigma^2$, $\beta = 2\mu/\sigma^2$ and $\theta = 2\lambda/\sigma^2$.

Again, no general form for $G(x)(\lambda)$ can be found, so numerical solutions seems to provide the way forward. However, it may be possible to prove qualitative results regarding the shape of the distribution of T .

3 Model 2: Continuous Proportion Portfolio Insurance

Black and Jones (1988) and Black and Perold (1992) discuss an investment strategy called Continuous Proportion Portfolio Insurance (CPPI) which is appropriate for funds which have some sort of minimum funding constraint imposed by either by law or by the trustees of the fund.

When the funding level is low ($A/L < M$) all assets should be invested in a low risk portfolio (relative to the M). As A/L rises above M any surplus and, perhaps more, should be invested in higher risk assets.

This is in contrast to the static investment strategy discussed in Section 2 which rebalances the portfolio continuously to retain the same proportion of assets in each asset class.

Suppose that we have two assets in which we can invest. Asset 1 is risk free and offers an instantaneous rate of return of δ_1 . Asset 2 is a risky asset with $d\delta_2(t) = \delta_2 dt + \sigma_2 dZ_t$. $\delta_1 < \delta_2$ and $\sigma_2^2 > 0$ (with $\sigma_2^2 > 0$). Since asset 2 is risky we have $\delta_2 > \delta_1$.

Let $p(t)$ be the proportion of assets at time t which are invested in asset 2 and let X_t be the funding level at time t . Under the static investment strategy $p(t) = p$ for all t . Under the CPPI strategy $p(t)$ depends on X_t only: $p(t) = 0$ whenever $X_t < M$; and $p(t) = p(X_t) > 0$ when $X_t > M$. A strategy which results in $p(t) > 1$ for some values of X_t allows for the risk-free asset to be sold short.

We consider the case $p(t) = (X_t - M)/X_t$. Then

$$\begin{aligned} dX_t &= p(t)X_t d\delta_2(t) + (1 - p(t))X_t \delta_1 dt + (k - \delta_v)Ldt - kX_t dt \\ &= (X_t - M)(\delta_2 dt + \sigma_2 dZ_t) + M\delta_1 dt + (k - \delta_v)Ldt \\ &\quad - k(X_t - M)dt - kMdt \end{aligned}$$

Hence

$$\begin{aligned} d(X_t - M) &= c \cdot dt - a(X_t - M)dt + \sigma_2(X_t - M)dZ_t \\ \text{where } a &= k - \delta_2 \\ c &= (k - \delta_v)L - (k - \delta_1)M \\ X_t - M &\sim \text{Inverse-Gamma}(\alpha - 1, \beta) \\ \text{where } \alpha &= 2\left(1 + \frac{a}{\sigma_2^2}\right) \end{aligned}$$

$$\begin{aligned}\beta &= \frac{2c}{\sigma_2^2} \\ \Rightarrow E[X_t - M] &= \frac{\beta}{\alpha - 2} \\ \text{Var}(X_t - M) &= \frac{\beta^2}{(\alpha - 2)^2(\alpha - 3)}\end{aligned}$$

Therefore we have

$$\begin{aligned}E[X_t] &= M + \frac{(k - \delta_v)L - (k - \delta_1)M}{k - \delta_2} \\ \text{Var}[X_t] &= \left[\frac{(k - \delta_v)L - (k - \delta_1)M}{k - \delta_2} \right]^2 \frac{\sigma_2^2}{2(k - \delta_2) - \sigma_2^2}\end{aligned}$$

provided $k - \delta_2 > \frac{1}{2}\sigma_2^2$

From these equations, we see that we require $c > 0$ to ensure that $X_t > M$ for all t almost surely (that is, the risk-free interest plus the amortization effort must be sufficient to keep the funding level above M). We also require $a > 0$ (that is, $k > \delta_2$) to ensure that X_t does not tend to infinity almost surely. Finally we can see that the variance will be infinite if $k - \delta_2 \leq \frac{1}{2}\sigma_2^2$.

4 Comparing models 1 and 2

Models 1 and 2 describe two quite different asset allocation strategies and it is, therefore, useful to be able to compare them and to decide which strategy is better and when. The following theorem answers this to a certain extent.

Theorem 4.1

Suppose that we have a risk-free asset (with $d\delta_1(t) = \delta_1 \cdot dt$) and a risky asset (with $d\delta_2(t) = \delta_2 \cdot dt + \sigma_2 dZ(t)$).

Under CPPI the mean funding level is $\mu = E[X_t]$ and its variance is $\sigma_C^2 = \text{Var}[X_t]$. Under a static investment strategy we invest a proportion p in the risky asset and $1 - p$ in the risk-free asset.

There exists p such that under the static investment strategy $E[X_t] = \mu$ (as with CPPI) and $\text{Var}[X_t] = \sigma_S^2 < \sigma_C^2$.

Proof See Cairns (1996) but note that the appropriate value of p is $(\mu - M)/\mu$.

Interpretation: In the variance sense, the static strategy is *more efficient* than CPPI: that is, given a CPPI strategy we can always find a static strategy which delivers the same mean funding level but a lower variance.

One example illustrating this result is plotted in Figure 1.

Under CPPI we have $\delta_1 = 0.02, \delta_2 = 0.05, \delta_v = 0.015, \sigma_2^2 = 0.15^2, L = 1, M = 0.7$ and $k = 0.1$. This gives rise to $E[X_t] = 1.28, \text{Var}[X_t] = 0.313^2$. The mean is relatively high because the valuation rate of interest δ_v appears very cautious. However, the use of such a cautious basis is not necessarily too far from regular practice.

Under the equivalent static strategy which has $E[X_t] = 1.28$ we invest 45.3% of the fund in asset 2 and 54.7% of the fund in the risk-free asset 1. The stationary variance of the fund size is then found to be $\text{Var}[X_t] = 0.243^2$.

The static variance is significantly less than that for CPPI. This is not too evident from Figure 1, but arises out of the fact that the CPPI density has a much fatter right hand tail. CPPI also gives a much more skewed distribution.

Now there are various reasons for why we may prefer CPPI to the static strategy. Principally this will happen when the objective of the pension fund is more than just to minimize the variance of the contribution rate. For example, there may be a penalty attached to a funding level which is below some minimum. In the example above, if this is anything below about 0.9 then CPPI may be favoured. More generally some utility functions may result in a higher expected utility for CPPI (in particular, those which penalize low funding levels).

Conversely there exist utility functions which result in optimal strategies which are the exact opposite of CPPI. For example, Boulier *et al.* (1995) maximize the function

$$V = \int_0^{\infty} \exp(-\beta s) C(t)^2 ds$$

where $C(t) = N + D(t, X_t)$ is the contribution rate at time t .

They found that the optimal strategy was to invest in risky assets when the funding level is low and to move into the risk-free asset as the funding level increases. The rationale behind this is that if there is no minimum funding constraint then: (a) one should try to reach a high funding level as quickly as possible, no matter how risky the strategy; and (b) when a high funding position is reached then this should be protected. Investing in a low risk strategy when the funding level is high will do two things: (a) protect the low contribution rate; and (b) reduce the risk that if too much surplus is generated then the benefits will have to be improved.

In practice, one may wish to combine these two extremes by having a bell shaped asset allocation: that is, one which moves into the risk-free asset if the funding level

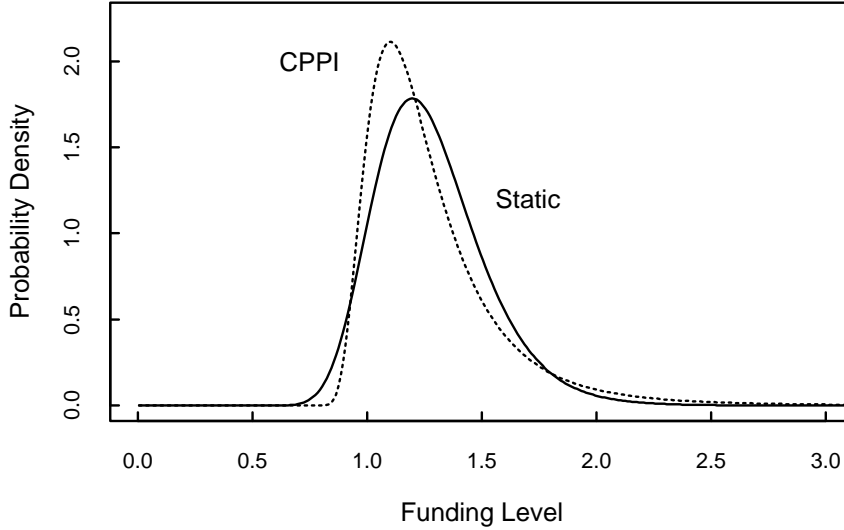


Figure 1: Comparison of the stationary densities for the Static and CPPI asset allocation strategies. Static (solid curve): $E[X_t] = 1.28$, $Var[X_t] = 0.243^2$. CPPI (dotted curve): $E[X_t] = 1.28$, $Var[X_t] = 0.313^2$.

approaches the minimum or if the funding level gets quite high and into more risky assets if the funding level lies between these two extremes.

5 Model 4: A generalization of CPPI

Section 3 described CPPI in its most basic form. Portfolio A was considered to be risk free for the purposes of minimum funding, while Portfolio B was a more risky portfolio offering higher expected returns. If the funding level (the A/L ratio) according to some prescribed basis lies below some minimum M then all assets would be invested on the low-risk asset A. If the A/L ratio is above M then a multiple c of the surplus assets over this minimum would be invested in the risky asset B. Given the existence of a risk-free asset A, and provided the level of adjustment for surplus or deficit is high enough then such a strategy ensures that the A/L ratio never falls below M , provided it starts above this level.

Here we will generalize this strategy to take account of the fact that often it is not possible to construct a completely risk-free portfolio (since the nature of the liabilities means that it is rarely possible for us to match them with appropriate

assets).

Suppose that we may invest in a range of n assets. The values of these assets all follow correlated Geometric Brownian Motion. Thus asset j produces a return in the time interval $[t, t+dt)$ of

$$d\delta_j(t) = \delta_j dt + \sum_{k=1}^n c_{jk} dZ_k(t)$$

where $Z_1(t), \dots, Z_n(t)$ are independent standard Brownian Motions.

At all times portfolio A invests a proportion π_j^A in asset j for $j = 1, 2, \dots, n$, with the portfolio being continually rebalanced to ensure that the proportions invested in each asset remain constant.

Portfolio B follows the same strategy but has a different balance of assets $\{\pi_j^B\}_{j=1}^n$. Portfolio B invests in what may be regarded as more risky assets than does portfolio A.

For portfolio A the return in the time interval $[t, t+dt)$ is

$$d\delta_A(t) = \sum_{j=1}^n \pi_j^A \left(\delta_j dt + \sum_{k=1}^n c_{jk} dZ_k(t) \right)$$

similarly for portfolio B the return in the time interval $[t, t+dt)$ is

$$d\delta_B(t) = \sum_{j=1}^n \pi_j^B \left(\delta_j dt + \sum_{k=1}^n c_{jk} dZ_k(t) \right)$$

The matrix $C = (c_{jk})$ is somewhat arbitrary but has the constraint that $CC^T = V$ where V is the symmetric covariance matrix for the n assets.

These equations can be condensed into the following forms:

$$\begin{aligned} d\delta_A(t) &= \delta_A dt + \sigma_{AA} dZ_A(t) + \sigma_{AB} dZ_B(t) \\ d\delta_B(t) &= \delta_B dt + \sigma_{BA} dZ_A(t) + \sigma_{BB} dZ_B(t) \end{aligned}$$

where $\delta_A = \sum_{j=1}^n \pi_j^A \delta_j$

$$\delta_B = \sum_{j=1}^n \pi_j^B \delta_j$$

$$\text{and if } S = \begin{pmatrix} \sigma_{AA} & \sigma_{AB} \\ \sigma_{BA} & \sigma_{BB} \end{pmatrix}$$

$$\text{then } SS^T = \begin{pmatrix} \pi_A^T V \pi_A & \pi_A^T V \pi_B \\ \pi_B^T V \pi_A & \pi_B^T V \pi_B \end{pmatrix}$$

Thus without loss of generality we may work with two assets 1 and 2 instead of the two portfolios A and B.

At any time a proportion of the fund $p(t)$ is invested in asset 2. Thus the return in the time interval $[t, t+dt)$ is

$$d\delta(t) = (1 - p(t))d\delta_1(t) + p(t)d\delta_2(t)$$

$$\text{where } d\delta_1(t) = \delta_1 dt + \sigma_{11}dZ_1(t) + \sigma_{12}dZ_2(t)$$

$$d\delta_2(t) = \delta_2 dt + \sigma_{21}dZ_1(t) + \sigma_{22}dZ_2(t)$$

In a continuous time stationary pension fund model there is a continuous inflow of contribution income $C(t)$ and a continuous outflow of benefit payments B . The contribution rate is made up of two parts: the normal contribution rate N ; and an adjustment for the difference between the funding level $X(t)$ and the target level of L . Thus $C(t) = N + k(L - X(t))$.

The stochastic differential equation governing the dynamics of the fund size is therefore

$$dX(t) = X(t)d\delta(t) + [N - B + k(L - X(t))]dt$$

Note that if δ_v is the valuation force of interest then N , B and L are related by the balance equation $0 = dL = \delta_v L dt + (N - B)dt$ which implies that $N - B = -\delta_v L$. Hence

$$dX(t) = X(t)d\delta(t) + [(k - \delta_v)L - kX(t)]dt$$

Generalizing the formulation of Black and Jones (1988) we suppose that

$$p(t) = \frac{p_0 + p_1 X(t)}{X(t)}$$

Then (abbreviating $X(t)$ by X and $dX(t)$ by dX etc.) we have

$$\begin{aligned}
dX &= (-p_0 + (1 - p_1)X)\{\delta_1 dt + \sigma_{11}dZ_1 + \sigma_{12}dZ_2\} \\
&\quad + (p_0 + p_1X)\{\delta_2 dt + \sigma_{21}dZ_1 + \sigma_{22}dZ_2\} \\
&\quad - kX dt + (k - \delta_v)L dt \\
&= [p_0(\sigma_{21} - \sigma_{11}) + ((1 - p_1)\sigma_{11} + p_1\sigma_{21})X]dZ_1 \\
&\quad + [p_0(\sigma_{22} - \sigma_{12}) + ((1 - p_1)\sigma_{12} + p_1\sigma_{22})X]dZ_2 \\
&\quad + [p_0(\delta_2 - \delta_1) + (k - \delta_v)L]dt \\
&\quad + [(1 - p_1)\delta_1 + p_1\delta_2 - k]X dt \\
&= (\alpha + \beta X + \gamma X^2)^{1/2}dZ_3 + \mu dt - \nu X dt
\end{aligned}$$

where $Z_3(t)$ is a standard Brownian Motion and

$$\begin{aligned}
\alpha &= p_0^2[(\sigma_{21} - \sigma_{11})^2 + (\sigma_{22} - \sigma_{12})^2] \\
\beta &= 2p_0\left[(\sigma_{21} - \sigma_{11})((1 - p_1)\sigma_{11} + p_1\sigma_{21})\right. \\
&\quad \left.+ (\sigma_{22} - \sigma_{12})((1 - p_1)\sigma_{12} + p_1\sigma_{22})\right] \\
\gamma &= [(1 - p_1)\sigma_{11} + p_1\sigma_{21}]^2 + [(1 - p_1)\sigma_{12} + p_1\sigma_{22}]^2 \\
\mu &= p_0(\delta_2 - \delta_1) + (k - \delta_v)L \\
\nu &= k - (1 - p_1)\delta_1 - p_1\delta_2
\end{aligned}$$

This stochastic differential equation for $X(t)$ is therefore in the correct form for Theorem 1.1. Thus the stationary distribution of $X(t)$ is

$$f_X(x) = k' \exp \left[2a \tan^{-1} \frac{x+b}{c} \right] (\alpha + \beta x + \gamma x^2)^{-1-\nu/\gamma}$$

for $-\infty < x < \infty$

$$\text{where } a = \frac{1}{\sqrt{4\alpha\gamma - \beta^2}} \left(\frac{\nu\beta}{\gamma} + 2\mu \right)$$

$$b = \frac{\beta}{2\gamma}$$

$$c = \frac{\sqrt{4\alpha\gamma - \beta^2}}{2\gamma}$$

This is true provided that it is not possible to synthesize a risk-free asset out of the two portfolios. If that is the case then we will have $4\alpha\gamma - \beta^2 = 0$.

An example of this is given in Figure 2. Here we have $\delta_1 = 0.02$, $\delta_2 = 0.05$, $\delta_\nu = 0.02$, $k = 0.1$, $L = 1$, $\sigma_{11} = 0.04$, $\sigma_{12} = \sigma_{21} = 0.08$ and $\sigma_{22} = 0.15$. The asset allocation strategy uses $p_0 = -0.8$ and $p_1 = 1$. This gives $E[X_t] = 1.11$ and $\text{Var}[X_t] = 0.439^2$. Figure 2 also plots the density for the equivalent static strategy. This strategy used a linear combination of portfolios 1 and 2 (with $p = 0.275$) and gives $E[X_t] = 1.11$ and $\text{Var}[X_t] = 0.343^2$. We see that generalized CPPI appears to have a similar effect to the more basic form: that is, the distribution has lower probabilities of low funding levels, a fat tail and is more skewed than the static strategy.

It should be noted that below a funding level of $M = -p_0/p_1$ the new CPPI strategy goes short in asset 2 and long in asset 1. Furthermore, there is nothing to stop the funding level going negative (although the probability that this happens in any one year is very small). This is because at that point the fund is long in asset 1 and short in asset 2. If asset 2 performs much better than asset 1 then the funding level will continue to move in a negative direction. In effect, when the funding level goes below M , the level of risk increases again. To avoid this problem, Cairns (1996) considers the case

$$p(t) = \left\{ \begin{array}{ll} \frac{p_0 + p_1 X_t}{X_t} & \text{when } X_t \geq -p_0/p_1 \\ 0 & \text{when } X_t < -p_0/p_1 \end{array} \right\}$$

This strategy remains wholly in asset 1 below the minimum and means that X_t will remain positive with probability 1.

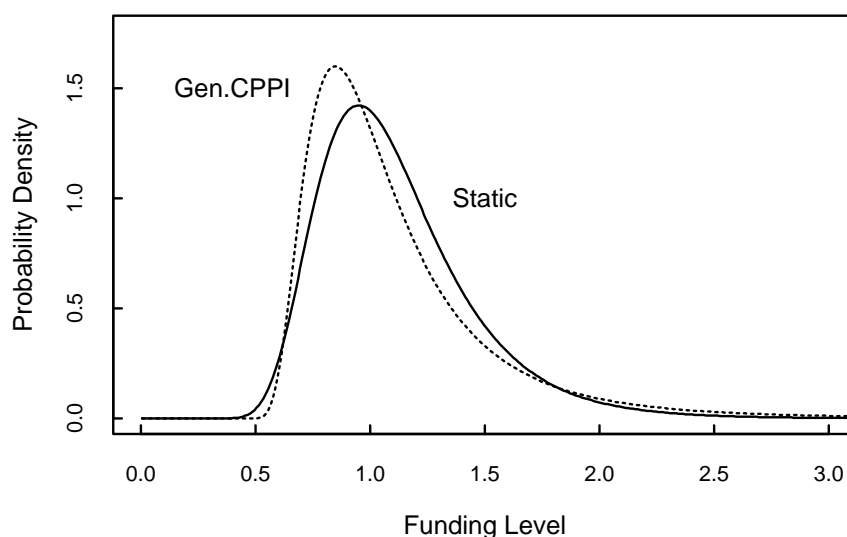


Figure 2: Comparison of the stationary densities for the Static and Generalized CPPI asset allocation strategies. Static (solid curve): $E[X_t] = 1.11$, $Var[X_t] = 0.343^2$. Generalized CPPI (dotted curve): $E[X_t] = 1.11$, $Var[X_t] = 0.439^2$.

6 References

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