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A FAMILY OF TERM-STRUCTURE MODELS FOR LONG-TERM RISK MANAGEMENT AND DERIVATIVE PRICING

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In this paper we propose a new family of term-structure models based on the Flesaker and Hughston (1996) positive-interest framework. The models are Markov and time homogeneous, with correlated Ornstein-Uhlenbeck processes as state variables. We provide a theoretical analysis of the one-factor model and a thorough emprical analysis of the two-factor model. This allows us to identify the key factors in the model affecting interest-rate dynamics. We conclude that the new family of models should provide a useful tool for use in long-term risk management. Suitably parameterized, they can satisfy a wide range of desirable criteria, including:

- sustained periods of both high and low interest rates similar to the cycle lengths we have observed over the course of the 20th century in the United Kingdom and the United States
- realistic probabilities of both high and low interest rates consistent with historical data and without the need for regular recalibration
- a wide range of shapes of yield curves, again consistent with what we have observed in the past and including the recent Japanese yield curve.

KEY WORDS: term-structure model, multifactor, positive interest, Ornstein-Uhlenbeck, time-homogeneous, Japan, dynamic financial analysis

1. INTRODUCTION

We propose a new family of models for the term structure of interest rates. A primary motivation for the development of these models was the perceived need for new models that combine the following two characteristics:

- reasonable, arbitrage-free dynamics in the short term
- realistic dynamics in the long term.

Recent years have seen considerable research and interest-rate frameworks dealing with the first of these characteristics; see, for example, Vasicek (1977), Cox, Ingersoll, and

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Ross (1985; hereafter CIR), Black and Karasinski (1991), Hull and White (1990), Heath, Jarrow, and Morton (1992; hereafter HJM), Brace, Gatarek, and Musiela (1997), Jamshidian (1997), and Duffie and Kan (1996) or the books by James and Webber (2000), Hunt and Kennedy (2000), Rebonato (2002), and Cairns (2004). Development of these models has been driven primarily by the need for models to price and hedge relatively short-term interest-rate derivatives.

A number of models also have been developed—mainly in an insurance context—that address the second characteristic (see, e.g., Wilkie 1995; Yakoubov, Teeger, and Duval 1999). These models are, strictly, arbitrage free but they are typically framed in discrete time and were not designed for short-term risk management or derivative pricing. The models were developed because of the clear need within life insurance and pensions to assess long-term asset and liability risks.

Until recently, little research has been conducted on the development of models that satisfy both of these characteristics. As interest rates in the early 2000s fell to historical lows, this work has become very important, particularly as a result of substantial increases in the value of interest-rate guarantees issued by some insurance companies on a variety of contracts (e.g., see Waters, Wilkie, and Yang 2003). Besides the insurance industry's need for good internal models, banks have also developed a need for good long-term interest-rate models as demand from insurers for certain long-maturity derivatives has increased. Additionally, changes in accounting practices have led to a requirement for interest-rate models that can deal with pricing (the so-called "fair value" of liabilities) as well as long-term risk management.

1.1. Data

Before we describe the main contents of this paper we will take a brief look at some historical data to provide us with some pointers toward desirable characteristics of an interest-rate model.

In Figures 1.1 and 1.3 we have plotted the development over time of short- and long-term interest rates in the United Kingdom and the United States of America. In Figures 1.2 and 1.4 short-term rates are plotted against long-term rates. We can make the following observations:

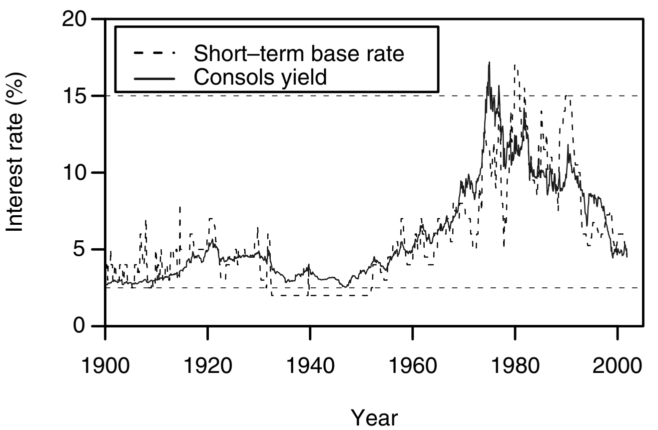

FIGURE 1.1. U.K. interest rates 1900–2002. Bank of England base rate (short-term) and U.K. consol's yields (perpetual bonds) plotted against time.

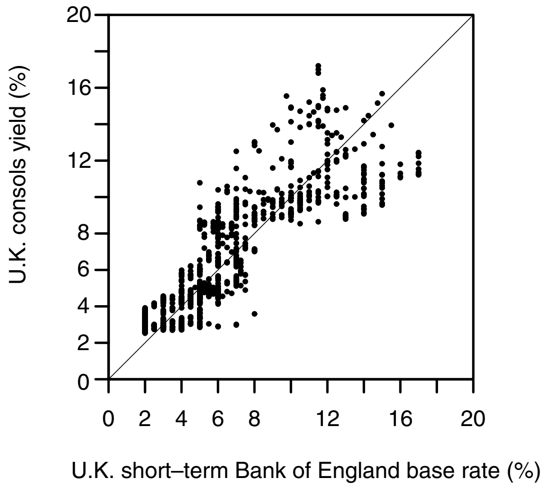

FIGURE 1.2. U.K. interest rates 1900–2002. Short-term versus long-term interest rates.

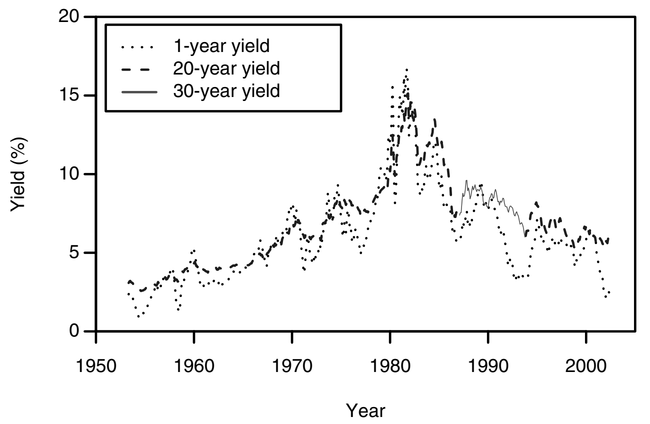

FIGURE 1.3. U.S. interest rates 1954–2002. One-year interest rates and long-bond yields. Long-bond yields are mostly 20-year par yields with 30-year yields to fill in a gap in the 20-Year data in the 1980s.

- Short- and long-term interest rates are highly, but imperfectly correlated in the medium and long term. In the shorter term the rates are still correlated, but to a smaller extent.
- Since the short-term and long-term interest rates are not perfectly correlated (comonotonic), the data are clearly inconsistent with the use of a one-factor, time-homogeneous model (as considered by Chan et al. 1992).
- Short-term interest rates are more volatile.
- Short- and long-term interest rates are subject to sustained periods of both high (e.g., 1970–1990 in the United Kingdom) and low interest rates (U.K.: 1900–1960). We can also remark that the existence of such long cycles in interest rates makes it essentially impossible to test for stationarity in the various series.
- Both short- and long-term interest rates have varied over a considerable range (U.K.: 2% to 17%; U.S.A.: 1% to 17%).

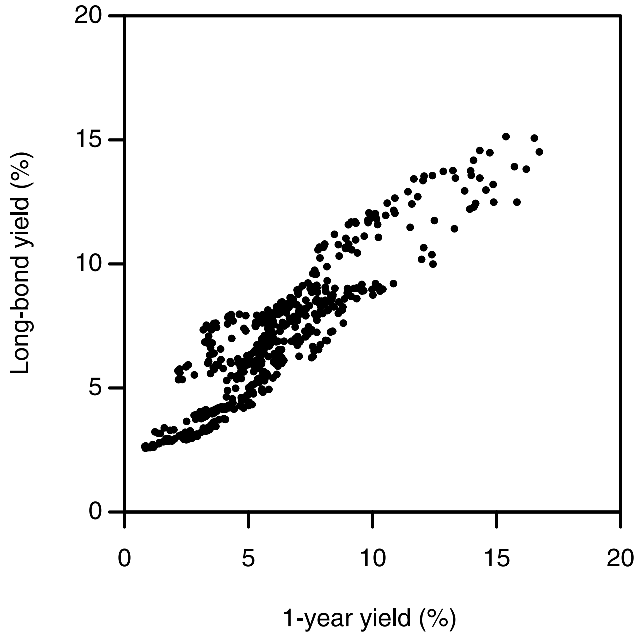

FIGURE 1.4. U.S. interest rates 1954–2002. Short-term versus long-term interest rates.

In general, although not always, continuous-time, arbitrage-free models include a degree of mean reversion that is too strong to allow for long cycles without compromising the short-term dynamics. A family of models that provides an exception to this is the multifactor CIR class (e.g., see Longstaff and Schwartz 1992; Duffie 2001, Chap. 7). However, these models have the problem that all interest rates other than the risk-free rate, $r(t)$, have strictly positive lower bounds, which compromises the pricing of derivatives that come into the money when interest rates are low. Additionally, this class of models cannot reproduce the sort of yield curve experienced in Japan in 2002 where interest rates out to maturities of several years were very low (similar to curve F in Figure 4.1; see Section 4).

Based on these observations we can draw up a list of desirable characteristics for a term-structure model, supplemented by others that fit in with the aim of using a model for long-term risk management.

1. The model should be arbitrage free to allow its use in derivative pricing and short-term dynamic hedging.
2. All interest rates should be positive and all rates should be able to take values arbitrarily close to zero (albeit with very small probability).
3. The model should be framed in continuous time. This allows implementation in discrete time with any length of time step, Δt, without the need to construct a new model each time we change Δt.
4. Short-term interest-rate dynamics should be consistent with what we observe in historical data.
5. The model should permit straightforward analytical or numerical calculation of bond and derivative prices.

6. Interest rates should be mean reverting. This requirement reflects the perception that whenever interest rates reach extreme levels the government will intervene to bring rates back to more reasonable levels rather than let rates continue unchecked to greater extremes.
7. The model should be flexible enough (a) to give rise to a range of different yield curve shapes (consistent with those we have observed in the past) and (b) to be able to deal with a variety of derivative contracts.
8. The model should give rise to sustained periods of both high and low interest rates of all maturities in a manner consistent with what we have observed in the past.
9. Interest rates with different terms to maturity should, with reasonable probability, be able to attain both high and low values consistent with what we have observed in historical data (e.g., the 2% to 17% observed in the U.K. data).

The final points in this list relate to consistency with the past. If we are concerned with long-term risk management, then it is appropriate to go beyond this by applying rigorous statistical methods to estimate model parameters, assess the goodness of fit, and quantify the degree of parameter uncertainty. The use of historical data alone can result in wide confidence intervals for some parameters—intervals that can sometimes be reduced by incorporating information from the derivatives market.

As with all modeling exercises, we are subject to both model risk and parameter uncertainty. In general, parameter uncertainty tends to be the more important source of error for long-term risk management (e.g., see Cairns 2000). This means that probability statements derived from the use of a single model and parameter set should be treated with caution. However, what is important in any modeling exercise is that the model gives rise to a wide and realistic range of future scenarios. For example, it is important that long cycles can occur and that interest rates can exceed 15% from time to time with reasonable probability rather than not at all. It is much less important to be able to say that a particular event will happen with probability exactly 0.01. It could equally be 0.005 or 0.04. The aim is then to identify which types of scenario pose the greatest risk to a financial institution and which do not, and then act accordingly to reduce these risks.

These final statistical issues are left for future work.

1.2. Outline of the Paper

We now move on to the construction of a family of models that address all of these desirable charateristics.

In Section 2 we review, briefly, the general positive-interest framework developed by Flesaker and Hughston (1996; hereafter FH), Rogers (1997), and Rutkowski (1997). We then use the FH approach to propose a general family of multifactor models for the term structure with a multivariate Ornstein-Uhlenbeck process, $X(t)$, as its driver. In particular, we will see that zero-coupon bond prices can be expressed in the time-homogeneous form

$$P(t, T) = \frac{\int_{T-t}^{\infty} H(u, X(t))\, du}{\int_0^{\infty} H(u, X(t))\, du}$$

for some function $H(u, x)$.

In Section 3 we prove certain theoretical properties of the one-factor version of the model. In particular, we prove its asymptotic similarity to the Black and Karasinski (1991) model when the risk-free rate $r(t)$ gets very small and to the Vasicek model when $r(t)$ gets very large.

In Section 4 we provide an extensive analysis of a two-factor version of the model. Here we make a variety of qualitative observations based on accurate numerical calculations of bond prices and interest rates. In particular, we aim to demonstrate that the new family of models satisfies all of the nine desirable characteristics if we choose our parameters in an appropriate way.

In Section 5 we propose a extension of the family of models analyzed in this paper similar to Duffie and Kan's (1996) exposition on affine term-structure models.

2. A MODEL FOR FIXED-INTEREST BOND PRICES

2.1. Background

We propose a family of multifactor models for the fixed-interest term structure that makes use of the framework developed by Flesaker and Hughston (1996), and later extended by Rutkowski (1997) and Rogers (1997).

We define $P(t, T)$ to be the price at time t of a zero-coupon bond that matures at time T. Under the FH framework the dynamics of these prices are linked to the progress of a family of martingales, $M(t, s)$, under some reference or pricing measure $\hat{P}$.

Let $M(t, s)$, for $0 \leq t \leq s < \infty$, be a family of strictly positive diffusion processes over the index s which are martingales with respect to t under some probability measure $\hat{P}$; that is, given s, for $t < u < s$, $E_{\hat{P}}[M(u, s) \mid \mathcal{F}_t] = M(t, s)$. Furthermore, we define $M(0, s) = 1$ for all s and we assume in this paper that for each s, $M(t, s)$ is a diffusion process adapted to a finite (say n) dimensional Brownian motion, $\hat{Z}(t)$ (under $\hat{P}$).

FH proposed that zero-coupon bond prices are defined by the equation

$$P(t, T) = \frac{\int_T^\infty M(t, s)\,\phi(s)\,ds}{\int_t^\infty M(t, s)\,\phi(s)\,ds} \tag{2.1}$$

for some deterministic function $\phi(s)$.

A somewhat more general and abstract form of this framework was proposed by Rutkowski (1997) and Rogers (1997). Rutkowski defined

$$P(t, T) = \frac{E_{\hat{P}}[A(T) \mid \mathcal{F}_t]}{A(t)}, \tag{2.2}$$

where $A(t)$ is a strictly positive supermartingale under the measure $\hat{P}$. For this reason we will often refer to $\hat{P}$ as the *pricing measure*. The FH form we use in this paper is obtained by setting $A(t) = \int_t^\infty M(t, s)\phi(s)\,ds$. Rutkowski and Rogers each demonstrated that models of this general type are arbitrage free. These developments were anticipated to some extent in earlier work by Constantinides (1992) where the "pricing kernel" formula (2.2) was introduced for the case where $\hat{P}$ is implicitly identified with the real-world measure P.

Where the objective is to price a derivative contract that pays $V(T)$ at time T, the price (using the Rutkowski 1997 formulation) at time t is

$$V(t) = \frac{E_{\hat{P}}[A(T)V(T) \mid \mathcal{F}_t]}{A(t)}. \tag{2.3}$$

We will make use of the FH formulation (eq. 2.1) in our development of the new family of models.

Since $M(0, s) = 1$ for all s we may infer that, if we choose to calibrate initial theoretical to observed prices, we require

$$\phi(s) = \frac{\partial}{\partial s} P(0, s) \tag{2.4}$$

up to a constant, nonzero scaling factor.

The instantaneous forward rates are given by

$$f(t, T) = -\frac{\partial}{\partial T} \log P(t, T) = \frac{M(t, T)\phi(T)}{\int_T^\infty M(t, s)\phi(s)\, ds}, \tag{2.5}$$

which implies that for the short rate we have

$$r(t) = f(t, t) = \frac{M(t, t)\phi(t)}{\int_t^\infty M(t, s)\phi(s)\, ds}. \tag{2.6}$$

Although we can write an expression for the short rate, $r(t)$, in this way, it is not possible, in general, to express the dynamics of $r(t)$ as a simple diffusion process, as we can for the Vasicek model and various other well-known models.

We can also write expressions for bond volatilities which enable us to link the model into the HJM framework. Since $M(t, T)$ is a strictly positive martingale under $\hat{P}$ for each T, we can write $dM(t, T) = M(t, T)\sigma(t, T)'\, d\hat{Z}(t)$, where $\hat{Z}(t)$ is a standard n-dimensional Brownian motion under $\hat{P}$ and $\sigma(t, T)$, for fixed T, is a previsible $n \times 1$ vector process. We now define the $n \times 1$ vector

$$V(t, T) = \frac{\int_T^\infty M(t, s)\, \phi(s)\sigma(t, s)\, ds}{\int_T^\infty M(t, s)\, \phi(s)\, ds}. \tag{2.7}$$

The dynamics of the zero-coupon bond prices can then be expressed in the form

$$\frac{dP(t, T)}{P(t, T)} = r(t)\, dt + S_P(t, T)'(d\hat{Z}(t) - V(t, t)\, dt) \tag{2.8}$$

where

$$S_P(t, T) = V(t, T) - V(t, t)$$

(see, e.g., Cairns 2004).

It follows that the vector $S_P(t, T)$ is the price volatility function with each of its n components defining the volatility of the price of a particular bond with respect to each of the n sources of uncertainty.

Since we have expressed the price dynamics in the way given above, we can immediately see that if

$$\tilde{Z}(t) = \hat{Z}(t) - \int_0^t V(s, s)\, ds,$$

then $dP(t, T) = P(t, T)(r(t)\, dt + S_P(t, T)'\, d\tilde{Z}(t))$.

Suppose that $\sigma(t, T)$ has been defined in such a way that the Novikov condition

$$E_{\hat{P}}\left[\exp\left(\frac{1}{2}\int_0^t \sum_{i=1}^n V_i(s, s)^2\, ds\right)\right] < \infty$$

is satisfied (e.g., see Karatzas and Shreve 1998). Then, by the Cameron-Martin-Girsanov (CMG) Theorem (e.g., see Karatzas and Shreve 1998), there exists a measure Q equivalent to $\hat{P}$ under which $\tilde{Z}(t)$ is an n-dimensional Brownian motion. Given the form of $dP(t, T)$

we can see that Q is the usual risk-neutral measure. In particular, if each $\sigma_i(t, T)$ for all t, $T > t$, is bounded, then we can see from equation (2.7) that $V_i(s, s)$ must be bounded, so the Novikov condition is satisfied.

We can also consider the dynamics of the forward-rate curve $f(t, T)$ under $\hat{P}$ and Q. Since $f(t, T) = M(t, T)\phi(T)/\int_T^\infty M(t, s)\,\phi(s)\,ds$, a straightforward application of Ito's Lemma and the Product Rule gives us

$$\begin{aligned} df(t, T) &= f(t, T)(\sigma(t, T) - V(t, T))'\{d\hat{Z}(t) - V(t, T)\,dt\} \\ &= f(t, T)(\sigma(t, T) - V(t, T))'\{d\tilde{Z}(t) - S_P(t, T)\,dt\}. \end{aligned}$$

2.2. A Specific Multifactor Model

We now take the general FH formulation and propose a specific model for the family of martingales $M(t, T)$. For the given choice of processes this allows us to develop straightforward formulas for bond prices.

The model for $M(t, T)$ is governed by the following assumptions:

$$\begin{aligned} M(0, T) &= 1 \quad \text{for all } T \\ dM(t, T) &= M(t, T)\sigma(t, T)'\,d\hat{Y}(t) = M(t, T)\sigma(t, T)'C\,d\hat{Z}(t) \\ &= M(t, T)\sum_{i=1}^{n}\sigma_i(t, T)\,d\hat{Y}_i(t), \end{aligned}$$

where

$$\begin{aligned} d\hat{Y}(t) &= C\,d\hat{Z}(t), \\ \hat{Y}(0) &= 0, \end{aligned}$$

and $\hat{Z}_1(t), \ldots, \hat{Z}_n(t)$ are n independent Brownian motions under $\hat{P}$. (For convenience of our subsequent development, the SDE for $M(t, T)$ is stated slightly differently from the previous section.) In this expression we choose the matrix C such that $CC' = (\rho_{ij})_{i,j=1}^n$ is an instantaneous correlation matrix. It follows then that each of the $\hat{Y}_i(t)$ is a Brownian motion under $\hat{P}$ with $d\langle\hat{Y}_i(t), \hat{Y}_j(t)\rangle = \rho_{ij}\,dt$.

Suppose now that $\sigma_i(t, T) = \sigma_i\exp[-\alpha_i(T - t)]$. Then

$$\begin{aligned} d\log M(t, T) &= \sigma(t, T)'d\hat{Y}(t) - \frac{1}{2}\sum_{i,j=1}^{n}\sigma_i(t, T)\sigma_j(t, T)\,d\langle\hat{Y}_i(t), \hat{Y}_j(t)\rangle \\ \Rightarrow \log M(t, T) &= \sum_{i=1}^{n}\sigma_i\int_0^t e^{-\alpha_i(T-s)}d\hat{Y}_i(s) - \frac{1}{2}\sum_{i,j=1}^{n}\rho_{ij}\sigma_i\sigma_j\int_0^t e^{-(\alpha_i+\alpha_j)(T-s)}ds \\ &= \sum_{i=1}^{n}\sigma_i e^{-\alpha_i(T-t)}\hat{X}_i(t) - \frac{1}{2}\sum_{i,j=1}^{n}\frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i+\alpha_j}e^{-(\alpha_i+\alpha_j)(T-t)}\left(1 - e^{-(\alpha_i+\alpha_j)t}\right), \end{aligned}$$

where $\hat{X}_i(t) = \int_0^t \exp[-\alpha_i(t - s)]\,d\hat{Y}_i(s)$ is an Ornstein-Uhlenbeck process with $\hat{X}_i(0) = 0$ and $d\hat{X}_i(t) = -\alpha_i\hat{X}_i(t)\,dt + d\hat{Y}_i(t)$. Because the $\hat{Y}_i(t)$ are correlated, so too are the $\hat{X}_i(t)$.

Instead of assuming that the function $\phi(s)$ can be arbitrarily specified (e.g., FH suggest equation 2.4), we define

$$\phi(s) = \phi\exp\left[-\beta s + \sum_{i=1}^{n}\sigma_i\hat{x}_i e^{-\alpha_i s} - \frac{1}{2}\sum_{i,j=1}^{n}\frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i+\alpha_j}e^{-(\alpha_i+\alpha_j)s}\right]$$

for some parameters ϕ, β, $\hat{x}_1, \ldots, \hat{x}_n$. (Of these parameters we will see that, provided $\phi \neq 0$, the actual value of ϕ is irrelevant. On the other hand β has an unambiguous interpretation, as we will see in Section 2.3, because it is equal to the constant long-term forward rate.) Then, for $t < s$,

$$\phi(s)M(t,s) = \phi \exp\left[-\beta s + \sum_{i=1}^{n} \sigma_i e^{-\alpha_i(s-t)} X_i(t) - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i+\alpha_j} e^{-(\alpha_i+\alpha_j)(s-t)}\right],$$

where $X_i(t) = \hat{x}_i \exp(-\alpha_i t) + \hat{X}_i(t)$. From this we can see that the complex choice for $\phi(s)$ ensures that several terms in $\phi(s)$ and $M(t,s)$ cancel. $X_i(t)$ is an Ornstein-Uhlenbeck process under $\hat{P}$ with $X_i(0) = \hat{x}_i$ and $dX_i(t) = -\alpha_i X_i(t)\,dt + d\hat{Y}_i(t)$.

It follows that

$$\begin{aligned} A(t,T) &= \int_T^{\infty} \phi(s)M(t,s)\,ds \\ &= \phi e^{-\beta t}\int_{T-t}^{\infty} H(u, X(t))\,du \end{aligned}$$

where

$$H(u,x) = \exp\left[-\beta u + \sum_{i=1}^{n} \sigma_i x_i e^{-\alpha_i u} - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i+\alpha_j} e^{-(\alpha_i+\alpha_j)u}\right].$$

We then have

$$P(t,T) = \frac{A(t,T)}{A(t,t)} = \frac{\int_{T-t}^{\infty} H(u, X(t))\,du}{\int_0^{\infty} H(u, X(t))\,du}. \tag{2.9}$$

Other specific models using Ornstein-Uhlenbeck processes as drivers within this positive-interest framework have been proposed by Rogers (1997). Models of the type given in equation (2.9) are also considered by Brody and Hughston (2001, 2002).

2.3. Forward Rates and Irredeemable Bond Yields

We will now look at forward rates for this model and demonstrate that $f(t,T) \to \beta$ as $T \to \infty$.

Applying the general formula in equation (2.5), we see that the forward-rate curve is

$$\begin{aligned} f(t,T) &= \frac{H(T-t, X(t))}{\int_{T-t}^{\infty} H(u, X(t))\,du} \\ &= \left\{\int_{T-t}^{\infty} \exp\left(-\beta\{u-(T-t)\} + \sum_{i=1}^{n} \sigma_i X_i(t)\left[e^{-\alpha_i u} - e^{-\alpha_i(T-t)}\right]\right.\right. \\ &\qquad \left.\left. - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i+\alpha_j}\left[e^{-(\alpha_i+\alpha_j)u} - e^{-(\alpha_i+\alpha_j)(T-t)}\right]\right) du\right\}^{-1} \\ &= \left\{\int_0^{\infty} \exp\left(-\beta v + \sum_{i=1}^{n} \sigma_i X_i(t) e^{-\alpha_i(T-t)}(e^{-\alpha_i v} - 1)\right.\right. \\ &\qquad \left.\left. - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i+\alpha_j} e^{-(\alpha_i+\alpha_j)(T-t)}\left[e^{-(\alpha_i+\alpha_j)v} - 1\right]\right) dv\right\}^{-1} \end{aligned} \tag{2.10}$$

$$(2.11)\qquad \Rightarrow r(t) = \left\{ \int_0^\infty \exp\left(-\beta v + \sum_{i=1}^n \sigma_i X_i(t)(e^{-\alpha_i v} - 1) - \frac{1}{2} \sum_{i,j=1}^n \frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i + \alpha_j} \left[e^{-(\alpha_i+\alpha_j)v} - 1 \right] \right) dv \right\}^{-1}.$$

If we look more closely at the formula for $f(t, T)$ we can see that as T tends to infinity, the terms in the summations tend to zero. This implies that as T tends to infinity, $f(t, T)$ tends to β; that is, β is the constant long-term forward rate. (This is consistent with a result of Dybvig, Ingersoll, and Ross 1994, who established that, under the assumption of no arbitrage, a model for the term-structure of interest rates must have a nondecreasing long-term *spot* rate. For a more rigorous proof of this result, see Hubalek, Klein, and Teichmann 2002.)

The par yield on irredeemable bonds (assuming continuous payment of coupons) is

$$(2.12)\qquad \rho(t) = \left[\int_0^\infty P(t, t+s)\, ds \right]^{-1} = \frac{\int_0^\infty H(u, X(t))\, du}{\int_0^\infty u H(u, X(t))\, du}.$$

We consider the process for $\rho(t)$ in a later section when we look at the qualitative properties of some specific models.

2.4. Equivalence of $\hat{P}$ and Q

Recall that $d\tilde{Z}_j(t) = d\hat{Z}_j(t) - V_j(t, t)\, dt$, where the $\tilde{Z}_j(t)$ and $\hat{Z}_j(t)$ are Brownian motions under the risk-neutral measure Q and the pricing measure $\hat{P}$ respectively. Here

$$V_j(t, t) = \frac{\int_0^\infty H(u, X(t))\bar{\sigma}_j(u)\, du}{\int_0^\infty H(u, X(t))\, du},$$

where

$$\bar{\sigma}_j(u) = \sum_{i=1}^n \sigma_i e^{-\alpha_i u} c_{ij}.$$

Since $H(u, x) > 0$ for all $u > 0$, $-\infty < x < \infty$, we have $V_j(t, t) < \sum_{i=1}^n |\sigma_i c_{ij}| < \infty$ for all t. This upper bound implies that $E_{\hat{P}}[\exp(\frac{1}{2}\int_0^t \sum_{j=1}^n V_j(s, s)^2\, ds)] < \infty$ (the Novikov condition), which means that $\hat{P}$ and Q are equivalent.

2.5. Time Homogeneity

From the form of $H(u, x)$ and $X(t) = (X_1(t), \ldots, X_n(t))'$ we can see that the $P(t, T)$ are Markov and time homogeneous. As such, the model plus knowledge of $X(t)$ gives us a set of theoretical prices that may differ from those observed. Under the original no-arbitrage FH framework, initial observed prices form part of the input (hence the earlier definition of $\phi(s)$ in equation (2.4)) but this results in the loss of time homogeneity. Each approach has its own merits. Here the intention is that the number of factors, n, should be large enough to ensure that once the $X_i(t)$ have been estimated there is a close correspondence (but probably not exact) between theoretical and observed prices for all t. It can then be argued that frictions in the market, such as transaction costs and buying and selling spreads, prevent exploitation of the price errors. In the context of short-term derivative pricing, the FH approach to $\phi(s)$ is more popular with practitioners. For many models, however, recalibration is theoretically inconsistent with model assumptions (e.g., here, recalibration implies that $\phi(t)$ is stochastic, whereas the model assumes it is deterministic). In other cases (e.g., market models) this inconsistency can be avoided. What is missing at present is a general theory of calibration and updating of information.

2.6. Practical Considerations

The structure of this model is such that only a limited number of random factors ($X_i(t)$ for $i = 1, \ldots, n$) need to be recorded in order for us to be able to reconstruct the evolution of the term structure through time, to calculate prices, returns on assets, and so on. From the computational point of view this offers an advantage over some models (e.g., some based on the HJM framework) that require a record of the entire forward rate curve at all times. This advantage is, of course, shared with many other models but it is important to record this fact. The time homogeneity of the model also results in computational gains, although mainly in the programming phase.

It is necessary to carry out numerical integration in order to compute bond prices and interest rates on a given date and given $\tilde{X}_1(t)$; see equation (2.9). However, this step can be done in a straightforward and accurate way because it only involves one-dimensional integration even for the multifactor version of the model.

Investigation of the properties of the model reveals that the inclusion of correlation (the ρ_{ij}) between the $dX_i(t)$ does not significantly affect the range of possible yields curves on a given date. Consequently the quality of fit on any given date is not substantially enhanced. It is only when we consider the dynamics of the model that the correlations come into play. First, when we analyze historical data, the estimated values of the $dX_i(t)$ may (and, indeed, do) exhibit cross-correlation. Second, such correlations will affect the future distribution of $X(T)$ and therefore will, for example, shift the balance of probabilities between certain shapes of yield curve. This final point is particularly important when we are considering derivative pricing, as in equation (2.3).

The $X_i(t)$, for $i = 1, \ldots, n$, follow standard, correlated Ornstein-Uhlenbeck processes under $\hat{P}$ (in particular, $X(t)$ is a multivariate Gaussian process under $\hat{P}$). It follows that, for low-dimensional models (e.g., $n = 1$ or 2), straightforward and accurate numerical computations are possible for—for example, European- and Bermudan-style derivative prices. Equally, for higher dimensional models, the multivariate normality of $X(t)$ makes the process particularly simple to simulate accurately under $\hat{P}$. The nature of the changes of measure for each of the $X_i(t)$ means that the $X_i(s)$ given $X_i(t)$ are no longer normally distributed under Q, making pricing under $\hat{P}$ much more attractive. For risk management purposes, we are often interested in the real-world measure P. If a constant market price of risk is employed relative to Q, then the same problem exists (that $X_i(s)$ is not normally distributed—although over a one-month period the normal approximation is reasonable). As an alternative we can employ (as we propose in Section 4) a constant change of drift between $\hat{Z}(t)$ and $Z(t)$, the P-Brownian motion. This ensures that the $X_i(t)$ still follow Ornstein-Uhlenbeck processes under P (now with nonzero means). A less desirable consequence, though, is that this does allow risk premia to become negative from time to time. The frequency of this clearly depends on the parametrization of the model and the size of the change of measure, with a low frequency being tolerable for the sake of ease of simulation of the $X_i(t)$.

3. PROPERTIES OF THE ONE-FACTOR MODEL

In Section 4 we will investigate the properties of multifactor (especially two-factor) models from an empirical perspective, with a view to establishing whether the model satisfies various desirable criteria.

In this section we look at theoretical properties of the one-factor model. We have as our driver the one-dimensional process $X(t)$ with SDEs

$$dX(t) = -\alpha X(t)\,dt + d\hat{Z}(t), \quad \text{under } \hat{P},$$

and

$$dX(t) = -\alpha X(t)\,dt + d\tilde{Z}(t) + V(t,t)\,dt, \quad \text{under } Q,$$

where $V(t,t)$ is given below.

We first define the functions (for $k = 0, 1, \ldots$)

$$I_k(x) = \frac{\partial^k}{\partial x^k}\int_0^\infty H(u,x)\,du = \int_0^\infty H(u,x)\sigma^k e^{-k\alpha u}\,du,$$

where

$$H(u,x) = \exp\left(-\beta u + \sigma x e^{-\alpha u} - \frac{\sigma^2}{4\alpha}e^{-2\alpha u}\right).$$

Then we have

$$r(t) \equiv r(X(t)) = \frac{H(0,X(t))}{\int_0^\infty H(u,X(t))\,du} = \frac{H(0,X(t))}{I_0(X(t))}$$

and

$$V(t,t) = \frac{\int_0^\infty H(u,X(t))\sigma e^{-\alpha u}\,du}{\int_0^\infty H(u,X(t))\,du} = \frac{I_1(X(t))}{I_0(X(t))}.$$

By application of Ito's lemma it is straightforward to show that

(3.1) $$dr(t) = \hat{m}(X(t))\,dt + \hat{s}(X(t))\,d\hat{Z}(t), \quad \text{under } \hat{P},$$

or

(3.2) $$dr(t) = \tilde{m}(X(t))\,dt + \tilde{s}(X(t))\,d\tilde{Z}(t), \quad \text{under } Q,$$

where

(3.3) $$\hat{s}(x) = \tilde{s}(x) = r(x)\left(\sigma - \frac{I_1(x)}{I_0(x)}\right),$$

(3.4) $$\hat{m}(x) = r(x)\left[-\alpha\sigma x + \frac{1}{2}\sigma^2 + \alpha x\frac{I_1(x)}{I_0(x)} - \frac{1}{2}\frac{I_2(x)}{I_0(x)} + \frac{I_1(x)^2}{I_0(x)^2} - \sigma\frac{I_1(x)}{I_0(x)}\right],$$

and

(3.5) $$\tilde{m}(x) = \hat{m}(x) + V(t,t)\hat{s}(x) = r(x)\left[-\alpha\sigma x + \frac{1}{2}\sigma^2 + \alpha x\frac{I_1(x)}{I_0(x)} - \frac{1}{2}\frac{I_2(x)}{I_0(x)}\right].$$

We will now consider how the dynamics of $r(t)$ depend on $X(t)$ when $X(t)$ tends to its extremes at $-\infty$ and $+\infty$. Although this means considering extremely rare events, the results allow us to compare the model with others that have known properties. This permits us then to "guess" at certain properties of the new model which depend on tail events.

First consider the limit as $x \to -\infty$. In order to make inferences about the drift and the volatility of $r(t)$ for large and negative $X(t)$ we need to understand the behavior of the $I_k(x)$ as $x \to -\infty$.

By substitution of $\exp(-\alpha u) = v$ we have

$$I_k(x) = \frac{\sigma^k}{\alpha} \int_0^1 v^{k+\frac{\beta}{\alpha}-1} \exp\left[\sigma x v - \frac{\sigma^2}{4\alpha} v^2\right] dv.$$

Now we make further substitutions: $y = -x$ and $w = \sigma y v$. Hence,

$$I_k(-y) = \frac{\sigma^k}{\alpha(\sigma y)^{k+\frac{\beta}{\alpha}}} \int_0^{\sigma y} w^{k+\frac{\beta}{\alpha}-1} \exp\left[-w - \frac{w^2}{4\alpha y^2}\right] dw.$$

It follows that

$$\alpha y^{k+\frac{\beta}{\alpha}} \sigma^{\frac{\beta}{\alpha}} I_k(-y) \to \int_0^{\infty} w^{k+\frac{\beta}{\alpha}-1} e^{-w} dw \quad \text{as } y \to +\infty \tag{3.6}$$
$$= \Gamma\left(k + \frac{\beta}{\alpha}\right).$$

From this we can infer first that

$$I_k(x) \sim \frac{\Gamma\left(k + \frac{\beta}{\alpha}\right)}{\alpha \sigma^{\beta/\alpha}(-x)^{k+\beta/\alpha}} \quad \text{as } x \to -\infty \tag{3.7}$$

and second that

$$\frac{I_k(x)}{I_0(x)} \sim \frac{\alpha \sigma^{\beta/\alpha}(-x)^{\beta/\alpha}}{\alpha \sigma^{\beta/\alpha}(-x)^{k+\beta/\alpha}} \times \frac{\Gamma\left(k + \frac{\beta}{\alpha}\right)}{\Gamma\left(\frac{\beta}{\alpha}\right)} \tag{3.8}$$
$$= \frac{1}{(-x)^k} \prod_{i=0}^{k-1} \left(\frac{\beta}{\alpha} + i\right) \quad \text{as } x \to -\infty,$$

since $\Gamma(z+1) = z\Gamma(z)$ for all $z \geq 1$.

Now recall that $r(x) = H(0, x)/I_0(x) = \exp(\sigma x - \sigma^2/4\alpha)/I_0(x)$. From (3.7), $\log I_0(x) = O(\log(-x))$ as $x \to -\infty$. It follows that

$$\log r(x) = \sigma x - \frac{\sigma^2}{4\alpha} - \log I_0(x) \tag{3.9}$$
$$\sim \sigma x \quad \text{as } x \to -\infty.$$

Next consider the volatility of $r(X(t))$ as $X(t) \to -\infty$. From (3.8) we know that $I_1(x)/I_0(x) \to 0$ as $x \to -\infty$. It follows that $\hat{s}(x)/r(x) = \tilde{s}(x)/r(x) \to \sigma$ as $x \to -\infty$ or

$$\tilde{s}(x) = \hat{s}(x) \sim \sigma r(x) \quad \text{as } x \to -\infty. \tag{3.10}$$

Finally, consider the drift of $r(X(t))$ as $X(t) \to -\infty$. In equation (3.5) the dominant term in the brackets is $-\alpha\sigma x$ as $x \to -\infty$ since $\alpha x I_1(x)/I_0(x) = O(1)$ and $\frac{1}{2} I_2(x)/I_0(x) = O((-x)^{-2})$ as $x \to -\infty$. For similar reasons, the dominant term in the brackets in equation (3.4) is also $-\alpha\sigma x$ as $x \to -\infty$. Since, in addition, $\sigma x \sim \log r(x)$ as $x \to -\infty$ (equation 3.9) we conclude that the dominant term in both $\hat{m}(x)$ and $\tilde{m}(x)$ is $-\alpha r(x) \log r(x)$ as $x \to -\infty$. That is,

$$\hat{m}(x) \sim -\alpha r(x) \log r(x) \tag{3.11}$$

$$\tilde{m}(x) \sim -\alpha r(x) \log r(x) \quad \text{as } x \to -\infty. \tag{3.12}$$

It follows from equations (3.10) and (3.12) that for values of $r(t)$ close to zero the dynamics of $r(t)$ are similar to those of the Black and Karasinski (1991) model.

We next look at the dynamics of $r(t)$ for large, positive values of $X(t)$. This time we use the substitution $v = \sigma x(1 - e^{-\alpha u})$, which gives us

$$I_k(x) = \frac{\sigma^{k-1} e^{\sigma x}}{\alpha x} \int_0^{\sigma x} \left(1 - \frac{v}{\sigma x}\right)^{\frac{\beta}{\alpha}+k-1} \exp\left[-v - \frac{\sigma^2}{4\alpha}\left(1 - \frac{v}{\sigma x}\right)^2\right] dv.$$

If we focus on the integral in this expression, we see that, as $x \to \infty$,

$$\int_0^{\sigma x} \left(1 - \frac{v}{\sigma x}\right)^{\frac{\beta}{\alpha}+k-1} \exp\left[-v - \frac{\sigma^2}{4\alpha}\left(1 - \frac{v}{\sigma x}\right)^2\right] dv \to \int_0^{\infty} e^{-v-\sigma^2/4\alpha} dv = e^{-\sigma^2/4\alpha}.$$

Hence,

$$I_k(x) \sim \frac{\sigma^{k-1} e^{\sigma x - \sigma^2/4\alpha}}{\alpha x} \quad \text{as } x \to \infty. \tag{3.13}$$

It follows trivially that

$$\frac{I_k(x)}{I_0(x)} \to \sigma^k \quad \text{as } x \to \infty. \tag{3.14}$$

We now note that, by reorganization of equation (3.4),

$$\hat{m}(x) = r(x)\left[\frac{(I_1(x) - \sigma I_0(x))}{I_0(x)}\left(\alpha x + \frac{I_1(x)}{I_0(x)}\right) + \frac{1}{2}\frac{\left(\sigma^2 I_0(x) - I_2(x)\right)}{I_0(x)}\right]. \tag{3.15}$$

From equation (3.14) we observe that $(I_k(x) - \sigma^k I_0(x))/I_0(x) \to 0$ as $x \to \infty$. This means that we need something more precise than we have in (3.14) to make useful inferences based on (3.15). In particular, (3.15) implies that it is appropriate to investigate the asymptotics of $I_k(x) - \sigma^k I_0(x)$. Thus,

$$\begin{aligned} I_1(x) - \sigma I_0(x) &= \frac{e^{\sigma x}}{\alpha \sigma x} \int_0^{\sigma x} \left(\frac{-v}{x}\right)\left(1 - \frac{v}{\sigma x}\right)^{\frac{\beta}{\alpha}-1} \\ &\quad \times \exp\left[-v - \frac{\sigma^2}{4\alpha}\left(1 - \frac{v}{\sigma x}\right)^2\right] dv \\ &\sim -\frac{e^{\sigma x}}{\alpha \sigma x^2} \int_0^{\infty} v \exp\left[-v - \frac{\sigma^2}{4\alpha}\right] dv \quad \text{as } x \to \infty \\ &= -\frac{e^{\sigma x - \sigma^2/4\alpha}}{\alpha \sigma x^2}. \end{aligned} \tag{3.16}$$

Similarly,

$$\begin{aligned} I_2(x) - \sigma^2 I_0(x) &= \frac{e^{\sigma x}}{\alpha x} \int_0^{\sigma x} \left(-\frac{2v}{x} + \frac{v^2}{\sigma x^2}\right)\left(1 - \frac{v}{\sigma x}\right)^{\frac{\beta}{\alpha}-1} \\ &\quad \times \exp\left[-v - \frac{\sigma^2}{4\alpha}\left(1 - \frac{v}{\sigma x}\right)^2\right] dv \\ &\sim -\frac{2e^{\sigma x - \sigma^2/4\alpha}}{\alpha x^2} \quad \text{as } x \to \infty. \end{aligned} \tag{3.17}$$

From (3.13) ($k = 0$) and (3.16) we can infer that

$$\frac{x(I_1(x) - \sigma I_0(x))}{I_0(x)} \to -1 \quad \text{as } x \to \infty$$

and from (3.13) ($k = 0$) and (3.17) we see that

$$\frac{x\big(I_2(x) - \sigma^2 I_0(x)\big)}{I_0(x)} \to -2\sigma \quad \text{as } x \to \infty.$$

Combining these asymptotics with equation (3.15), it follows that

$$\hat{m}(x)/r(x) \to -\alpha \quad \text{as } x \to \infty. \tag{3.18}$$

Also

$$\frac{\tilde{m}(x)}{r(x)} = \frac{\hat{m}(x)}{r(x)} - \frac{I_1(x)}{I_0(x)}\left(\frac{I_1(x) - \sigma I_0(x)}{I_0(x)}\right).$$

But $I_1(x)/I_0(x) \to \sigma$ and $(I_1(x) - \sigma I_0(x))/I_0(x) \to 0$ as $x \to \infty$. Thus we also have

$$\tilde{m}(x)/r(x) \to -\alpha \quad \text{as } x \to \infty. \tag{3.19}$$

Finally, consider the volatility of $r(t)$ (see equation (3.3)) for large, positive $X(t)$. Recall that

$$r(x) = \frac{H(0, x)}{I_0(x)} = \frac{e^{\sigma x - \sigma^2/4\alpha}}{I_0(x)}.$$

Incorporating (3.13) ($k = 0$) this implies that

$$r(x) \sim \alpha\sigma x \quad \text{as } x \to \infty,$$

which is consistent with (3.18) and (3.19).

Therefore,

$$\hat{s}(x) = \tilde{s}(x) = r(x)\left(\frac{\sigma I_0(x) - I_1(x)}{I_0(x)}\right) \to \alpha\sigma \quad \text{as } x \to \infty.$$

So we can see that at its extremes the one-factor model can act like the Black and Karasinski (1991) model when $r(t)$ is small and like the Vasicek (1977) model when $r(t)$ is large. The latter observation suggests that the present model does not share the problem that the Black and Karasinski model has with infinite futures prices (see, e.g., Hunt and Kennedy 2000, p. 322) and infinite expected values for the accumulated value of a cash investment (see, e.g., James and Webber 2000, p. 231).

4. MULTIFACTOR MODELS

In this section we discuss a variety of aspects of the model including qualitative issues relating to calibration of the model parameters. In particular, we aim to establish that the new family of models is suitable for a variety of applications including long-term risk management.

We argue that the α_i are central to the validity of the family of multifactor models. In particular, the smallest of the α_i serves two key purposes. First, it allows long-dated bond yields to take a wide range of values with reasonable probability. Second, a small value of α_i means that $X_i(t)$ reverts very slowly. The consequent long cycles experienced by $X_i(t)$ feed through to sustained periods of both high and low interest rates. Both of these characteristics can be clearly observed in historical data. For example, in

Figure 1.1 we can see that long-dated bond yields have, over a period of 100 years, ranged from below 2.5% to over 15%. Additionally, in many countries there have been periods of several decades when interest rates were low and stable and other sustained periods when interest rates were high and more volatile.

In the numerical examples that follow we will concentrate on the two-factor version of the model. This allows us to visualize certain features of the multifactor version of the model through the use of contour plots. However, it should be borne in mind that other analyses (e.g., Cairns 1998; Feldman et al. 1998; Rebonato 1998) suggest that three or more factors might be appropriate.

4.1. Yield Curves

A wide variety of yield curve shapes can be generated depending on the number of factors. As an example consider the two-factor model with

$$\alpha_1 = 0.6, \quad \alpha_2 = 0.06, \quad \sigma_1 = 0.6, \quad \sigma_2 = 0.4, \quad \rho_{12} = -0.5, \quad \beta = 0.04. \tag{4.1}$$

In Figure 4.1 we can see typical spot-rate curves (i.e., $R(t, T) = -(T-t)^{-1}\log P(t, T)$) for this model and parameter set. Thus we have the classic rising and falling yield curves along with humped, dipped, and flat curves. Of particular note, perhaps, is curve F, which starts and remains close to 0 for some time before gradually picking up. This sort of curve is similar to that which has been typical in Japan for some time (2002). This is a feature that some other positive-interest models, such as the positive affine term-structure models, cannot produce. The key to this type of behavior was indirectly highlighted in

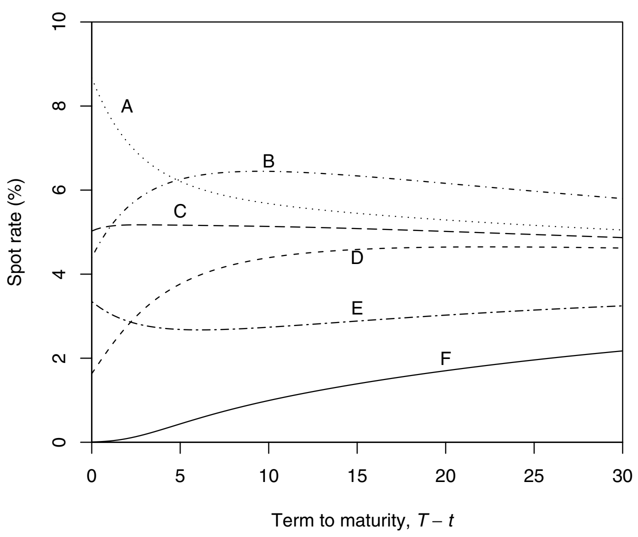

FIGURE 4.1. Two-factor model. Sample spot-rate curves at time t, $R(t, T)$ for six sets of values for $X(t) = (X_1(t), X_2(t))$. Parameter values are $\alpha = (0.6, 0.06)'$, $\sigma = (0.6, 0.4)'$, $\rho_{12} = -0.5$, and $\beta = 0.04$. Curve. A: $X(t) = (1, 3)'$; curve B: $X(t) = (-1, 5)'$; curve C: $X(t) = (0, 3)'$; curve D: $X(t) = (-2, 3)'$; curve E: $X(t) = (1, -1)'$; curve F: $X(t) = (-8, -4)'$. We can see that the two-factor model can produce a wide range of shapes for the spot-rate curve.

our analysis of the one-factor model, where, as $r(t)$ gets small, both its drift and volatility decline to zero. Besides the present model, it is similar to the Black-Karasinski model.

We can also observe from Figure 4.1 that even though all of the spot-rate curves converge ultimately to 4% (β) they still show considerable diversity at $T - t = 30$ years. This is a direct consequence of the low choice of $\alpha_2 = 0.06$. The smaller this value is the greater the variation we see in spot rates and other rates of interest at long maturity dates. Long-dated par yields tend to show up a greater degree of variation, and, indeed, do not converge to the same value. This amount of variation is exactly what we observe in historical data (e.g., see Figures 1.1 and 1.3).

In Figure 4.2 we give contour plots for the risk-free rate, $r(t)$, the 20-year spot and forward rates, $R(t, t+20)$ and $f(t, t+20)$, and the par yield on consols (irredeemable bonds), $\rho(t)$. In each plot, the ellipses centered on (0, 0) give the 90% and 99% unconditional confidence regions for $X(t)$ under $\hat{P}$. This figure tells us how the four rates of interest depend on $X_1(t)$ and $X_2(t)$. We can see that $f(t, t+20)$ depends almost entirely on $X_2(t)$. In contrast, $r(t)$ is most dependent on $X_1(t)$ although it also depends to some extent on $X_2(t)$. In between these two we can see that $R(t, t+20)$ and $\rho(t)$ are primarily determined by $X_2(t)$ but have a small dependence on $X_1(t)$.

4.2. Short- and Long-Term Variability: Approximations

Recall that $f(t, T)(X(t)) = H(T-t, X(t))/\int_{T-t}^{\infty} H(u, X(t))\,du$ and the consols' yield is $\rho(t)(X(t)) = \int_0^{\infty} H(u, X(t))\,du/\int_0^{\infty} uH(u, X(t))\,du$. Without loss of generality suppose that $t = 0$ and define

$$I_0(T, x) = \int_T^{\infty} H(u, x)\,du$$

and

$$I_{1i}(T, x) = \frac{\partial I_0}{\partial x_i}(T, x) = \int_T^{\infty} \sigma_i e^{-\alpha_i u} H(u, x)\,du.$$

Also note that

$$\frac{\partial H}{\partial x_i}(T, x) = \sigma_i e^{-\alpha_i T} H(T, x).$$

Then

$$\frac{\partial f}{\partial x_i}(0, T)(x) = I_0(T, x)^{-2}\left[\frac{\partial H}{\partial x_i}(T, x) I_0(T, x) - H(T, x)\frac{\partial I_0}{\partial x_i}(T, x)\right] \tag{4.2}$$

$$= f(0, T)(x)\sigma_i e^{-\alpha_i T} - f(0, T)(x)\frac{I_{1i}(T, x)}{I_0(T, x)}. \tag{4.3}$$

Now we will make a crude approximation by ignoring the $\rho_{ij}\sigma_i\sigma_j$ terms in $H(T, x)$, $I_0(T, x)$, and $I_{1i}(T, x)$. Thus, for $x = 0$,

$$H(T, 0) \approx e^{-\beta T}$$

$$I_0(T, x) \approx \int_T^{\infty} e^{-\beta u}\,du = \frac{e^{-\beta T}}{\beta}$$

$$I_{1i}(T, x) \approx \int_T^{\infty} \sigma_i e^{-\alpha_i u} e^{-\beta u}\,du = \frac{\sigma_i e^{-(\beta+\alpha_i)T}}{\beta + \alpha_i}.$$

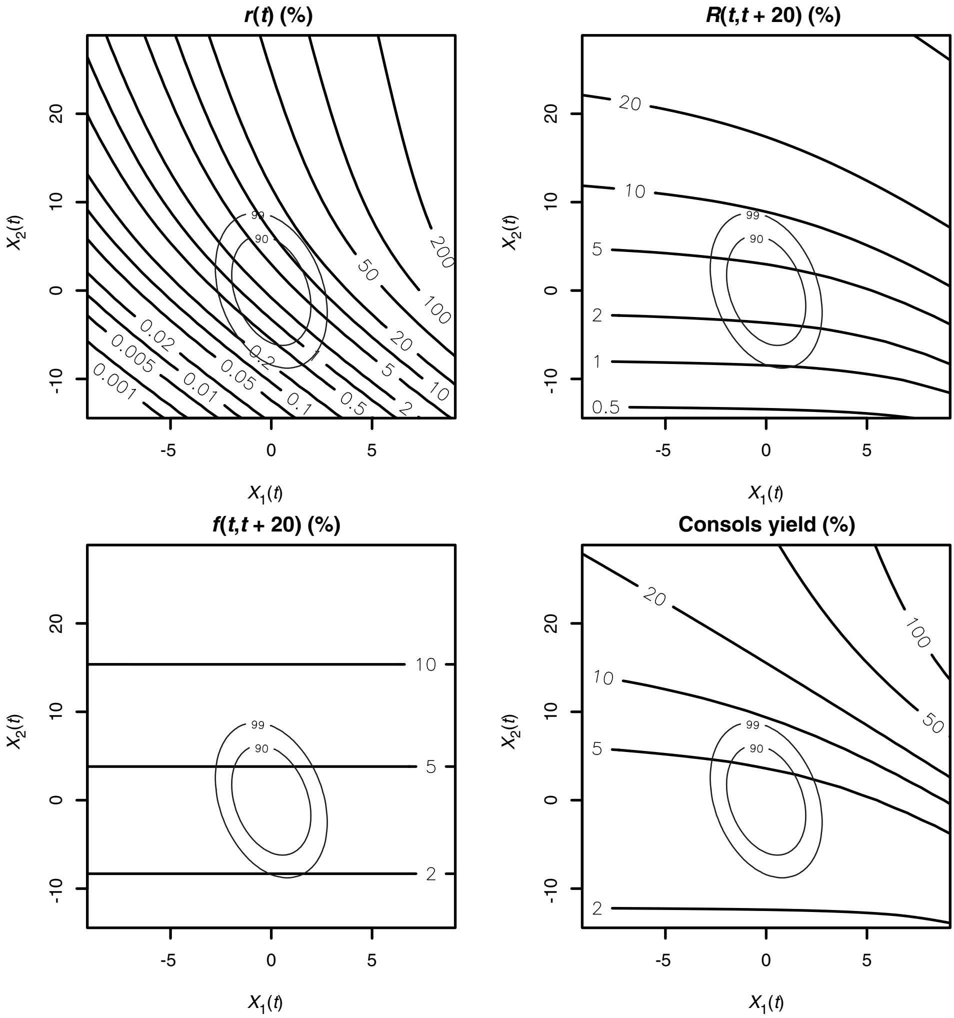

FIGURE 4.2. Dependence of different interest rates on $X_1(t)$ and $X_2(t)$ for the parameters $\alpha = (0.6, 0.06)'$, $\sigma = (0.6, 0.4)'$, $\rho_{12} = -0.5$, and $\beta = 0.04$. Interest rates considered are the risk-free rate, $r(t)$; the 20-year spot rate, $R(t, t + 20)$; the 20-year forward rate, $f(t, t + 20)$; and the par yield on consols (irredeemable bonds), $\rho(t)$. Each graph is a contour plot with lines connecting pairs of points $(X_1(t), X_2(t))$ with the same rate of interest. The ellipses give 90% and 99% confidence regions for $X(t)$ under $\hat{P}$; $r(t)$ has a strong dependence on both $X_1(t)$ and $X_2(t)$; $f(t, t + 20)$ has almost no dependence on $X_1(t)$. The consols' yield and $R(t, t + 20)$ depend more on $X_2(t)$ than on $X_1(t)$.

It follows that

$$(4.4) \qquad \frac{\partial f}{\partial x_i}(0, T)(0) \approx f(0, T)(0)\left[\sigma_i e^{-\alpha_i T} - \frac{\sigma_i e^{-(\beta+\alpha_i)T}/(\beta + \alpha_i)}{e^{-\beta T}/\beta}\right]$$

$$= f(0, T)(0)\frac{\alpha_i \sigma_i e^{-\alpha_i T}}{\beta + \alpha_i}.$$

If we apply the same approximation to $f(0, T)(0)$, we have $f(0, T)(0) \approx \beta$.

It follows that, as a crude approximation,

$$f(t, T)(X(t)) \approx \beta\left(1 + \sum_{i=1}^{n} d_i(T-t)X_i(t)\right)$$

where

$$d_i(\tau) = \frac{\alpha_i \sigma_i e^{-\alpha_i \tau}}{\beta + \alpha_i}. \tag{4.5}$$

Now recall that the local volatility of each $X_i(t)$ is 1 with correlations ρ_{ij}, so the local variance of $f(t, T)$ if $X(t) \approx 0$ will be approximately

$$\beta^2 \sum_{i,j=1}^{n} d_i(T-t)d_j(T-t)\rho_{ij}. \tag{4.6}$$

The unconditional covariance of $X_i(t)$ and $X_j(t)$ is $\rho_{ij}/(\alpha_i + \alpha_j)$. It follows that the unconditional variance of $f(t, T)(X(t))$ will be approximately

$$\beta^2 \sum_{i,j=1}^{n} \frac{d_i(T-t)d_j(T-t)\rho_{ij}}{\alpha_i + \alpha_j}. \tag{4.7}$$

Suppose that $\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$. The effect of the $1/(\alpha_i + \alpha_j)$ components means that the process $X_n(t)$ (smallest α_i) will have proportionately a greater effect on long-term variability than on short-term variability.

We can also see from the definitions of the $d_i(\tau)$ that the process $X_n(t)$ will have proportionately a greater effect on long-dated interest rates than on short-dated rates.

We can show in a similar fashion that the par yield on irredeemable bonds is approximately

$$\rho(t) \approx \beta\left[1 + \sum_{i=1}^{n} e_i X_i(t)\right], \tag{4.8}$$

where

$$e_i = \frac{\alpha_i \sigma_i \beta}{(\beta + \alpha_i)^2}.$$

Hence its short-term variance will be approximately

$$\beta^2 \sum_{i,j=1}^{n} e_i e_j \rho_{ij}$$

and its unconditional variance will be approximately

$$\beta^2 \sum_{i,j=1}^{n} \frac{e_i e_j \rho_{ij}}{\alpha_i + \alpha_j}.$$

As a final comment, refer to equations (4.4) and (4.5), which show that, at least as a first approximation, changes in the forward rate curve are made up of a combination of exponentials with rates $\alpha_1, \ldots, \alpha_n$. This means that over the short term we can generate the usual range of changes in the shape of the forward-rate curve, such as changes in slope, humps, and twists (depending on how many factors there are).

We have stressed that these are crude approximations. However, they are useful as part of a subjective calibration of the model. For example, we can specify the level of variability in short- and long-dated interest rates in both the short term and the long term. These approximations can be used to get a first estimate of the parameter values that will achieve these aims.

4.3. The Volatility Term Structure

We now consider the full term structure of volatility.

Recall from equation (4.3) that the volatility of $f(t, T)$ with respect to changes in $X_i(t)$ is

$$v_i(t, T)(x) = \frac{\partial f}{\partial x_i}(t, T)(x) = f(t, T)(x)\left[\sigma_i e^{-\alpha_i(T-t)} - \frac{I_{1i}(T-t, x)}{I_0(T-t, x)}\right]. \tag{4.9}$$

Now the $X_i(t)$ will be correlated, so that these volatility curves can be thought of as dependent. Recall that we defined the matrix C so as to satisfy $CC' = (\rho_{ij})$. In particular, we now choose C to be lower triangular (thus defining C uniquely), resulting in, for each i,

$$dX_i(t) = -\alpha_i X_i(t)\,dt + \sum_{j=1}^{i} c_{ij}\,d\hat{Z}_j(t).$$

We can then see that the *independent* volatility curve for $f(t, T)$ corresponding to $d\hat{Z}_j(t)$ is

$$\tilde{v}_j(t, T)(x) = \sum_{i=j}^{n} c_{ij} v_i(t, T)(x).$$

We will now illustrate the volatility curves for the parameter set given in equation (4.1). In Figure 4.3 we plot volatility curves for a typical rising yield curve (with a slight hump) and an inverted yield curve.

We can make the following empirical observations:

- Volatility is higher when interest rates are higher. This is merely a reflection of the common factor of $f(t, T)$ in equation (4.9).
- Changes in $X_2(t)$ have, proportionately, a greater effect on longer dated forward rates. This backs up earlier observations.
- The independent volatility curves are, perhaps, more informative. Typically, for the chosen decomposition for C, $\tilde{v}_2(t, T)$ provides us with a change in the slope or level of $f(t, T)$, whereas $\tilde{v}_1(t, T)(x)$ provides us with a twist or a hump.

4.4. Bond Volatilities

In equation (2.8) we noted that the SDE for $P(t, T)$ is

$$dP(t, T) = P(t, T)[r(t)\,dt + S_P(t, T)'(d\hat{Z}(t) - V(t, t)\,dt)],$$

where here we have

$$\begin{aligned}
S_P(t, T) &= V(t, T) - V(t, t), \\
V_j(t, T) &= \frac{\int_{T-t}^{\infty} H(u, X(t))\bar{\sigma}_j(u)\,du}{\int_{T-t}^{\infty} H(u, X(t))\,du} = \frac{\sum_{i=1}^{n} c_{ij} I_{1i}(T-t, X(t))}{I_0(T-t, X(t))}, \\
\sum_{k=1}^{n} c_{ik}c_{jk} &= \rho_{ij}, \\
\bar{\sigma}_j(u) &= \sum_{i=1}^{n} c_{ij}\sigma_i e^{-\alpha_i u}.
\end{aligned} \tag{4.10}$$

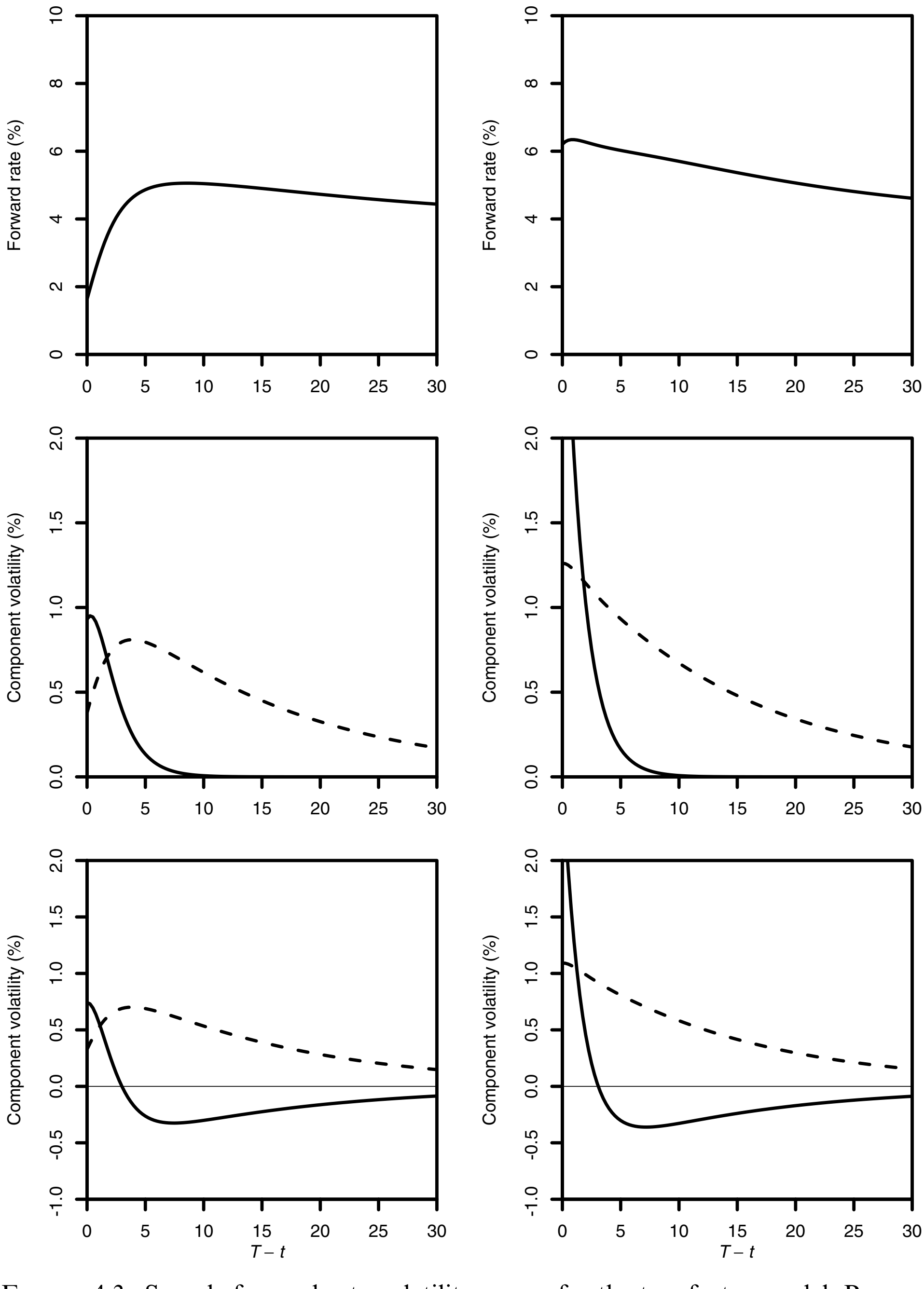

FIGURE 4.3. Sample forward-rate volatility curves for the two-factor model. Parameter values are $\alpha = (0.6, 0.06)'$, $\sigma = (0.6, 0.4)'$, and $\rho_{12} = -0.5$, and $\beta = 0.04$. State variable $X(t)$ in the left-hand column: $X_1(t) = -2$, $X_2(t) = 3$; in the right-hand column: $X_1(t) = 0$, $X_2(t) = 4$. Top row: forward-rate curve, $f(t, T)(X(t))$. Middle row: volatility with respect to short-term changes in $X_1(t)$, and $X_2(t)$; the solid curve is the volatility function $v_1(t, T)(X(t))$ (see the main text); the dashed curve is $v_2(t, T)(X(t))$. Bottom row: independent volatility curves with solid curve $\tilde{v}_1(t, T)(X(t))$ and dashed curve $\tilde{v}_2(t, T)(X(t))$.

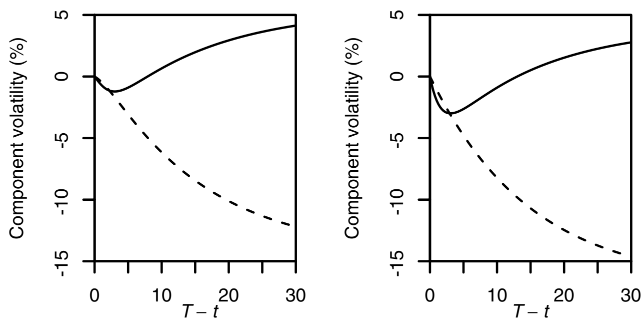

FIGURE 4.4. Sample zero-coupon price volatility curves for $S_{P1}(t, T)$ (solid curve) and $S_{P2}(t, T)$ (dashed curve) for the two-factor model; see equations (4.10). Parameter values are $\alpha = (0.6, 0.06)'$, $\sigma = (0.6, 0.4)'$, $\rho_{12} = -0.5$, and $\beta = 0.04$. State variable $X(t)$ in the left-hand graph: $X_1(t) = -2$, $X_2(t) = 3$; in the right-hand graph: $X_1(t) = 0$, $X_2(t) = 4$.

It follows that the overall squared volatility of $P(t, T)$ is

$$\sum_{j=1}^{n} S_{Pj}(t, T)^2 = \sum_{j=1}^{n}(V_j(t, T) - V_j(t, t))^2$$
$$= \sum_{i,k=1}^{n} \rho_{ik}\Delta_i(t, T)\Delta_k(t, T),$$

where

$$\Delta_i(t, T) = \frac{I_{1i}(T - t, X(t))}{I_0(T - t)} - \frac{I_{1i}(0, X(t))}{I_0(0)}.$$

As illustrative examples, the independent volatility curves $S_{Pj}(t, T)$ for the same two cases as in the previous section are plotted in Figure 4.4.

We can see that typically the sign of the volatility curves is the opposite of that observed in the corresponding forward-rate volatility curves (although the zero-coupon bond price volatilities are more closely linked to spot-rate volatilities).

4.5. Real-World Probabilities and Risk Premiums

Earlier we defined the risk-neutral equivalent measure Q by introducing the drift $V(t, t)$; that is, under Q, the $d\tilde{Z}_j(t) = d\hat{Z}_j(t) - V_j(t, t)\,dt$ are independent standard Brownian motions. This leaves us with the usual dynamics for the $P(t, T)$ under Q: $dP(t, T) = P(t, T)[r(t)\,dt + S_P(t, T)'\,d\tilde{Z}(t)]$. The nature of $V(t, t)$ means that there is no particular gain to be made from looking directly at Q: pricing is much easier under $\hat{P}$ than under Q where the the $X_i(t)$ are no longer normally distributed.

If the model is to be used, for example, as a tool in long-term risk management (e.g., in dynamic financial analysis or pension-plan asset allocation), then we need to consider model dynamics under the real-world measure P. Therefore, we need to model the market prices of risk which take us from the risk-neutral measure Q to the real-world measure P. We introduce appropriate drifts $\gamma_j(t)$, which satisfy the Novikov condition and define

$Z(t)$ to be a standard Brownian motion under P with $dZ(t) = d\tilde{Z}(t) + \gamma(t)dt = d\hat{Z}(t) - V(t, t)\,dt + \gamma(t)\,dt$.

As part of our modeling assumptions we propose that $\gamma(t) = V(t, t) + \theta$ for some constant vector θ. This component of the model is a pragmatic choice that gives $dZ(t) = d\hat{Z}(t) + \theta\,dt$ and ensures that the $X_i(t)$ are normally distributed under P as well as $\hat{P}$ but with nonzero means. Thus

$$\begin{aligned} dX_i(t) &= -\alpha_i X_i(t)\,dt + \sum_{j=1}^{n} c_{ij}(dZ_j(t) - \theta_j\,dt) \\ &= \alpha_i(\mu_i - X_i(t))\,dt + \sum_{j=1}^{n} c_{ij}\,dZ_j(t) \end{aligned}$$

where

$$\mu_i = -\alpha_i^{-1} \sum_{j=1}^{n} c_{ij}\theta_j.$$

In practical terms it is useful to specify the μ_i first and then derive the θ_j. Thus

$$\theta = -C^{-1}\mathrm{diag}(\alpha_i)\mu,$$

where $\mathrm{diag}(\alpha_i)$ is the diagonal matrix with elements $(\alpha_1, \ldots, \alpha_n)$.

Let us now return to the SDE for $P(t, T)$.

$$\begin{aligned} dP(t, T) &= P(t, T)[r(t)\,dt + S_P(t, T)'\,(dZ(t) - \gamma(t)\,dt] \\ &= P(t, T)[(r(t) - S_P(t, T)'(V(t, t) + \theta))\,dt + S_P(t, T)'\,dZ(t)]. \end{aligned}$$

It follows that the risk premium on $P(t, T)$ (i.e., the excess expected return over the risk-free rate, $r(t)$) is

$$-S_P(t, T)'(V(t, t) + \theta). \tag{4.11}$$

Sample contour plots for risk premiums are given in Figure 4.5 for the 30-year zero-coupon bond. In this set of plots we illustrate how risk premiums depend on the parameter vector μ (equivalently, θ).

- The case where $\mu = (-2, 6)'$ gives a reasonable outcome. By overlaying the unconditional confidence region for $X(t)$ we can see which values are most likely under P. Most of the time, then, the 30-year risk premium will be positive. However, from time to time, corresponding to periods when interest rates are low, it will become negative. This is a plausible scenario. For example, in the U.K. in 2002 low interest rates resulted in unprecedented demand for long-dated bonds from life insurers and pension plans in an attempt to protect guaranteed fixed liabilities.
- Other aspects of this choice for μ are reasonable by considering the confidence region relative to lines A and B. Most of the region lies to the left of line A, indicating that the spot-rate curve is typically (but not always) rising from 0 to 20 years. Rising curves are what we expect to see, although this is, of course, connected to positive risk premiums. Most of the region also lies above line B, which indicates that spot rates are typically falling between 20 and 30 years to maturity. This decline is a feature of the study by Brown and Schaefer (2000). This persistent decline at the long end of the yield curve reflects stronger demand for long-dated bonds over medium-dated bonds under most interest-rate scenarios, primarily again from life insurers and pension plans.

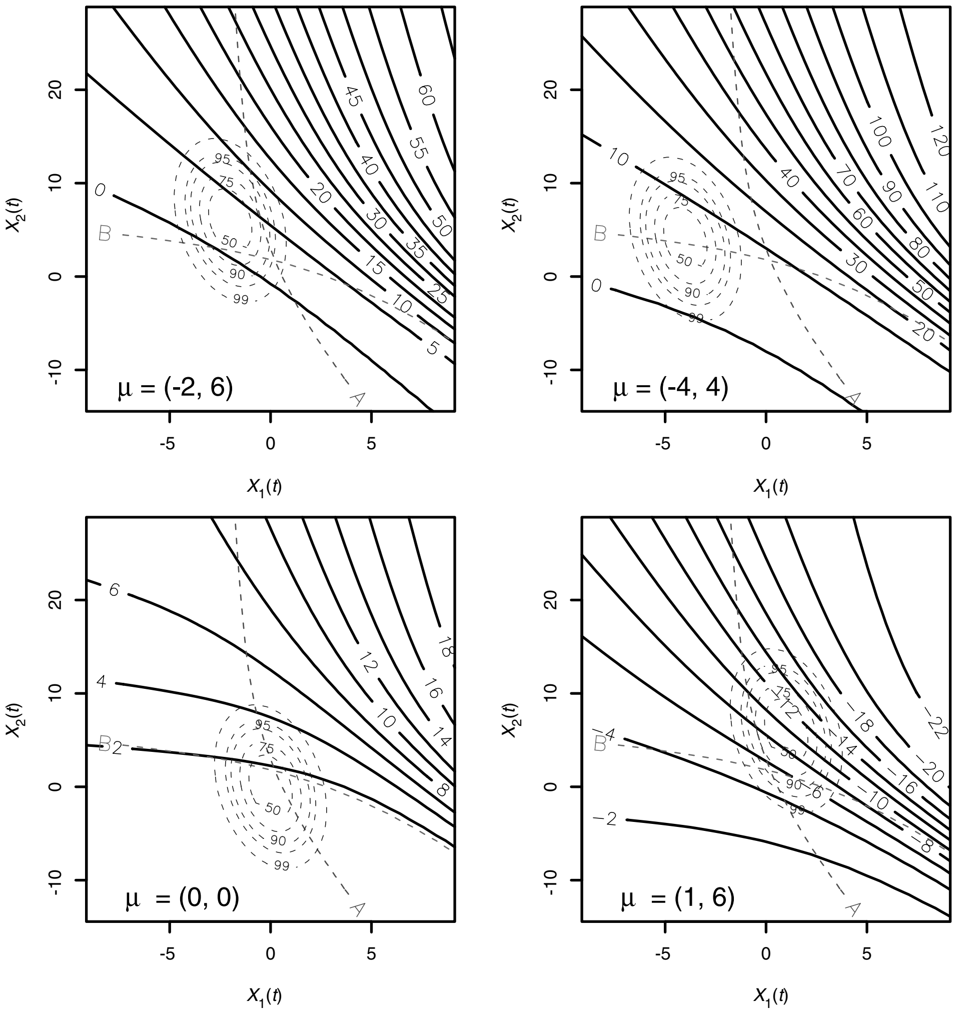

FIGURE 4.5. Risk premiums on the 30-year zero-coupon bond as a function of the state variables $X_1(t)$ and $X_2(t)$ when $\alpha = (0.6, 0.06)'$, $\sigma = (0.6, 0.4)'$, $\rho_{12} = -0.5$, and $\beta = 0.04$. The four plots show how the risk premium depends on the unconditional mean, μ, of $X(t)$ under the real-world measure P. The solid lines give contours (in percentage) of points with equal risk premiums. The ellipses give the 50%, 75%, 90%, 95%, and 99% confidence regions for $X(t)$ under P. Dashed lines A and B give an indication of the shape of the spot-rate curve. To the left (right) of A, $r(t) - R(t, t + 20)$ is negative (positive). Above (below) B, $R(t, t + 20) - R(t, t + 30)$ is positive (negative). The risk premium can be seen to be quite sensitive to the choice of μ. Some choices for μ give intuitively sensible risk premiums; others do not.

Other choices for μ have positive and negative aspects.

- The case $\mu = (-4, 4)$ gives more positive risk premiums. On the other hand, the 20–30 year section of the spot-rate curve is falling as often as it is rising. The nature of the model (in particular, the low value of α_2) means that $X(t)$ could easily spend

50 years or more above line B, giving the impression that spot rates normally fall between 20 and 30 years. The 99% confidence region lies entirely to the left of line A, meaning that $r(t)$ is almost always lower than $R(t, t+20)$. This seems unlikely.

- The case $\mu = (0, 0)'$ has mainly small positive risk premiums with rising spot rates from 20 to 30 years but a significant chance that $r(t)$ is higher than $R(t, t+20)$. In general, interest rates will be relatively low. (See, also, Figure 4.2.)
- The case $\mu = c(1, 6)'$ looks very poor. Risk premiums are all significantly negative and, connected to this, the yield curve is typically falling.

In Figure 4.6 we look at the term structure of the risk premium for our standard parameter set and $\mu = (-2, 6)'$. Two sets of values for $X(t)$ are given $(-2, 0)'$ and $(1, 3)'$. For reference, the spot-rate curves, $R(t, T)$, are plotted (top left), so that we can see that we have one low and rising curve and one high and falling. In the top right-hand plot we can see how the risk premium (equation 4.11) varies with term to maturity. In general, for this parameter set and choice for μ we find that the risk-premium term structure tends to peak at around 5 years. We can also see that, as in Figure 4.5, the risk premium depends on $X(t)$. We note also that when interest rates are very low (particularly short-dated rates), the risk premium can become slightly negative, as indicated in the earlier discussion. In the lower plots we can see the full dependence of the 5-year and 30-year zero-coupon bond risk premiums on $X_1(t)$ and $X_2(t)$. The pattern of dependence is similar but 5-year risk premiums are consistently higher for this choice of parameters.

4.6. Simulation

In Section 4.5 we proposed a particular type of transformation, as part of the model, to take us from $\hat{P}$ to the real-world measure P. This choice makes simulation of sample paths of $X(t)$ simple because we can exploit the known distributional properties of the correlated Ornstein-Uhlenbeck process to simulate exactly in discrete time.

As an example, a typical sample path is represented in Figure 4.7. Here we plot the risk-free rate, $r(t)$, and the consols' rate, $\rho(t)$. Also plotted are the corresponding sample paths for $X_1(t)$ and $X_2(t)$.

Note that $X_2(t)$ has a sustained peak at A and a sustained trough at B, which are reflected in peaks and troughs in the two interest rates. We can clearly see the link between both interest rates and $X_2(t)$. Since $X_2(t)$, through the low value of α_2, has long cycles, we see that both rates of interest are also subject to long cycles of both high and low values. This type of behavior is similar to that observed in both the U.K. and U.S. data in Figures 1.1 and 1.3. We can look at this simulation path by eye and can note that even with as much as 100 years of data we could easily interpret the series as being nonstationary when, in fact, the underlying model is stationary. This serves to highlight the fallacy of trying to prove or disprove stationarity.

The shorter term volatility in $X_1(t)$ is mainly reflected in $r(t)$. This volatility also affects $\rho(t)$ but to a lesser extent and we can consequently observe that $\rho(t)$ is less volatile.

In Figure 4.8 we give a scatter plot of $r(t)$ versus $\rho(t)$ over the 400 years of the simulation, corresponding to Figure 1.2. Most of the points lie to the left of the diagonal $r(t) = \rho(t)$, indicating, as we remarked earlier, that the yield curve is generally upward sloping (but possibly with a hump). We can note that the regression slope would be less than 1. Further empirical analysis of the model has indicated that this slope is very much dependent on the value of ρ_{12}, the correlation between $dX_1(t)$ and $dX_2(t)$. The more negative is ρ_{12}, the steeper is the regression slope.

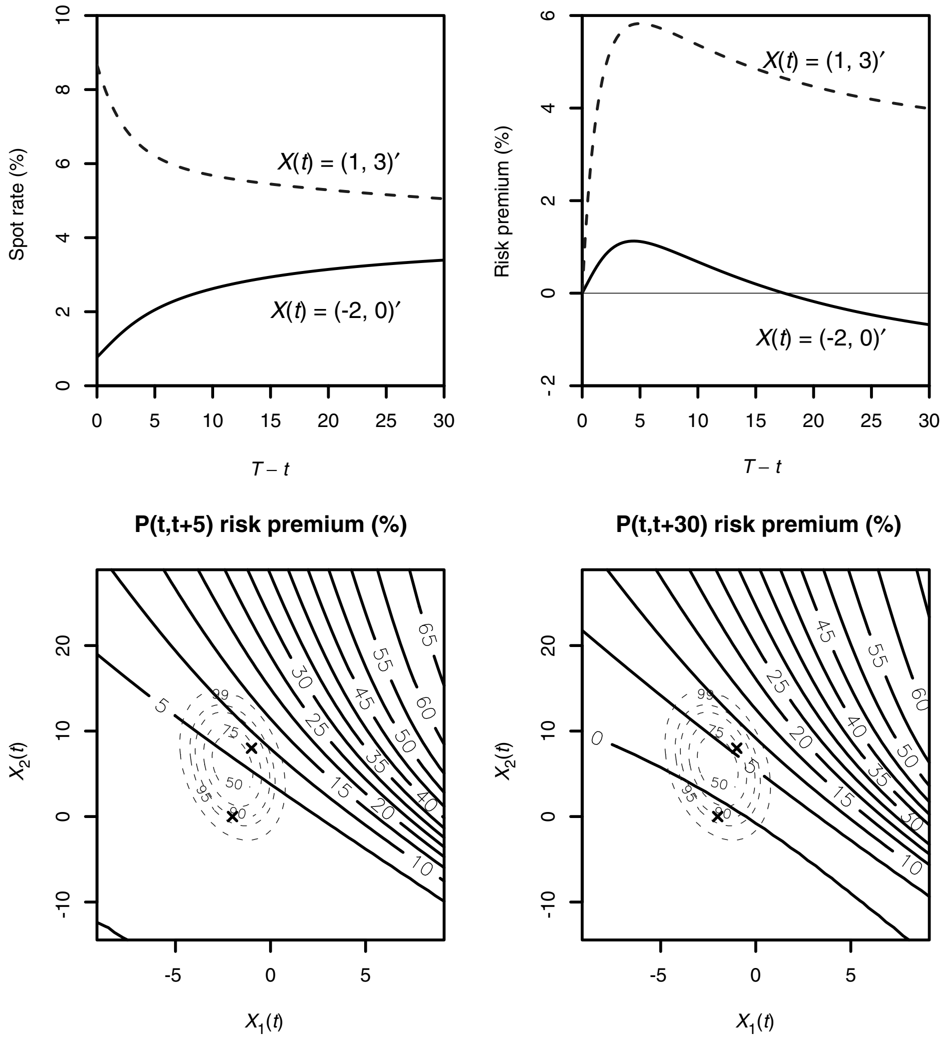

FIGURE 4.6. Risk premiums on zero-coupon bonds related to term to maturity. Parameter values are $\alpha = (0.6, 0.06)'$, $\sigma = (0.6, 0.4)'$, $\rho_{12} = -0.5$, $\beta = 0.04$, and $\mu = (-2, 6)'$. Top left: spot-rate curves for $X(t) = (-2, 0)'$ (solid curve) and $X(t) = (1, 3)'$ (dashed curve). Top right: corresponding risk-premium term structure for zero-coupon bonds. Bottom left: contour plot of the risk premium for 5-year zero-coupon bonds. The crosses mark the positions for $X(t)$ in the upper plots. The ellipses give confidence regions for $X(t)$. Bottom right: contour plot of the risk premium for 30-year zero-coupon bonds. (For a description of the contour plots, see the Figure 4.5 caption.)

5. EXTENSIONS TO A WIDER FAMILY: THE INTEGRATED AFFINE (IA) CLASS

We have worked here with a pricing formula that can be expressed in the form

$$P(t, T) = \frac{A(t, T)}{A(t, t)} = \frac{\int_T^\infty e^{B(t,u)+C(t,u)'X(t)}\,du}{\int_t^\infty e^{B(t,u)+C(t,u)'X(t)}\,du}, \tag{5.1}$$

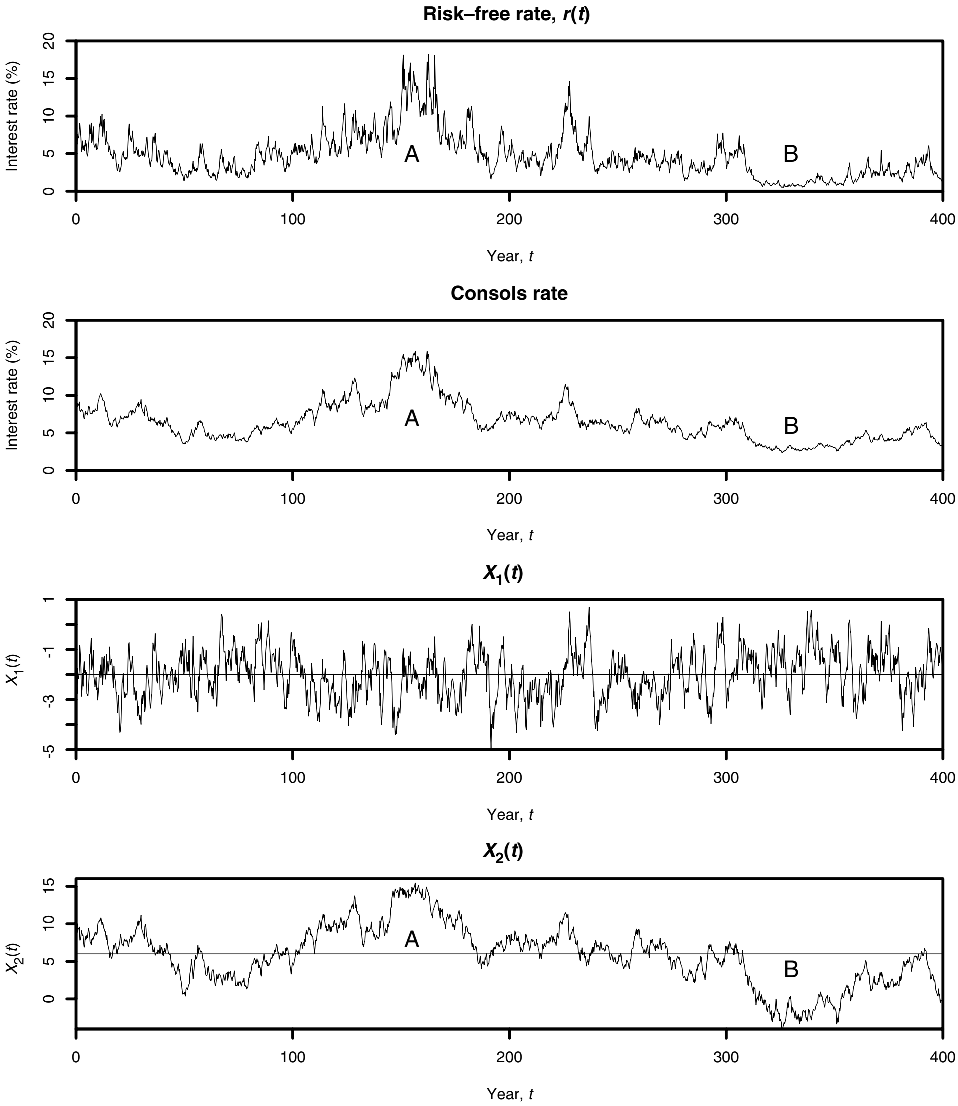

FIGURE 4.7. Simulation example over 400 years for $\alpha = (0.6, 0.06)'$, $\sigma = (0.6, 0.4)'$, $\rho_{12} = -0.5$, $\beta = 0.04$, and $\mu = (-2, 6)'$. In the plots for $X_1(t)$ and $X_2(t)$ the horizontal line indicates the mean-reversion level, μ_i, for $X_i(t)$ under P. In the various plots, A and B correspond to a peak and a trough in $X_2(t)$. We can see sustained periods of both high (at A) and low (at B) interest rates.

where

$$e^{B(t,u)+C(t,u)'X(t)} = \phi e^{-\beta t} H(u-t,\ X(t)) \qquad (\phi = 1, \text{say}).$$

Thus

$$B(t,u) = -\beta u - \frac{1}{2}\sum_{i,j} \frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i + \alpha_j} e^{-(\alpha_i+\alpha_j)(u-t)}$$

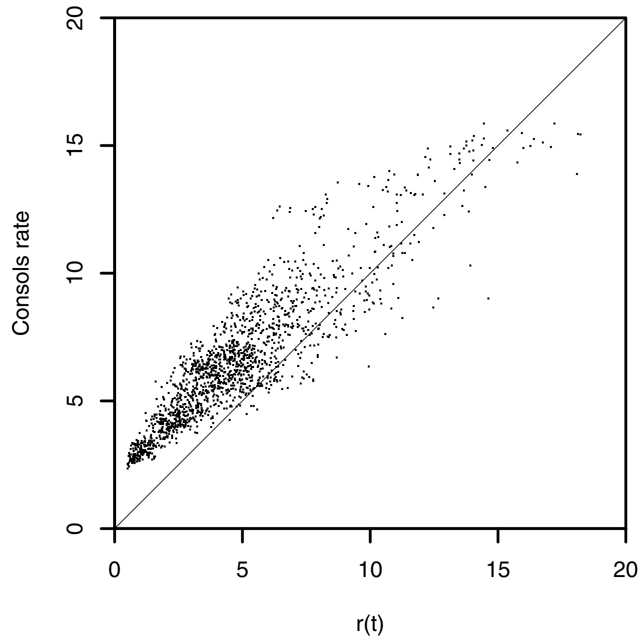

FIGURE 4.8. Simulation example for $\alpha = (0.6, 0.06)'$, $\sigma = (0.6, 0.4)'$, $\rho_{12} = -0.5$, $\beta = 0.04$, and $\mu = (-2, 6)'$. Scatter plot of $r(t)$ versus the consols' rate, $\rho(t)$. The diagonal line is $r(t) = \rho(t)$. Most points lie to the left of the diagonal, indicating that we generally have a rising yield curve.

and

$$C_i(t, u) = \sigma_i e^{-\alpha_i(u-t)}.$$

This suggests an interesting extension to a much wider class of *integrated affine* (IA) *models*. For example, in the one-factor case we can propose a model for $P(t, T)$ as in equation (5.1) but where $dX(t) = \alpha(\mu - X(t))\,dt + \sigma\sqrt{X(t)}\,d\hat{Z}(t)$; see Brody and Hughston 2001, 2002. It is then necessary to establish the relevant functional forms for $B(t, u)$ and $C(t, u)$ to ensure that $\exp[B(t, u) + C(t, u)X(t)]$ is a $\hat{P}$-martingale (equivalent to $\phi(u)M(t, u)$ in Section 2).

Equally, this can be extended to multifactor models. It is well known that the standard, multifactor affine term-structure models $P(t, T) = \exp[\tilde{B}(t, T) + \tilde{C}(t, T)' Y(t)]$ exist for a limited range of diffusion processes for $Y(t)$ (see, e.g., Duffie and Kan 1996). In the context of the IA models we ask the question: *For what n-dimensional diffusion processes, $X(t)$, do there exist deterministic functions $B(t, u)$ and $C(t, u)$ such that* $\exp[B(t, u) + C(t, u)' X(t)]$ *is a martingale for all u?*

The answer to this is as we might expect: $X(t)$ is limited to the same range of processes established by Duffie and Kan (1996). Proof of this follows a similar argument to that of Duffie and Kan with some small differences.

As with the affine term-structure class of models it is expected that the extension proposed here will give access to a wider range of volatility term structures.

6. CONCLUSIONS

We have proposed in this paper a new family of time-homogeneous multifactor models for the term structure of interest rates.

The primary motivation for this is to provide a powerful tool for long-term risk management and the pricing of long-term interest-rate derivatives. Theoretical aspects of the model were investigated first. More important though, we have conducted a thorough numerical analysis of the two-factor model in order to establish the suitability of the wider family for long-term risk management. These investigations have demonstrated that the two-factor model, with a suitable choice of parameters, can satisfy all of the desirable characteristics set out in Section 1. In particular, the model can

- produce sustained periods of both high and low interest rates
- produce wide ranges of values for both short- and long-term interest rates, consistent with historical data
- in extreme cases, give rise to yield curves similar to those experienced in Japan in 2002.

We have not attempted statistical fitting of the model to historical and market data. This is a a considerable piece of research in its own right and is the subject of ongoing work.

Bond and derivative prices need to be calculated numerically. However, we have shown that bond prices require a simple one-dimensional numerical integration procedure. In the case of derivative pricing we will be able to exploit the normal distribution of the state variable, $X(t)$, to develop efficient pricing procedures.

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