Euler–Maclaurin Expansion and Woolhouse's Formula

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Actuarial functions such as the expected present values of life annuities are generally tabulated for integer ages and terms, assuming that the cash flows they represent are payable annually. Thus, actuarial tables would often give, for integer ages x

$$\ddot{a}_{x:\overline{n}|} = \sum_{t=0}^{n-1} v^t \, {}_t p_x, \tag{1}$$

which is the expected present value of an annuity of \$1 per year, payable *yearly* in advance to a person now age x for n years or until they die, whichever happens first: but they would not usually give

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1}{m} \sum_{t=0}^{nm-1} v^{t/m} \, {}_{t/m} p_x, \tag{2}$$

which is the expected present value of an annuity of \$1 per year, payable *m times a year* in advance to a person now age x for n years or until they die, whichever happens first [2, 4]. Nor would the expected present values of annuities payable continuously usually be tabulated, for example,

$$\overline{a}_{x:\overline{n}|} = \int_0^n v^t \, {}_t p_x \, \mathrm{d}t. \tag{3}$$

Annuities and life insurance premiums are often paid more frequently than annually (monthly is common) so $\ddot{a}^{(m)}_{x:\overline{n}|}$, and similar values arise often in practice. Annuities payable continuously are most often used as approximations to annuities payable very frequently (for example, daily or weekly). Nowadays, their expected present values can easily be computed with a spreadsheet, maybe with approximate values of the survival probabilities ${}_{t/m} p_x$ at noninteger terms t/m. However, simple approximations in terms of annuities with annual payments are available, based on the Euler–Maclaurin expansion and Woolhouse's formula, and these are still in use today.

The Euler–Maclaurin formula is a series expansion correcting for the error in approximating the integral of a suitably differentiable function by the trapezium rule; that is, by the integral of a piecewise linear approximating function. Suppose the problem is to find

$$\int_a^b f(x) \, \mathrm{d}x. \tag{4}$$

Dividing the range of integration into N equal steps of length $h = (b - a)/N$, the trapezium approximation is

$$\int_a^b f(x) \, \mathrm{d}x \approx h \left(\sum_{i=0}^{N} f(a + ih) - \frac{f(a) + f(b)}{2} \right). \tag{5}$$

Suppose that $f(x)$ is differentiable $2k$ times, for a positive integer k. The Euler–Maclaurin formula gives correction terms up to the $(2k - 1)$th derivative, as follows

$$\begin{aligned}\int_a^b f(x) \, \mathrm{d}x &= h \left(\sum_{i=0}^{N} f(a + ih) - \frac{f(a) + f(b)}{2} \right) \\ &\quad + \sum_{j=1}^{k-1} \frac{B_{2j}}{(2j)!} h^{2j} \left(f^{(2j-1)}(a) - f^{(2j-1)}(b) \right) \\ &\quad - (b - a) \frac{B_{2k}}{(2k)!} h^{2k} f^{(2k)}(\xi),\end{aligned} \tag{6}$$

where B_i is the ith Bernoulli number [1], and in the final (error) term ξ is some number in (a, b). Explicitly, the first few terms of this expansion are

$$\begin{aligned}&h \left(\sum_{i=0}^{N} f(a + ih) - \frac{f(a) + f(b)}{2} \right) \\ &\quad + \frac{h^2}{12}(f'(a) - f'(b)) - \frac{h^4}{720}(f''(a) - f''(b)) + \cdots\end{aligned} \tag{7}$$

Choosing $f(t) = v^t \, {}_t p_x$ and $h = 1$, we see that the left-hand side is $\overline{a}_{x:\overline{n}|}$, and, ignoring terms in first and higher derivatives on the right-hand side, we have the often-used approximation

$$\begin{aligned}\overline{a}_{x:\overline{n}|} &\approx \ddot{a}_{x:\overline{n+1}|} - \tfrac{1}{2}\left(1 + v^n \, {}_n p_x\right) \\ &= \ddot{a}_{x:\overline{n}|} - \tfrac{1}{2}\left(1 - v^n \, {}_n p_x\right).\end{aligned} \tag{8}$$

If we include the terms in the first derivative, we have the 'corrected trapezium rule' [3], which we can use with

$$\frac{\mathrm{d}}{\mathrm{d}t} v^t \, {}_t p_x = -(\delta + \mu_{x+t}) \, v^t \, {}_t p_x. \tag{9}$$

Woolhouse's formula can be obtained from the Euler –Maclaurin expansion by choosing $h = 1$ and $h =$

$1/m$ in equation (6), and equating the right-hand sides of the two expressions. This gives

$$\frac{1}{m}\sum_{t=0}^{nm} f(t/m) = \sum_{t=0}^{n} f(t) - \left(\frac{1}{2} - \frac{1}{2m}\right)(f(n) + f(0)) + \sum_{j=1}^{k-1} \frac{B_{2j}}{(2j)!}\left(1 - m^{-2j}\right) \times \left(f^{(2j-1)}(0) - f^{(2j-1)}(n)\right) + \text{ error term} \quad (10)$$

$$= \sum_{t=0}^{n} f(t) - \frac{m-1}{2m}(f(n) + f(0)) + \frac{m^2-1}{12m^2}\left(f'(0) - f'(n)\right) - \frac{m^4-1}{720m^4}\left(f''(0) - f''(n)\right)\cdots \quad (11)$$

Almost trivially, the Euler–Maclaurin expansion can be recovered from Woolhouse's formula by letting $m \to \infty$. Choosing $f(t) = v^t\,{}_tp_x$, we see that the left-hand side of equation (11) is $\ddot{a}^{(m)}_{x:\overline{n}|} + (1/m)v^n\,{}_np_x$ and the summation on the right-hand side is $\ddot{a}_{x:\overline{n}|} + v^n\,{}_np_x$. Discarding the terms in first- and higher-order derivatives and simplifying, we have

$$\ddot{a}^{(m)}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - v^n\,{}_np_x\right). \quad (12)$$

Approximations to the expected present values of other annuities, such as those payable in arrears, for the whole of life, and so on, can be derived from this.

References

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