Siberian Mathematical Journal, Vol. 1, No. 0, pp. 1960
Original Russian Text Copyright © 1960 *Foss S. G., Denisov D. È.*

ON TRANSIENCE CONDITIONS FOR MARKOV CHAINS

S. G. Foss and D. È. Denisov UDC 519.21

1. Introduction

Let $X = \{X_n\}_{n\ge 0}$, $X_0 = \text{const}$ be a (generally) time-inhomogeneous Markov chain taking values in a measurable space $(\mathscr{X}, \mathscr{B})$. Let $L : \mathscr{X} \to [0,\infty)$ be an unbounded measurable function; i.e., the complete preimage $L^{-1}([c,\infty))$ of the set $[c,\infty)$ is nonempty for every $c > 0$. We study conditions that guarantee the a.s. convergence of $L(X_n)$ in n to infinity, i.e., such conditions that the equation

$$\mathbf{P}(\lim_{n\to\infty} L(X_n) = \infty) = 1 \tag{1.1}$$

holds for every initial X_0. Equation (1.1) implies transience of the set $\{x : L(x) < N\}$ for every $N > 0$.

As far as we know, general conditions for transience have been studied only in the case of countable Markov chains (i.e., chains with a countable state space $\mathscr{X}$) and under the additional assumption that the values of jumps are bounded. The most general assertion in this case is seemingly the following theorem of [1, p. 31, Theorem 2.2.7].

Theorem 1.1. *Let X be a homogeneous Markov chain with values in a countable set $\mathscr{X}$ which forms a single class of communicating states. Assume that there exist a function $L : \mathscr{X} \to [0,\infty)$, an integer-valued function $v : \mathscr{X} \to \{1, 2, \dots\}$, and numbers $\varepsilon > 0$, $N > 0$, $d > 0$ such that*

(a) $\sup_x v(x) < \infty$,

(b) *for all $x, y \in \mathscr{X}$,*

$$|L(x) - L(y)| > d \quad \textit{implies} \quad p_{x,y} \equiv \mathbf{P}(X_1 = y \mid X_0 = x) = 0 \tag{1.2}$$

(c) *for every $x \in \mathscr{X}$ such that $L(x) \ge N$,*

$$\mathbf{E}\{L(X_{v(x)}) - L(X_0) \mid X_0 = x\} \ge \varepsilon. \tag{1.3}$$

Then the Markov chain X is transient, i.e., for every state $x \in \mathscr{X}$

$$\tau(x) \equiv \min\{n \ge 1 : X_n = x \mid X_0 = x\}$$

is an improper random variable.

Remark 1. If we assume in addition that, for every initial state $X_0 = x$ in the set $\{x : L(x) < N\}$,

$$\min\{n \ge 1 : X_n \ge N\} < \infty \quad \text{a.s.}, \tag{1.4}$$

then (1.1) holds. For (1.4) to hold, it suffices for the set $\{x : L(x) < N\}$ to be finite.

Remark 2. Among the conditions of Theorem 1.1, the most restrictive ones are (1.2) and (1.3). We consider here a more general situation where (1.2) does not hold. On the other hand, to simplify exposition, we assume (1.3) to hold with $v(x) \equiv 1$.

In order to understand what kind of additional restrictions on jump increments should be made in the absence of (1.2), let us consider two examples of homogeneous Markov chains. For $x \in \mathscr{X}$, denote by Δ_x a random variable with distribution $\mathbf{P}(\Delta_x \in B) = \mathbf{P}(L(X_1) - L(x) \in B \mid X_0 = x)$. We will use the notation $a^+ = \max(a, 0)$ and $a^- = -\min(a, 0)$.

This research was partly supported by the A. M. Liapunov French–Russian Institute of Computer Science and Applied Mathematics (Grant 97–07) and INTAS (Grant 97–1625).

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 1, No. 0, pp. ??–??, *******–*******, 1960. Original article submitted October 9, 2000.

Example 1. Let $\mathscr{X} = \{0, 1, 2, \dots\}$ be the set of nonnegative integers, $L(x) = x$, and X the homogeneous Markov chain with transition probabilities $p_{i,j} = \mathbf{P}(X_1 = j \mid X_0 = i)$ given by

$$p_{0,0} = p_{0,1} = 1/2,$$

and for $i = 1, 2, \dots$

$$p_{i,i+1} = 1 - p_{i,0} \in (0, 1), \quad p_{i,j} = 0 \quad \text{for} \quad j \neq 0, j \neq i + 1.$$

This chain is irreducible and aperiodic, and the probability of failure to return to 0 is equal to the infinite product

$$\prod_{i=0}^{\infty} p_{i,i+1},$$

which is positive if and only if the series $\sum_{i=0}^{\infty} p_{i,0}$ converges. Note that the condition $p_{i,0} = o(i^{-1})$ (equivalent to uniform integrability of the random variables Δ_i^-) is necessary but not sufficient for the convergence of the series. If we assume that the sequence $p_{i,0}$ is nonincreasing then the convergence of the above series is equivalent to

$$\int_1^{\infty} \frac{dt}{h(t)} < \infty, \tag{1.5}$$

for every nondecreasing function $h(t)$ such that $h(i) = \frac{1}{p_{i,0}}$ for each integer i. A condition of the form (1.5) will appear later in the statement of the main theorem. Note also that (1.5) necessarily implies the "positive drift" condition (1.3) for a sufficiently large N.

This example shows that, generally speaking, in the presence of positive drift, the uniform integrability of the negative parts of the jumps Δ_x^- does not guarantee transience; we need something more.

Consider now the second example which shows the necessity of restrictions (to be given below) on the distributions of the positive parts of the jumps Δ_x^+.

Example 2. As in the first example, assume that $L(x) = x$ and that a homogeneous Markov chain is defined on the state space $\{0, 1, 2, \dots\}$. Let $\alpha > 1$, $\beta > 1$ be given numbers and let $k \equiv k(\alpha, \beta) = \max\{i : \frac{\alpha}{[i^\beta]} \geq 1\}$. Here $[x]$ represents the positive part of x.

Let the transition probabilities of the chain be as follows:

$$p_{0,0} = p_{0,1} = /2, \quad p_{i,i-1} = 1 - p_{i,i+1} = 1/3 \quad \text{for all} \quad i = 1, \dots, k$$

$$p_{i,l(i)} = 1 - p_{i,i-1} = \alpha i^{-\beta} \quad \text{for} \quad i = k + 1, k + 2, \dots \qquad \text{and} \qquad l(i) = i + [i^\beta],$$

$$p_{i,j} = 0 \quad \text{for} \quad j \neq i - 1, \ j \neq l(i).$$

The Markov chain is irreducible and aperiodic and all states communicate with each other. It is easy to see that $\mathbf{E}\Delta_i \geq \min(1/3, \alpha - 1) \equiv \varepsilon > 0$ for all $i \geq 1$. Negative jumps are bounded and, consequently, any condition like (1.5) holds. On the other hand, since $\beta > 1$, the Markov chain is recurrent. Indeed, fix any integer $N \geq k$. Assume that the chain starts from the state $X_0 = i > N$. We will show that, with probability one, it visits the state N in a finite time. Starting from any state $i > N$, there is a probability

$$\prod_{j=N+1}^{i} p_{j,j-1} \geq \prod_{j=N+1}^{\infty} p_{j,j-1} \equiv \gamma > 0,$$

that the chain hits N by a sequence of (unit) jumps to the left only; in the event of any jump to the right, say, to i_1, there is again a probability at least γ that the chain will hit N by a finite sequence of jumps to the left only, etc. In other words, we consider a sequence of (dependent) trials, each of which has a probability of success at least γ, irrespective of past history. Hence, the probability of an eventual success is equal to one.

On the other hand, if the Markov chain starts from a state $X_0 = x \le N$, then, with probability one, it eventually hits the interval $[N, \infty)$. It follows that the state N (and, therefore, the chain itself) is recurrent. Indeed a more precise analysis shows that the Markov chain is recurrent for $\beta = 1$, too.

In the second example, the positive drift $\mathbf{E}\Delta_i \ge \varepsilon > 0$ is provided by the jumps of the random variables $\{\Delta_x^+\}$ which are not uniformly integrable. Therefore, in the formulation of the transience criterion, it seems natural to require some condition such as uniform integrability on the positive parts of jumps or, more generally, a condition which guarantees a positive drift due to bounded positive jumps.

2. The Main Statement and Its Proof

Let $(\mathscr{X}, \mathscr{B})$ be a measurable space, $\{X_n \equiv X_n^{(x)}\}_{n\ge 0}$ a $\mathscr{X}$-valued time-inhomogeneous Markov chain with an initial value $X_0 = x = \text{const}$ and with transition probabilities

$$P(y, n, B) = \mathbf{P}(X_{n+1} \in B \mid X_n = y), \quad B \in \mathscr{B},\ y \in \mathscr{X}.$$

To simplify exposition, we further assume that the Markov chain can be represented as a *stochastic recursive sequence*

$$X_{n+1} = f_n(X_n, \xi_n), \quad n \ge 0,$$

where $\{\xi_n\}$ is a sequence of independent identically distributed (i.i.d.) random variables (r.v.'s) which are distributed uniformly on $[0, 1]$ and each $f_n : \mathscr{X} \times [0, 1] \to \mathscr{X}$ is a measurable function. This assumption is not too restrictive; for example, it holds if the sigma-algebra $\mathscr{B}$ is countably generated (see, for instance, [2]).

For $m = 0, 1, 2, \ldots$, denote by $\{X_{m+n}^{(x,m)}\}_{n\ge 0}$ the Markov chain that starts at the time instant m from the state $X_m^{(x,m)} = x$ and is defined by the recursive equations

$$X_{m+n+1}^{(x,m)} = f_{m+n}\big(X_{m+n}^{(x,m)}, \xi_{m+n}\big) \quad \text{for} \quad n = 0, 1, \ldots .$$

Note that, for every $n \ge 0$,

$$\mathbf{P}\big(X_{m+n+1}^{(x,m)} \in B \mid X_{m+n}^{(x,m)} = y\big) = P(y, m + n, B).$$

In particular, $X_n^{(x)} = X_n^{(x,0)}$ and $P(x, m, B) = \mathbf{P}\big(X_{m+1}^{(x,m)} \in B\big)$.

Next, let $L : \mathscr{X} \to [0, \infty)$ be a measurable function, $\Delta_{x,m} = L\big(X_{m+1}^{(x,m)}\big) - L(x)$ and, for $N > 0$,

$$\tau_{x,m}(N) = \min\big\{n \ge 1 : L\big(X_{m+n}^{(x,m)}\big) \ge N\big\}.$$

Theorem 2.1. *Suppose that there exist numbers $N > 0$, $\varepsilon > 0$, $M > 0$ and a measurable function $h : [0, \infty) \to [1, \infty)$ such that*

(1) $\tau_{x,m}(N) < \infty$ *a.s. for all* $x \in \mathscr{X}$ *and* $m \ge 0$*;*

(2) *for all* $m = 0, 1, 2, \ldots$ *and for all* $x \in \mathscr{X}$ *such that* $L(x) \ge N$*,*

$$\mathbf{E}\{\Delta_{x,m} \cdot I(\Delta_{x,m} \le M)\} \ge \varepsilon;$$

(3) *the integral* $\int_1^\infty (h(t))^{-1}\, dt$ *converges and, for* $t \ge 1$*, the function* $g(t) = \frac{h(t)}{t}$ *is concave and nondecreasing;*

(4) *the family of the random variables* $\{h(\Delta_{x,m}^-); m \ge 0, L(x) \ge N\}$ *is uniformly integrable.*

Then, for all $x \in \mathscr{X}$ *and* $m \ge 0$*,*

$$\mathbf{P}\big(\lim_{n\to\infty} L\big(X_{m+n}^{(x,m)}\big) = \infty\big) = 1. \tag{2.1}$$

REMARK 3. Condition (2) of Theorem 2.1 is fulfiled if the following hold:
(1) $\mathbf{E}\Delta_{x,m} \geq 2\varepsilon$ for all $m \geq 0$ and $x \in \mathscr{X}$ such that $L(x) \geq N$;
(2) the family of random variables $\{\Delta^+_{x,m}; m \geq 0, L(x) \geq N\}$ is uniformly integrable.

REMARK 4. In an unpublished paper by S. G. Foss (1995), the assertion of Theorem 2.1 was proved in the particular case $h(t) = \max(1, t^2)$.

REMARK 5. In condition (3) of Theorem 2.1, the nondecrease and concavity assumptions on g are imposed for simplicity of formulation. They are technical and could be naturally weakened. However, it seems to us that the convergence of the integral in (3) alone is insufficient for Theorem 2.1 to hold.

REMARK 6. Conditions (3) and (4) are fulfiled, provided that, for some $\beta > 0$, the $(1+\beta)$th moments of the random variables $\Delta^-_{x,m}$ are uniformly bounded, i.e.,

$$\sup_{m\geq 0, L(x)\geq N} \mathbf{E}\{\Delta^-_{x,m}\}^{1+\beta} < \infty.$$

Indeed, we can take the function $h(t) = t^{1+\gamma}$ for $t \geq 1$, where γ is an arbitrary number in the interval $(0, \min(1, \beta))$. Note also that we can take $h(t), t \geq 1$, to be any of the following functions: $t(\log(1+t))^{1+\beta}$, $t\log(1+t)(\log(1+\log(1+t)))^{1+\beta}$, etc.

REMARK 7. It can also be shown that condition (4) of Theorem 2.1 may be weakened to the following:

$$\sup_{m\geq 0, L(x)\geq N} \mathbf{E}\, h(\Delta^-_{x,m}) < \infty.$$

Before starting the proof of Theorem 2.1, we prove the following preliminary lemma.

Lemma 2.1. *Suppose that conditions* (1) *and* (2) *of Theorem* 2.1 *hold. Then, for every nonnegative integer m and for every $x \in \mathscr{X}$,*

$$\mathbf{P}\big(\limsup_{n\to\infty} L\big(X^{(x,m)}_{m+n}\big) = \infty\big) = 1. \tag{2.2}$$

PROOF OF THE LEMMA. Take an arbitrary $b > N$. First, we show that, for arbitrary nonnegative integers m and l and for every $x \in \mathscr{X}$,

$$\mathbf{P}\big(\exists n \geq l : L\big(X^{(x,m)}_{m+n}\big) \geq b\big) = 1. \tag{2.3}$$

Without loss of generality, we can only consider the case $m = 0$. Take $d = \varepsilon/2$; then

$$\mathbf{P}(\Delta_{y,n} \geq d) \geq \mathbf{P}(M \geq \Delta_{y,n} \geq d) \geq \mathbf{E}\bigg\{\frac{\Delta_{y,n}}{M}\mathbf{I}(M \geq \Delta_{y,n} \geq d)\bigg\} \geq \frac{d}{M}.$$

Put $H = \max\big(l, [\frac{b-N}{d}] + 1\big)$ and $\nu_1 = \min\big\{n \geq 1 : L(X^{(x)}_n) \geq N\big\}$. Note that, by condition (1), the random variable ν_1 is finite a.s. Moreover,

$$\mathbf{P}\big(L\big(X^{(x)}_{\nu_1+H}\big) \geq b\big) \geq \left(\frac{d}{M}\right)^H \equiv \delta > 0.$$

Further, for $i = 1, 2, \dots$, define the random variables

$$\nu_{i+1} = \min\big\{n \geq \nu_i + H + 1 : L\big(X^{(x)}_n\big) \geq N\big\}$$

and note that all of them are finite a.s. For $i = 1, 2, \dots$, consider the sequence of the events $B_i = \big\{L(X^{(x)}_{\nu_i+H}) \geq b\big\}$ and the increasing sequence of the sigma-algebras $\mathscr{F}_i = \sigma(\nu_i, \xi_1, \dots, \xi_{\nu_i})$. Then $\mathbf{P}(B_i \mid$

$\mathscr{F}_i) \geq \delta > 0$ a.s. for all i. Hence, $\mathbf{P}(\overline{B}_i \mid \overline{B}_1 \dots \overline{B}_{i-1}) \leq 1 - \delta < 1$ for all $i = 2, 3, \dots$, where $\overline{B}_i$ is the complement of the event B_i. Therefore,

$$\mathbf{P}\left(\bigcap_{i=1}^{\infty} \overline{B}_i\right) \leq \mathbf{P}\left(\bigcap_{i=1}^{n} \overline{B}_i\right) \leq (1-\delta)^n \to 0$$

as $n \to \infty$, and at least one of the events B_i occurs a.s. Thus, (2.3) holds.

Take now $b_n = N + n$ and put

$$\mu_{n+1} = \min\bigl\{i \geq \mu_n + 1 : L\bigl(X_{m+i}^{(x,m)}\bigr) \geq b_{n+1}\bigr\}.$$

By (2.3), all of the random variables μ_n are finite a.s. Thus,

$$L\bigl(X_{m+\mu_n}^{(x,m)}\bigr) \to \infty \quad \text{a.s. as} \quad n \to \infty, \tag{2.4}$$

The proof of the lemma is complete.

Proof of Theorem 2.1. The proof is the same for all $m \geq 0$, therefore we only consider the case $m = 0$.

Take an arbitrary $x \in \mathscr{X}$ and $C > 0$. Put

$$Y_n = \int\limits_{1+((L(X_n^{(x)})-N)^+)/C}^{\infty} \frac{dt}{h(t)}.$$

It suffices to prove that, for a certain C,

$$\text{the sequence } \{Y_n\}_{n\geq 0} \text{ forms a positive supermartingale.} \tag{2.5}$$

Indeed, if (2.5) holds, then (by the familiar theorem) the sequence $\{Y_n\}$ converges a.s. But, by Lemma 2.1, there exists a subsequence $\{\mu_n\}$ such that Y_{μ_n} converges to zero a.s. Hence, $Y_n \to 0$ a.s., and the latter is equivalent to the convergence $L\bigl(X_n^{(x)}\bigr) \to \infty$ a.s.

Now we prove (2.5). Since $\{X_n\}$ is a Markov chain, it suffices to show that the inequality

$$\mathbf{E}\bigl\{Y_{n+1} - Y_n \mid X_n^{(x)}\bigr\} \leq 0 \tag{2.6}$$

holds a.s. for all n.

The proof of (2.6) is the same for all n. Therefore, we only consider the case $n = 0$.

The inequality $\mathbf{E}\{Y_1 - Y_0\} \leq 0$ is clear if x is such that $L(x) \leq N$. In what follows, we assume that $z \equiv L(x) - N = \text{const} > 0$. Let

$$A(x) = Y_1 - Y_0 = \int\limits_{1+(z+\Delta_x)^+/C}^{1+z/C} \frac{dt}{h(t)},$$

where $\Delta_x \equiv \Delta_{x,0}$.

We need to introduce positive constants r, R, and C that satisfy certain restrictions.

First, choose R and r. Let

$$D = \sup_{L(x)\geq N} \mathbf{E}\{\Delta_x^-\} \quad \text{and} \quad T(\alpha) = \sup_{t\geq 0} \frac{g(1+\alpha t)}{g(1+t)}, \quad \alpha > 1.$$

Condition (4) of Theorem 2.1 implies finiteness of D, while concavity of g guarantees finiteness of $T(\alpha)$ for each $\alpha > 1$ and the convergence $T(\alpha) \to 1$ as $\alpha \to 1$.

Choose $R \in (0,1)$ small enough for the following inequality to hold:

$$\frac{h(1+2R)-h(1)}{h(1)} D \leq \frac{\varepsilon}{2}, \tag{2.7}$$

and choose $r \in (0,1)$ small enough that, for $\alpha = \frac{1+r}{1-r}$, the inequality holds:

$$(\alpha T(\alpha) - 1)D \leq \varepsilon/2. \tag{2.8}$$

Note that, by condition (3) of Theorem 2.1, for all $\alpha > 1$ and for all $t > 0$,

$$1 \leq \frac{h(1+\alpha t)}{h(1+t)} = \frac{1+\alpha t}{1+t}\frac{g(1+\alpha t)}{g(1+t)} \leq \alpha T(\alpha).$$

Therefore, (2.8) implies the inequality

$$\frac{h(1+(1+r)u) - h(1+(1-r)u)}{h(1+(1-r)u)} D \leq \frac{\varepsilon}{2} \tag{2.9}$$

for all $u > 0$ (it suffices to put $t = 1-r)u$).

Now, choose C so large that the following relations hold:

$$C > \frac{\max(M, 1+R(1+r))}{Rr} \tag{2.10}$$

and

$$K\mathbf{E}\{h(|\Delta_x|\mathbf{I}(\Delta_x < -rRC)\} \leq \varepsilon r R/12, \tag{2.11}$$

where $K = \int_1^\infty (h(t))^{-1}\,dt$.

We now start the main argument. There are two possibilities: either $0 < z \leq RC$ or $z > RC$.

Consider the first case. Represent $\mathbf{E}A(x)$ as

$$\mathbf{E}A(x) = E_1 + E_2 + E_3,$$

where

$$E_1 = \mathbf{E}\{A(x)\mathbf{I}(\Delta_x > M)\}, \quad E_2 = \mathbf{E}\{A(x)\mathbf{I}(0 \leq \Delta_x \leq M)\},$$
$$E_3 = \mathbf{E}\{A(x)\mathbf{I}(\Delta_x < 0)\}.$$

Note that, by the monotonicity of h, $A(x)$ admits the following estimates from above:

$$A(x) \leq \begin{cases} 0, & \text{if } \Delta_x > M, \\ \frac{z-(z+\Delta_x)^+}{Ch(1+(z+\Delta_x)^+/C)} \leq \frac{-\Delta_x}{Ch(1+R+Rr)}, & \text{if } 0 \leq \Delta_x \leq M, \\ \int\limits_{1+(z+\Delta_x)^+/C}^{1+z/C} \frac{du}{h(u)} \leq \frac{-\Delta_x}{Ch(1)}, & \text{if } \Delta_x < 0. \end{cases}$$

Therefore,

$$E_2 \leq \frac{-1}{Ch(1+(1+r)R)} \mathbf{E}\{\Delta_x \mathbf{I}(\Delta_x \leq M) - \Delta_x \mathbf{I}(\Delta_x < 0)\}$$

and

$$E_3 \leq \frac{-1}{Ch(1)} \mathbf{E}\{\Delta_x \mathbf{I}(\Delta_x < 0)\}.$$

Hence,

$$C\mathbf{E}A(x) \le \frac{-\varepsilon}{h(1+(1+r)R)} + \left(\frac{1}{h(1)} - \frac{1}{h(1+(1+r)R)}\right)\mathbf{E}\{-\Delta_x \mathbf{I}(\Delta_x < 0)\}$$
$$\le \frac{1}{h(1+(1+r)R)}\left(-\varepsilon + \frac{h(1+(1+r)R) - h(1)}{h(1)} D\right).$$

By (2.7) and the monotonicity of h, the RHS of the latter inequality is negative. Therefore, $\mathbf{E}A(x) < 0$.

Consider now the second case. Represent $\mathbf{E}A(x)$ as a sum of four terms

$$\mathbf{E}A(x) = E_1 + E_2 + E_3 + E_4,$$

where

$$E_1 = \mathbf{E}\{A(x)\mathbf{I}(\Delta_x > M)\}, \quad E_2 = \mathbf{E}\{A(x)\mathbf{I}(0 \le \Delta_x \le M)\},$$
$$E_3 = \mathbf{E}\{A(x)\mathbf{I}(-rz \le \Delta_x < 0)\}, \quad E_4 = \mathbf{E}\{A(x)\mathbf{I}(\Delta_x < -rz)\}.$$

For $\Delta_x \ge -rz$, the integral $A(x)$ admits the following upper estimates:

$$A(x) \le \begin{cases} 0, & \text{if } \Delta_x > M, \\ \frac{-\Delta_x}{Ch(1+(z+M)/C)} \le \frac{-\Delta_x}{Ch(1+(z+rz)/C)}, & \text{if } 0 \le \Delta_x \le M, \\ \frac{-\Delta_x}{Ch(1+(z-rz)/C)}, & \text{if } -rz \le \Delta_x < 0. \end{cases}$$

Note that $E_1 \le 0$. Furthermore,

$$E_2 \le \frac{-1}{Ch\big(1+\frac{z+rz}{C}\big)}\mathbf{E}\{\Delta_x \mathbf{I}(0 \le \Delta_x \le M)\},$$
$$E_3 \le \frac{-1}{Ch\big(1+\frac{z-rz}{C}\big)}\mathbf{E}\{\Delta_x \mathbf{I}(\Delta_x < 0)\}.$$

Hence,

$$E_1 + E_2 + E_3 \le \frac{-\varepsilon}{Ch\big(1+\frac{z+rz}{C}\big)} + \frac{h\big(1+\frac{z+rz}{C}\big) - h\big(1+\frac{z-rz}{C}\big)}{h\big(1+\frac{z+rz}{C}\big)h\big(1+\frac{z-rz}{C}\big)C} D. \tag{2.12}$$

Using (2.9) for $u = z/C$, we get

$$E_1 + E_2 + E_3 \le \frac{-\varepsilon}{2Ch\big(1+\frac{z+rz}{C}\big)}. \tag{2.13}$$

It remains to estimate the summand E_4:

$$E_4 = \mathbf{E}\left(\int_{1+(z+\Delta_x)^+/C}^{1+z/C} \frac{du}{h(u)}\mathbf{I}(\Delta_x < -rz)\right) \le K \cdot \mathbf{P}(\Delta_x < -rz)$$
$$\le K \cdot \mathbf{E}\left\{\frac{h(|\Delta_x|)}{h(rz)}\mathbf{I}(\Delta_x < -rz)\right\} \le \frac{K}{h(rz)}\mathbf{E}\{h(|\Delta_x|)\mathbf{I}(\Delta_x < -rRC)\}.$$

By (2.11), the RHS of the latter inequality is not greater than $\varepsilon rR/12h(rz)$.

Equation (2.10) implies that $g(rz) \ge g(1+z(1+r)/C)$. Indeed, the function g is nondecreasing and, since $z > RC$,

$$rz - (1+z(1+r)/C) = z\frac{rC-(1+r)}{C} - 1 > RrC - R(1+r) - 1 > 0.$$

Furthermore, since $R < 1$, $r < 1$, and $z > RC$, the following inequality holds: $3rz/rRC > 1 + (z + rz)/C$. Indeed,

$$\frac{3z}{RC} - \frac{z(1+r)}{C} = \frac{z(3 - R(1+r))}{RC} > \frac{z}{RC} > 1.$$

Therefore,

$$E_4 \leq \frac{\varepsilon r R/12}{rRC(1 + (z + rz)/C)g(rz)/3} \leq \frac{\varepsilon}{4Ch(1 + z(1+r)/C)}. \tag{2.14}$$

Equations (2.13) and (2.14) now imply the estimate we need:

$$\mathbf{E}A(x) = E_1 + E_2 + E_3 + E_4 \leq \frac{-\varepsilon}{4Ch(1 + z(1+r)/C)} < 0.$$

Thus, for all $x \in \mathscr{X}$

$$\mathbf{E}A(x) \leq 0.$$

Theorem 2.1 is proved.

The authors would like to thank the referee for useful remarks, D. A. Korshunov and N. I. Chernova for fruitful discussions, and Stan Zachary for the significant improvement of the style of the English translation.

References

1. *Fayolle G., Malyshev V. A., and Menshikov M. V.*, Topics on the Constructive Theory of Countable Markov Chains, Cambridge Univ. Press, Cambridge (1995).
2. *Kifer Yu.*, Ergodic Theory of Random Transformations, Birkhäuser, Boston (1986).