WHAT ENTAILMENT CAN DO FOR TYPE THEORY

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Abstract. Logics of entailment were developed for philosophical reasons before recent work in type theory. Yet there is a confluence between them. This confluence is evident at the logic **B+**, which was proposed in 1972 as the minimal relevant logic. Yet a few years later an essentially equivalent system was independently invented as a theory of (non-Curry) intersection (and union) types. This paper suggests several ways in which research in relevant logics can be adapted to produce results (and perhaps even insights) for type-theorists. Conversely the profound technical understanding that has produced filter models for functional theories definitively illumines the semantics of relevant logics.

I. Introduction

Logics of relevant entailment were developed independently of (and mainly prior to) recent work in the theory of types. Yet there are amazing (and amazingly unintended) points of resonance. It is my aim here to concentrate on several such points. The logic **B+** is a (minimal) entailment; from a not very different perspective, it is a system of intersection (and union) types. Work on entailment furnishes (at the minimum) some equipment for type theorists' tool bags; and perhaps even for their philosophical finery. I call attention to three points of contact. First, Dunn-style algebraic analysis applies immediately to **B+**. Second, R.T. Brady's analysis of *depth relevance* applies also to type theories. I had intended to show also how the Dunn-Mints Gentzen systems for **R+** (and their kin, Belnap Display Logics) are quickly adaptable to **B+** (and hence to type theories); but time (and space) did not permit. Finally, Meyer-Routley on *Boolean* extensions of entailment logics works also from type-theoretic perspective, producing from **B+** the system **CB**.

In [1], Barendregt, Coppo and Dezani (henceforth BCD) set out the following postulates on an "intersection" theory of types **ITD**. Primitive are type variables, ω (universal type), $\leq$ (type inclusion), $\cap$ (type intersection), and $\rightarrow$. The type symbols are the result of closing variables and ω (and perhaps other constants) under $\rightarrow$ and $\cap$. The postulates of **ITD** are, for all type symbols A, B, B', C, C',

$$C \leq C;\ C \leq \omega;\ \omega \leq \omega\rightarrow\omega;\ C \leq C \cap C;\ B\cap C \leq B;\ B\cap C \leq C;$$
$$(B\rightarrow A) \cap (B\rightarrow C) \leq B \rightarrow (A\cap C);\ B \leq C \leq A \Rightarrow B \leq A;$$
$$B \leq B',\ C \leq C' \Rightarrow B\cap C \leq B'\cap C';\ B' \leq B,\ C \leq C' \Rightarrow B\rightarrow C \leq B'\rightarrow C';$$

"Lord," said I to myself on beholding these postulates,[1] "this is just the minimal relevant logic **B+** that I proposed with Routley in [2] and [3]."[2] More accurately, it is the $\rightarrow$, $\wedge$, **T** fragment **B∧T** of **B+**,[3] which may be formulated with the following axiom schemes and rules:[4]

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[1] Thanks to J.R. Hindley, who informed me of the BCD work.
[2] Thus Hindley is misleading when his title of [12] suggests that **ITD** is not a **logic**. It is just a much **weaker** logic than those to which the **SK**-chauvinists are accustomed!
[3] **B∧T** may be pronounced, "bat". The "Church" constant **T** was not mentioned in [3], though it is familiar from other works on relevant logic. It is to be assigned the value "true" at all points in every **B+** model.
[4] For ease in reading formulas, binary connectives are ranked o, $\wedge,\vee,\rightarrow,\supset,\leftrightarrow$ in order of **increasing** scope; note that $\rightarrow$ **precedes** $\supset$; associate **equal** connectives to the **left;** as above, $\Rightarrow$ is a **metalogical** *if.*

$A \to A$; $A \to \mathbf{T}$; $\mathbf{T} \to (\mathbf{T} \to \mathbf{T})$; $A \wedge B \to A$; $A \wedge B \to B$; $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$;
$A \to B \Rightarrow (B \to C) \to (A \to C)$; $B \to C \Rightarrow (A \to B) \to (A \to C)$
$A \to B$ and $A \Rightarrow B$; A and $B \Rightarrow A \wedge B$

Why should we believe that **ITD** and **B∧T** are the **same** system? We make the obvious identifications of **T** with ω, of ∧ with ∩, and of **provable** → in **B+** with **ITD** ≤. A quick deductive induction shows that every **ITD** theorem holds on **B+** translation. The most elegant[5] way to prove the converse is via a "metavaluation" v, with values in {1,0}. Define v on all formulas of **B∧T** thus:

If A is a sentential variable, $v(A) = 0$; $v(\mathbf{T}) = 1$;
$v(A \wedge B) = 1$ iff $v(A) = v(B) = 1$;
$v(A \to B) = 1$ iff (i) $A \leq B$ in **ITD** and (ii) $v(A) = 0$ or $v(B) = 1$.

It is readily verified that all theorems of **B∧T** take the value 1 on v. But in view of clause (i) in the truth-condition on →, this means that their counterparts are theorems of **ITD** as well. So

Observation 1. A is a theorem of **ITD** iff it is on translation a theorem of **B∧T**.

Since by Observation 1 we have **ITD = B∧T** (near enough), I shall go on to formulate all of **B+** (and of **B+T**) as relational systems. Get **B+T** by adding the following to **ITD**:

$A \leq A \vee B$; $B \leq A \vee B$; $(A \to C) \wedge (B \to C) \leq A \vee B \to C$; $A \wedge (B \vee C) \leq (A \wedge B) \vee C$

Shrink **B+T** back to **B+** by dropping **T** (a.k.a. ω) and its associated axioms. Note that the first 3 added axioms give **B+** lattice properties; while the last makes these *distributive* lattice properties.

II. Filters, theories, and the algebra of entailment

[1] presents its main results as providing a **filter lambda model.** But, as Dunn's thesis taught us,[6] "filter" is just Algebraese for what logicians call a "theory". So it would seem that we have a **theory lambda model** as well. This explains a lot. For one thing, it explains why Routley and I were right to call Combinatory Logic the "key to the Universe" in work cited above. For **CL** is, after all, just λ in another guise.[7] For our algebraic postulates in [2], and our relational semantical postulates in [3], separated **various** substructural logics on straightforward properties of combinators. Thanks to [1], we know **why** we were right. The semantics of relevant logics is, at root, a semantics of **B+** theories. Particular additional axioms (e. g., those for the system **R** of [4] and [5]) put constraints on the theories admitted into the semantics, **exactly corresponding** to the combinators that have these axioms as their "types".

Let us delve deeper. The intentions of type theorists in [1] and related papers were certainly very different from Routley's and mine in [2] and [3]. Ours were to present a semantics for a class of entailment logics, of which **B+** turned out to be a "natural" minimal positive one. By contrast [1] begins from λ-models, and shows how to **interpret** the λ-calculus in filters over these models. It is then a wonderful (and wonderfully unintended) coincidence that their insights and ours led (via Observation 1) to what is essentially the same formal system.

III. Brady's depth relevance

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[5] Belnap claims that an elegant proof is one where you feel cheated at every step!
[6] See [4] for the main results. Dunn's thesis, which concentrates on the algebra of **R**, is "The algebra of intensional logics," University of Pittsburgh. (University Microfilms, Ann Arbor, 1966). The extension of Dunn's algebraic methods to other relevant logics in [2] is non-trivial.
[7] [6] takes **CL** as the "assembly language" for λ. But the λ-**CL** relation is delicate. See [13] and [14].

One of the guiding ideas behind entailment systems is that the **premises** of a good argument shall be relevant to its conclusion. This is partially spelled out in [4] as a Variable-Sharing Principle.

(VSP) If $A \rightarrow B$ is provable then A and B share a variable.

The rationale for (VSP) is first spelled out in [4] on p. 33, which notes "commonality of meaning is carried in propositional logic by commonality of propositional variables." Although unwelcome to classical and intuitionist **SK-**chauvinists, with their attachment to theorems like $p\rightarrow(q\rightarrow q)$ and $p\wedge{\sim}p\rightarrow q$, (VSP) makes sense as a requirement on a decent implication. That it holds of the strong relevant logic **R** (and hence of all its subsystems) is quickly demonstrable using the following DeMorgan monoid **CR6**.[8] Read the lattice operations of **CR6** off the following Hasse diagram:

$$\begin{tikzcd}[column sep=small]
& \mathbf{T}\ (= {\sim}\mathbf{F}) \arrow[d, no head] & \\
& \mathbf{f}\ ({\sim}\mathbf{t}) \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbf{a}\ (= {\sim}\mathbf{a}) \arrow[dr, no head] & & \mathbf{b}\ (= {\sim}\mathbf{b}) \arrow[dl, no head] \\
& \mathbf{t}\ ({\sim}\mathbf{f}) \arrow[d, no head] & \\
& \mathbf{F}\ (= {\sim}\mathbf{T}) &
\end{tikzcd}$$

Thus **CR6** can be split into an interior "diamond" {**f, a, b, t**} and a "head and tail" {**T, F**}. Note the pair {**t, T**} of "true" constants, with their negations {**f, F**}. Do not be deceived. Thinking of **CR6** as a "matrix", all elements except **F** count as *designated,* even **f**. Note finally that the "middle" elements **a** and **b** are *fixed points* for ~, being *their own* negations. Here is the → table for **CR6**:

→	**T**	**f**	**a**	**b**	**t**	**F**
T	**T**	**F**	**F**	**F**	**F**	**F**
f	**T**	**t**	**F**	**F**	**F**	**F**
a	**T**	**a**	**a**	**F**	**F**	**F**
b	**T**	**b**	**F**	**b**	**F**	**F**
t	**T**	**f**	**a**	**b**	**t**	**F**
F	**T**	**T**	**T**	**T**	**T**	**T**

Note that $x \leq y$ iff $\mathbf{t} \leq x\rightarrow y$, which is standard for DeMorgan monoids. Put otherwise, if x *fails* to be $\leq y$, then $x\rightarrow y = \mathbf{F}$ (which is *not* standard, but which holds here because **F** is the *only* undesignated element). Further standard DeMorgan facts appear in the **t** row and in the **f** column. For $\mathbf{t} \rightarrow x = x$, while $x \rightarrow \mathbf{f} = {\sim}x$, always. Finally **CR6** is *rigorously compact.* Not only does $\mathbf{F}\rightarrow x = \mathbf{T}$ and $x\rightarrow\mathbf{T} = \mathbf{T}$ (standard), but *otherwise* $x\rightarrow\mathbf{F} = \mathbf{F}$ and $\mathbf{T}\rightarrow x = \mathbf{F}$ (special).

Shallow Relevance theorem. (Belnap) VSP holds for **R** and all its subsystems.
Proof. Suppose that A and B share no variable. Set $I(p) = \mathbf{a}$ for each variable p in A, and set $I(q)=\mathbf{b}$ for each variable q in B. Extending I homomorphically to all subformulas of A and B, it is

[8] I invented the "crystal lattice" **CR6**, as an economical alternative to the 8 element matrix $\mathbf{M_0}$ of [4], which Belnap originally used to show (VSP) for **E** and **R**. Cf. also [15], p. 250.

immediate on inspection of the tables that I(A) = **a** and I(B) = **b**, whence v(A→B) = **F**. Transposing, since **CR6** is a matrix for **R**, if A→B is provable then A,B share a variable.

The shallow relevance theorem is as good as we shall do for the strong relevant logic **R**. But as logics weaken, theorems strengthen. What Brady noted in [18] was that at the minimal level of the system that he calls **DR** (as for all of its subsystems, including **B+**), the theoremhood of A→B requires variable-sharing *at some level* of subformula nesting within →'s. Here are a couple of **R** theorems that are refuted on Brady's criteria.

Thm **CI**. p → ((p→q) → q)

Thm **WI**. (p→q)∧p → q

Brady *refutes* these formulas in **B+** by relativizing **CR6** valuations to levels, whence valuations at outer levels in → formulas are determined by valuations at inner levels. Consider **CI**. Here is its construction tree:

```
                 .  CI (0)
                / \
     (1) p .      . (p→q) → q (1)
                   /  \
  (2) p→q .         . q (2)
              / \
   (3) p .      . q (3)
```

Numbers in parentheses record the *depth of nesting* of each *subformula occurrence* in the tree. And each depth d gives rise to a corresponding interpretation I_d in **CR6** with I_d(C→D) = I_{d+1}(C) → I_{d+1}(D). So here is the Brady refutation of **CI**.

Depth	**p**	**q**	**p→q**	**(p→q)→q**	**CI**
3	**b**	**b**			
2		**b**	**b**		
1	**a**			**b**	
0					**F**

On the other hand, truth-functional particles like ∧ and ~ do *not* increase depth; e.g., I_d (C∨D) = I_d(C) ∨ I_d(D). We now give the Brady refutation of **WI**.

Depth	**p**	**q**	**p→q**	**(p→q)∧p**	**WI**
2	**b**	**b**			
1	**b**	**a**	**b**	**b**	
0					**F**

Deep Relevance theorem. (Brady) Suppose A→B is a theorem of **B+**. Then there is a depth n at which A and B share a variable.
Proof. See Brady [18], whose content he will contribute to the 2nd volume of [15].

IV. The semantics of B+

I review the relational semantics of **B+**.[9] A **B+-frame**[10] is a triple **K** = <0,K,R>, where K is a non-empty set (of "worlds", or "$\vee$-prime theories"), $0 \in K$ (0 is a "real world", or "preferred theory"),[11] and R is a **ternary** ("accessibility") relation on K. We define for all $x,y \in K$,

D≤. $x \leq y =_{df} R0xy$

and we set out the following **monotonicity** postulates:

p1. $x \leq x$

p2. $x \leq y$ and $Ryzw \Rightarrow Rxzw$

p3. $y \leq z$ and $Rxzw \Rightarrow Rxyw$

p4. $w \leq z$ and $Rxyw \Rightarrow Rxyz$

We may think of ≤ as **set inclusion** on theories; and Rxyz says that whenever $A \rightarrow B \in x$ and $A \in y$ then $B \in z$; i. e., if a *major premiss* of →E is in x and its *minor premiss* is in y then the *conclusion* of →E is in z.[12] Note immediately that ≤ is transitive, by D≤ and p3. It is also reflexive, by p1.[13]

Let F be the set of **B+T** formulas. An **interpretation** I is a function $F \times K \rightarrow \{t,f\}$, subject to the following conditions:

H. $x \leq y$ and $I(A,x) = t \Rightarrow I(A,y) = t$ (Heredity condition)

TT. $I(\mathbf{T},x) = t$, for all $x \in K$

T∧. $I(A \wedge B,x) = t$ iff $I(A,x)= t$ **and** $I(B,x)= t$

T∨. $I(A \vee B,x) = t$ iff $I(A,x)= t$ **or** $I(B,x)= t$

T→. $I(A \rightarrow B,x)= t$ iff for all $y,z \in K$ ($Rxyz \Rightarrow$ ($I(A,y) = t \Rightarrow I(B,z) = t$))

In view of the postulates, we may confine the heredity condition H to sentential variables p, q, r, etc., whence an easy induction via the truth-conditions shows that it holds for **all** formulas A.

A formula A is **verified** on an interpretation I just in case $I(A,0) = t$. In particular $B \leq C$ is verified on I, and we may say that B **I-entails** C, just in case $B \rightarrow C$ is verified on I. And A is **B+ valid** iff A is verified on all interpretations in all **B+** frames. The following is a principal result of [3] (and holds *mutatis mutandis* for other positive relevant logics, adding additional semantical postulates that correspond combinatorially to further logical axioms):

Soundness and completeness for B+. A is a theorem of **B+** iff A is **B+** valid.

It was a satisfying accident that **B∧T** is *the* theory of intersection types. It would be equally satisfying, and greatly to be desired, if **B+T** could be taken as the theory of intersection and union types. Yet this satisfaction (at least in the sense that I personally wish it) is thus far ours only in part. True, the Torino group and their allies (of whom I trust that I am one) have various interesting results modeling this or that modification of λ in this or that modification of **B+**.

Still, our thinking has not been quite clear. What, after all, does finding a model of λ amount to? The grounding intuitions of Church and Curry lay in a space of 1-place functions, closed under *application* of functions (*qua* functions) to functions (*qua* arguments). It was widely (and with

[9] Restall in his Ph.D. thesis [10] uses **DW+** as his **name** for **B+**. He also applies the "simplified semantics" of [11], which (at least for my money) is just Boolean semantics under another name.

[10] Routley and I have used **B+ms** (for **B+ model structure**) previously. But **frame** seems now the **entrenched** name for the notion.

[11] That there is only **one** preferred theory 0 produces what [15] calls a "reduced" semantics.

[12] Restall in [10] follows Dunn in writing Ryxz where I have always written Rxyz. Boo! Hiss!

[13] These accord with our intended interpretation of ≤ as ⊆ .

reason) held that these intuitions made no sense; the fact that clever people like Kleene and Rosser made a habit of proving candidate systems *inconsistent* did not help.

Amazingly, sense has now been made of the Church-Curry ideas, thanks to even more clever people like Dana Scott. What has to be modeled, first and foremost, is *functional application.* What does duty for this binary operation is *relational image.* Consider, for example, the modeling of λ in **B∧T** theories. Suppose that x and y are theories, and ask what it is to *apply* x to y. The answer is that application is *simulated* in the fusion xoy, defined as

Do. $x o y = \{B : \exists A(A \rightarrow B \in x \text{ and } A \in y\}$

So viewed, x is working like a *relation.* Which relation? The relation that holds between A and B iff $A \rightarrow B$ is in the theory x. And y is working like a *set* of arguments. Briefly, in the notation of *Principia Mathematica,* we are *asking for* the result x'y of applying the function x to the argument y. But what we are *getting* is the *image* x"y of all the B's related by x to members of y.

So viewed, that **B∧T** theories supply a model of λ seems to me *magic.* The reason that they do lies in some magical lemmas proved in [1] and elsewhere. And what makes these lemmas go through is the *weakness* of **B∧T**. As a system in which one can prove hardly anything, **B∧T** provides a lot of control over such deductions as there are.

A further point is of significance. We think of the elements that model λ for **B∧T** as any old theories. But our semantic viewpoint for *all* of **B+T** favors ∨-prime theories. Reason: we prefer those theories that *respect* intended truth-functional particles. And only if a theory x is ∨-prime does it satisfy the condition, for all formulas B and C,

T∨. $B \vee C \in x$ iff $B \in x$ or $C \in x$.

But now we come to a problem with our previous trick of using fusion o to simulate application. It is straightforward to show that the class of **B+T** theories is closed under fusion. But it is *not* straightforward to show (nor is it even *true*) that the class of ∨-prime theories is closed under fusion. This is a great impediment (as Dezani and her colleagues have noted in various *counterexamples*) to doing the same tricks at the **B+** level that worked at the **B∧T** level.

V. Booleanizing B+

It is an interesting question how *negation* should be handled in a theory of types. It is no less interesting a question when the handling system is a logic of relevant entailment. (Or, truth to tell, *any* logic.) About all that is generally agreed, where negation is concerned, is that it is a unary operation on sentences. (Or, maybe, on *propositions.*)

Still, there are various plans that we might follow in equipping **B+** with a suitable *not.* In [15] we opted for a *minimally truth-functional* sort of ~, identifying the system **B** as one that satisfies all versions of double negation and DeMorgan laws. But transposition holds only in *rule* form. And excluded middle and non-contradiction theorems are just *absent.*

Well, that's one policy. I investigate in this paper a considerably more vertebrate one, adding rather the *maximally truth-functional* Boolean negation ¬ to **B+**. The method is that used to add ¬ to **R+** in [8]. The result will be a system **CB** (for "classical **B+**").[14] Now that we are in an

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[14] **CB** was introduced as **CB+** in my 1995 technical report, "Types and the Boolean logic **CB+**." It has been available since then from our website, but it is so cunningly hidden that you'll never find it! Accordingly I have incorporated large parts of that technical report in this paper.

explicitly Boolean context, we can as in [9] greatly simplify our presentation. We add new definitions

D$\vee$. $A\vee B =_{df} \neg(\neg A\wedge\neg B)$.

D**T**. **T** =df $p \vee \neg p$ (for an arbitrary sentential variable p)

D**F**. **F** =df $\neg$**T**

D$\supset$. $A\supset B =_{df} \neg A\vee B$

D$\leftrightarrow$. $A\leftrightarrow B =_{df} (A\to B)\wedge(B\to A)$

Saving D$\leq$, we replace all of p1-p4 with

p0. $x \leq y$ iff $x = y$

Any frame satisfying p0 will be a **CB** frame. It is trivial that any **CB** frame is a **B+** frame (since p1-p4 are automatically satisfied).

We extend the modelling conditions above, for all interpretations I and $x \in K$, with

T$\neg$. $I(\neg A,x) = t$ iff $I(A,x) = f$.[15]

All else being as above, a formula A is **CB valid** iff A is verified on all interpretations I in all **CB** frames. The following **semantical entailment** theorem is useful:

SemEnt. Let I be an interpretation in a **B+** frame **K** = <0,K,R>. Then for all A,B,

$I(A\to B,0) = t$ iff $\forall x\in K\ (I(A,x)= t \Rightarrow I(B,x) = t\ )$.[16]

Proof. See [3].

Conservative extension theorem. For all positive formulas A, A is **B+ valid** iff A is **CB valid.**

Proof.[17] Trivial from left to right. From right to left, we prove the contrapositive. Suppose then that there is a **B+ frame K** = <0,K,R> and an interpretation I therein such that I(A,0) = f. We show how to **massage K** to produce a **CB** frame **K' =** <0',K', R'> and an interpretation I' therein such that I' (A, 0') = f. Let 0' be an element not in K, and let K' = K $\cup$ {0'}. Let R' **agree** with R on elements of K; let R'0'xy iff x = y, for all elements $x,y \in K'$; and let R'xyz **fail** otherwise. For all atomic terms p and elements x of K, set I' (p,x) = I(p,x); and let I'(p,0') = I(p,0). Complete the definition of I' by imposing T$\to$, T$\wedge$, T$\neg$. We leave to the reader the verification, which is as in [8], that for all positive formulas B and $x \in K$,

(a) I (B, x) = I' (B, x), and

(b) I (B, 0) = I' (B, 0').

This ends the proof of the theorem.

IV. Soundness and completeness theorems for CB. We give now axioms and rules for **CB.** A **Boolean tautology** shall be any formula A that is a substitution instance of a truth-table tautology in $\wedge$, $\neg$.[18] Our axioms and rules for **CB** shall then be the following:

c1. $A \to B$, whenever $A \supset B$ is a Boolean tautology

c2. $(A\to B) \supset (C\to A\to(C\to B))$ (Prefixing)

c3. $(A\to B) \supset (B\to C\to(A\to C))$ (Suffixing)[19]

c4. $A\wedge(A\to B) \supset B$ (Assertion)

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[15] Note that, in view of p0, the condition H is now redundant. Put otherwise, **every assignment** of truth-values **to** variables **at** worlds in K is extended to an **interpretation** I by T$\to$, T$\wedge$, T$\neg$.

[16] The import of SemEnt is that $A\to B$ is **verified** iff B is true at **all worlds** at which A is true.

[17] Using the simplified semantics, Restall proves this theorem **very easily** in [10].

[18] And, by extension, in the **defined** Boolean particles $\vee$, $\supset$, **T**, **F**.

[19] We strengthen the prefixing and suffixing **rules** of [3] to corresponding material **axioms**. Restall reminds me that Giambrone and I claimed in [17] that c1, c4, c6 and $\supset$E would suffice for conservative extension, added to a positive relevant basis. If so, I cannot reconstruct now what I had then in mind.

c5. $(A\to B)\wedge(A\to C) \to (A \to B\wedge C)$ ($\to\wedge$I)
c6. $A \to (B \to \mathbf{T})$ ($\mathbf{T}$ triviality)
c7. $A \to (\mathbf{F} \to B)$ ($\mathbf{F}$ absurdity)
$\supset$E. $A \supset B$ and $A \Rightarrow B$ ($\supset$E rule)

Here are some theorem and rule schemes of **CB** that can be derived from these axioms:

b0. A, when A is a Boolean tautology
b1. $A\wedge B\to C \supset A\wedge\neg C\to\neg B$ (Antilogism)
b2. $A \leftrightarrow \neg\neg A$, etc. (From c1)
b3. $(A\to C)\wedge(B\to D)\to(A\wedge B\to C\wedge D)$ (From c5, c1, c3, $\supset$E)
b4. $(B\to C)\wedge(A\to B) \supset (A\to C)$ (From c3, c1, c4, $\supset$E)
$\wedge$I. A and B $\Rightarrow A\wedge B$ (From b0, $\supset$E)
$\to$E. $A\to B$ and $A \Rightarrow B$ (From c4, $\wedge$I, $\supset$E)

We will show that our suggested axioms c1-c7 and rule $\supset$E suffice exactly for the formulas valid in all **CB** frames. The following is by straightforward deductive induction:
Soundness lemma. If A is a theorem of **CB** then A is **CB valid**.

For completeness, we must as usual do more work. Let L be the set of all formulas in the vocabulary of **CB**, and let S and T be non-empty subsets of L. We say

$\leq$E. T is **S-closed** iff whenever $A\to B \in S$ then if $A \in T$ then $B \in T$
$\wedge$I. T is **$\wedge$-closed** iff whenever $A\in T$ and $B\in T$ then $A\wedge B \in T$
Th. T is an **S-theory** iff T is both S-closed and $\wedge$-closed
Rg. T is **S-regular** iff $S \subseteq T$ [20]
$\to$E. T is **detached** iff whenever $A\to B \in T$ then if $A \in T$ then $B \in T$ [21]
$\vee$E. T is **$\vee$-prime** iff whenever $A\vee B \in T$ then either $A \in T$ or $B \in T$
$\neg$E. T is **$\neg$-consistent** iff whenever $A \in T$ then $\neg A \notin T$
$\neg$I. T is **$\neg$-complete** iff whenever $A \notin T$ then $\neg A \in T$
Ok. T is **S-ok** iff T is a $\neg$-consistent $\vee$-prime S-theory
Nr. T is **S-normal** iff T is both S-regular and S-ok

In studies of the relational semantics of entailment (cf. [3], [5], [15], [16]), **$\vee$-prime S-theories** have played a central role for very many choices of relevant (and sometimes irrelevant) logic S. So shall they here, with the extra proviso that (since inconsistency is a Boolean catastrophe) it is the **S-ok** theories that must be preferred for **CB.** These correspond, in classical perspective, exactly to **maximal consistent** theories.[22] But as **CB** is a **relevant** logic (and a **minimal** one to boot), one ought **not** to expect **every** S-ok theory T to contain **all** of **CB.** So long as our theories **respect** entailment, it is indifferent that they do not always **assert** the laws of logic. Cf. [16] for related discussion.

Lemma on CB-theories. I) Let S be an arbitrary **CB**-theory. Then

(i) A is a **Boolean tautology** $\Rightarrow A \in S$
(ii) S is **materially detached**; i.e., if $A\supset B \in S$ then if $A \in S$ then $B \in S$

II) Suppose moreover that S is regular. Then

(iii) $A\to B \in S \Rightarrow (C\to A)\to(C\to B) \in S$ (Prefixing)

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[20] In view of our main interest here, I sometimes use **regular** to mean simply **CB-regular**.
[21] Thus **detachment** is being closed under **your own** modus ponens. By contrast, as Belnap once put it, S-closure is being closed under (usually) **somebody else's** modus ponens.
[22] We use below the **Boolean** fact that a $\neg$-consistent non-empty theory T is $\vee$-prime iff it is $\neg$-complete.

(iv) $A \to B \in S \Rightarrow (B \to C) \to (A \to C) \in S$ (Suffixing)

(v) $A \to B \in S \Rightarrow \neg B \to \neg A \in S$ (Transposition)

Proof. **T** ∈ S. Suppose A is a Boolean tautology. Then $\mathbf{T} \to A \in$ **CB**, by c1; whence A∈S, by ≤E. This is (i). For (ii), suppose both of $A \supset B$, $A \in S$; by ∧I, $(A \supset B) \wedge A \in S$; but this entails B, whence B ∈ S by ≤E. For (iii), since S is regular it contains the **CB**-theorem $A \to B \supset (C \to A) \to (C \to B)$, whence if $A \to B \in S$ then $(C \to A) \to (C \to B) \in S$ by the ⊃E principle (ii). (iv) and (v) are similar, ending the proof of the lemma.

We now define the **fusion** o of subsets S, T of L as suggested above by

Do. SoT =df $\{B: \exists A\, (A \to B \in S \wedge A \in T)\}$

Fusion lemma. Let **0** be any regular **CB-**theory. Then if S and T are **0**-theories so is SoT.
Proof. We must show (a) SoT is **0**-closed and (b) SoT is ∧-closed. For (a) suppose B∈SoT. By Do we have $A \to B \in S$ for some $A \in T$. To show **0**-closure, we may also assume $B \to C \in$ **0**. We must show C ∈ SoT. But since **CB** ⊆ **0**, we have by (iii) of the "theories" lemma that $(A \to B) \to (A \to C) \in$ **0** also. As S is a **0**-theory it is **0**-closed; whence $A \to C \in S$ and C ∈ SoT, as desired. For (b), suppose B',B" ∈ SoT, and let B = B'∧B". We must show that B∈SoT. By definition there are A',A" ∈ T such that both $A' \to B'$ and $A'' \to B''$ belong to S, whence $(A' \to B') \wedge (A'' \to B'') \in S$. By b3 above this **CB**-entails (and hence **0**-entails) $A \to B$, setting A = A'∧A". But A∈T (by ∧I), whence B∈SoT. SoT is thus a **0**-theory. QED.

For completeness, I adapt the methods of [3]. Let **CB** ⊆ **0**,[23] where **0** is a **CB-**theory. By the calculus **K** of **0-theories,** I mean the structure

D**K**. **K** = <K, ⊆ , o, **0** >,

where K is the collection of all non-empty **0-theories**, ⊆ is set-inclusion, and o is the binary operation defined on arbitrary sets of formulas by Do. Note that ∧-closed T is a **0-theory** iff 0oT = T.[24]

By the fusion lemma, the K of D**K** is closed under o. We now define a ternary R on K by

DR. Rabc iff aob ⊆ c.

We next note that, on **0-theories,** ≤ turns out as advertised to be **exactly** ⊆. I. e.,
Subset lemma. Where **0** is a regular **CB-**theory and b,c are **0-**theories R**0**bc iff b ⊆ c.
Proof. Since b is a **0-**theory we have R**0**bc iff b = **0**ob ⊆ c. QED.
Theorem on B+ frames. Let **0** be any regular **CB-**theory, and let **K** be the associated calculus of **0-**theories. Then < **0,** K, R> is a **B+** frame, where R is defined by DR.
Proof. It suffices to verify p1-p4 above, all of which are trivial by the Subset lemma just above.

While our theorem is pleasant, it is not yet good enough for the **CB** completeness proof at which we aim.
Happily, we may help ourselves to a good deal of Boolean reasoning in the **CB** context, which makes things simpler than usual when proving relevant metatheorems.
Where S ⊆ L and A ∈ L, we say that A is (**materially**) **deducible from** S just in case there is a finite sequence $A_1, \ldots, A_n$ of formulas of L such that

(i) A_n is A, and

(ii) for $1 \le k \le n$, each A_k is either

[23] I. e., **0** is fixed as a **regular CB-**theory.

[24] I. e., **0** is a **left identity** for o in **K**. For **0**oT ⊆ T because ≤E requires **0-**closure of the **0-**theory T. Conversely, if A∈T then since $A \to A \in$ **CB**⊆ **0,** we have A ∈ **0**oT. So T ⊆ **0**oT as well.

(a) a member of S, or
(b) a Boolean tautology, or
(c) such that some A_h is $A_i \supset A_k$, h,i < k.

We write S $\models$ A iff A is materially deducible from S. Writing S,A for S $\cup$\{A\} henceforth, we have the

Boolean deduction theorem. S, A $\models$ B iff S $\models$ A$\supset$B. **Proof** as usual.

We now give instructions to build a **CB frame** to refute an arbitrary non-theorem A. First

0-theorem. Let A $\notin$ **CB.** Then there is a **CB-normal theory 0** such that A $\notin$ **0.**

Proof. By familiar Henkin-style construction. Set T_0 = \{B: **CB,** $\neg$A $\models$ B \}. Enumerate the formulas of L, and let W_{i+1} = \{B: T_i , A_{i+1} $\models$ B\}, for each i from 0 on. If W_{i+1} is $\neg$-consistent let $T_{i+1} = W_{i+1}$. Otherwise let $T_{i+1} = T_i$. We assert, first, that each T_i is $\neg$-consistent. Surely T_0 is, for otherwise we should have by the Boolean deduction theorem that **CB** $\models$ $\neg\neg$A, whence by double negation A $\in$ **CB**, contradicting our hypothesis. After that, $\neg$-consistency is preserved at each step. Finally, define **0** as the union of the T_i , i $\in$ **N**. Clearly **0** contains **CB.** So it is regular. **0** is closed under $\supset$E (since all of the T_i are); whence it is also **$\wedge$-closed** and **0-closed** (the latter by regularity and axiom c4). Finally we must show **0** to be $\vee$-prime. But if B$\vee$C $\in$ **0** and B $\notin$ **0**, then the attempt to add B, when its turn came up, produced inconsistency. By the deduction theorem $\neg$B $\in$ **0** already, whence C $\in$ **0** by $\supset$E. End of proof.

By a similar argument we have the

Theory extension lemma. Let **0** be a regular **CB-**theory. Let x be a **0**-theory, C $\notin$ x. Then there is a **0**-ok theory x' such that (a) x $\subseteq$ x' and (b) $\neg$C $\in$ x'.

Proof. Where y $\subseteq$ L write y$\wedge$A for \{A$\wedge$B: B $\in$ y\}. Write y $\leq$ D if for some conjunction B of members of y, we have B$\rightarrow$D $\in$ **0**. Set x_0 = \{B: x$\wedge\neg$C $\leq$ B \}. Clearly x_0 is $\wedge$-closed and **0**-closed; so it is a **0-**theory. It is also $\neg$-consistent; for otherwise we should have x$\wedge\neg$C $\leq$ **F**, whence x $\leq$ C after all by the antilogism principle b1. Enumerate the formulas of L again as A_1 , A_2 ,..., and form theories x_i as before on the recipe

If $x_{i-1} \wedge A_i \leq$ **F** then $x_i = x_{i-1}$ else x_i = \{B: $x_{i-1} \wedge A_i \leq B$\}.

Again each x_i is a **0**-theory. Let the union of the x_i be x'. Evidently x' is $\wedge$-closed and **0**-closed; so it is a **0-**theory as well. It is $\neg$-consistent (since consistency was preserved at each step). Finally, x' is $\neg$-complete; for if we did **not** add A_i, when its turn came up, the reason lies in $x_{i-1} \wedge A_i \leq$ **F**. Whence $x_{i-1} \leq \neg A_i$ already, appealing again to b1. So x' is **0**-ok. And C $\notin$ x', as $\neg$C was added at stage 0. End of proof.

Having chosen a **0** to refute A, we now let K' be the set of all **0-**ok theories. Consider now the structure **K' =** < **0**, K', R'>, where R' is the restriction to K' of the R defined by DR above on **all** **0-**theories. This structure **K'** will be a **canonical A-refuting frame**. The **canonical interpretation I' in K'** sets

DI'. I'(B,x') = t iff B $\in$ x', for all x' $\in$ K', B $\in$ L.

Canonical Frame theorem. Each canonical A-refuting frame **K' = < 0,** K',R'> is a **CB-frame.**

Proof. We need only verify p0, which comes here to x' = **0**ox' $\subseteq$ y' iff x' = y', for all x', y' $\in$ K'. This is obvious r. to l. From l. to r., assume x' $\subseteq$ y'. As x', y' are classically maximally consistent, they must **differ** on at least one formula B if distinct. But if B$\in$x' and $\neg$B$\in$y', evidently the subset relation does not hold. Conclusion: R'**0**x'y' $\Rightarrow$ x' = y', as required. End of proof.

We shall now require a couple of lemmas before proving our next theorem. Fixing a regular **CB**-theory **0** in context, for B∈L let [B] be the **principal 0**-theory determined by B; i. e., [B] = {C: B → C ∈ **0**}.

Priming lemma. Let **0** be a regular **CB-**theory, and let x' be a **0-**ok theory. Suppose B → C ∉ x'. Then C ∉ the **0-**theory y = x' o [B]. Moreover there is a **0-ok** y' such that y ⊆ y' but C ∉ y'.

Proof. [B] is clearly a **0**-theory, being ∧-closed on applying axiom c5 and **0-**closed because if D→E ∈ **0** and B→D ∈ **0** then B→E ∈ **0** as well, by the transitivity theorem b4. Then by the fusion lemma x' o [B] = y is also a **0-**theory. This y is moreover a ¬-consistent theory without C. (Verify.) Apply the theory extension lemma to get a y' satisfying the last sentence of the theorem. QED.

Squeezing lemma. Let **0** be a regular **CB**-theory. Let y be a **0**-theory, and let x' and z' be **0**-ok theories such that x'oy ⊆ z'. Then there is a **0**-ok theory y' such that (a) y ⊆ y' and (b) x'oy' ⊆ z'.

Proof. Again write B ≤ C for B→C ∈ **0**; and write z∧A for {B: ∃C∈z (C∧A ≤ B)}. Enumerate the formulas of L as A_1, A_2 ,..., and set y_0 = y. We define the y_i recursively for each i from 1 on. First, set y_i" = $y_{i-1} \wedge A_i$. Then if x'oy_i" ⊄ z' then $y_i = y_{i-1}$ else $y_i = y_i$". Let y' be the union of the y_i. By construction, x'oy' ⊆ z'. Moreover y' is a **0**-theory (since we have kept it **0**-closed and ∧-closed at every stage in the construction). We wish to show y' ¬-consistent. Suppose not. Then **F**, when its turn came up, was added.[25] But this cannot be, since x' is (among other things) a **CB**-theory. Accordingly, since **T** ∈ x' we have **F**→**F** ∈ x' as well by axiom c7 and ≤E, whence **F** ∈ x'oy' ⊆ z', contradicting the ¬-consistency of z'. Finally, we need to show y' ¬-complete. So suppose B ∉ y'. Then B, when its turn came up, could not be "squeezed" into y'. This means that, at that stage in the construction, there was a C ∈ y' such that C∧B → ¬Z ∈ x', where Z ∈ z'. (Recall that z' is ¬-complete.) If we had similar problems with ¬B there would be a C' ∈ y' such that C'∧¬B → ¬Z' ∈ x', where Z' ∈ z'. But then on elementary distributive lattice moves we would have C∧C'∧(B∨¬B) → ¬Z ∨ ¬Z' ∈ x'. As the antecedent of this conditional is certainly in y', while its consequent is **not** in the prime theory z', we should then have a violation (which we **don't**) of x'oy' ⊆ z'. Conclusion: we do add ¬B when we cannot add B to y', whence y' is ¬-complete (and hence **0**-ok). This ends the proof of the squeezing lemma.

Canonical Interpretation theorem. Let I' be a canonical interpretation in the A-refuting canonical frame **K' =** < **0**, K', R'>. Then I' is an interpretation; i. e., it satisfies the truth-conditions T→, T∧, T¬ above.[26]

Moreover it is an A-refuting interpretation, since A ∉ **0** means I(A,**0**) = f.

Proof. T∧ holds of **all** theories. And T¬ holds of **0-**ok ones, since ∨-primeness implies ¬-completeness. We turn to T→, which is more fun. We must show, for all B,C ∈ L and all x' ∈ K',

T→. B→C ∈ x' iff ∀y'∈K'∀z'∈K'(x'oy' ⊆ z' ⇒ (B ∈ y' ⇒ C ∈ z'))

From l. to r., T→ is pretty obvious; if B→C ∈ x' and B ∈ y' then by Do we have C ∈ x'oy'; whence if x'oy' ⊆ z' we also have C ∈ z'. To show the converse, assume B→C ∉ x'. We need to find **0-**ok theories y',z' such that B∈y' but C∉z'. By the Priming lemma above there is at least a **0**-theory y and a **0**-ok theory z' such that B ∈ y , x'oy ⊆ z', but C ∉ z'. Apply then the Squeezing lemma to find the **0**-ok theory y' such that y ⊆ y', x'oy' ⊆ z', completing the verification of T→. Thus I' is indeed an interpretation, and an A-refuting one, ending the proof.

Completeness theorem for CB. All **CB-**valid formulas A are theorems of **CB.**

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[25] Or, as Abraham put it, any trouble is BIG trouble when one goes Boolean!

[26] Since we are Boolean, we need not fret about the heredity condition H, which has become trivial by the Frame theorem just proved.

Proof. By contraposition. If A isn't a theorem, then there is a canonical interpretation I' in a canonical frame **K'** such that I(A,0) = f. End of argument.

In a companion paper, I shall discuss modelling λ and **CL** in **CB-**theories.[27]

References for "Relevant" types: (Some of these are incomplete, but time is up. I'll finish.)

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