1 Gaussian Elimination

**PROCEDURE FOR GAUSSIAN ELIMINATION**

Any matrix can be reduced to row echelon form by carrying out the following procedure. (Roughly speaking we find a leading 1 in each column and transform each entry in the column under this 1 to 0.)

**STEP 1.** Find the leftmost column which does not consist entirely of zeros.

**STEP 2.** By interchanging rows if necessary obtain a nonzero entry \((a)\) say\) at the top of the column found in step 1.

**STEP 3.** Divide the first row by \(a\) to obtain a leading 1.

**STEP 4.** Add suitable multiples of the first row to the rows below so that all the entries below the leading 1 become 0.

**STEP 5.** Ignore the first row of the matrix and repeat the above procedure on the matrix which remains. Continue in this way until the entire matrix is in echelon form.

**PROCEDURE FOR SOLVING SYSTEMS OF EQUATIONS**

To solve a system of linear equations \(AX = B\):

**STEP 1.** Form the *augmented matrix* \((A|B)\).

**STEP 2.** Reduce to row echelon form as above.

**STEP 3.** Ignore rows of zeroes.

**STEP 4.** If the last row has the form \((0 \ldots 0|1)\), then the system is inconsistent.

**STEP 5.** Otherwise, the system is consistent, but may be *underdetermined*, ie have more unknowns (columns) then equations (rows). In that case, insert new rows of the form \((0\ldots0\ 0\ldots0|\alpha_j)\), where \(\alpha_1,\ldots,\alpha_k\) are *parameters*, between existing rows in such a way as to keep the matrix in row echelon form.

**STEP 6.** Now do further row operations to make the matrix to the left of the \(\mid\) equal to the identity: \((I|C)\). The new column matrix \(C\) is the *general solution* of the system. Note
that $C$ involves $k = n - m'$ parameters, where $n$ is the number of unknowns, and $m'$ is the number of nonzero rows remaining after step 3 above. Substituting arbitrary real numbers for the parameters will give a particular solution of the system of equations.

**EXAMPLE**

Find the general solution of the system of equations

\[
\begin{align*}
    w + x - y + 4z &= 2 \\
    2w + 2x - y + 7z &= 1 \\
    3w + 3x - 2y + 11z &= 3,
\end{align*}
\]

First, form the augmented matrix (step 1):

\[
\begin{pmatrix}
    1 & 1 & -1 & 4 & 2 \\
    2 & 2 & -1 & 7 & 1 \\
    3 & 3 & -2 & 11 & 3
\end{pmatrix}
\]

Now try to get into row echelon form (step 2). First try to get 0’s in the first column, below the 1 in the top left corner ($R_2 \rightarrow R_2 - 2 \times R_1$, $R_3 \rightarrow R_3 - 3 \times R_1$):

\[
\begin{pmatrix}
    1 & 1 & -1 & 4 & 2 \\
    0 & 0 & 1 & -1 & -3 \\
    0 & 0 & 1 & -1 & -3
\end{pmatrix}
\]

Continue: $R_3 \rightarrow R_3 - R_2$:

\[
\begin{pmatrix}
    1 & 1 & -1 & 4 & 2 \\
    0 & 0 & 1 & -1 & -3 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Delete the row of zeros (step 3).

\[
\begin{pmatrix}
    1 & 1 & -1 & 4 & 2 \\
    0 & 0 & 1 & -1 & -3
\end{pmatrix}
\]

Note that there is no row of the form $0 0 0 0 1$ (step 4), so the system is *consistent* (i.e., solutions exist).
Now insert two extra rows with parameters, keeping the matrix in row echelon form (step 5).

\[
\begin{pmatrix}
1 & 1 & -1 & 4 & 2 \\
0 & 1 & 0 & 0 & \beta \\
0 & 0 & 1 & -1 & -3 \\
0 & 0 & 0 & 1 & \alpha
\end{pmatrix}
\]

Now apply further row operations (first \(R3 \rightarrow R3 + R4\), \(R1 \rightarrow R1 - 4 \times R4\), then \(R1 \rightarrow R1 + R3\), and finally \(R1 \rightarrow R1 - R2\)) to get the identity matrix to the left of the bar (step 6).

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 2 + \beta - 3\alpha \\
0 & 1 & 0 & 0 & \beta \\
0 & 0 & 1 & 0 & -3 + \alpha \\
0 & 0 & 0 & 1 & \alpha
\end{pmatrix}
\]

So the general solution is: \(w = 2 + \beta - 3\alpha\), \(x = \beta\), \(y = -3 + \alpha\), \(z = \alpha\). One particular solution, for example, is \(w = 3\), \(x = 1\), \(y = -3\), \(z = 0\), obtained by setting \(\alpha = 0\) and \(\beta = 1\).

**PROCEDURE FOR INVERTING A MATRIX**

To invert an \(m \times m\) matrix \(A\), first form an augmented matrix of size \(m \times 2m\) \((A|I)\), where \(I\) is the identity. Next apply row operations to obtain \(I\) to the left of the bar: \((I|C)\). The resulting \(m \times m\) matrix \(C\) to the left of the bar is \(A^{-1}\).

What happens if \(A\) is not invertible? Then this procedure will fail: putting the matrix into echelon form by row operations will result in a row of the form \((0 \ldots 0|* \ldots *)\), so it will never be possible to get \(I\) to the left of the bar by row operations. In other words, the Gaussian elimination process will not only invert \(A\) when it is invertible, it will detect whether or not a given matrix is invertible.