Decompositions of finitely generated and finitely presented groups

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Abstract. In this paper we discuss the splitting or decomposing of finitely generated groups into free products, free products with amalgamation or HNN extensions and we discuss the JSJ decomposition of finitely presented groups.

1 Introduction

The decompositions of a finitely generated group into a free product of freely indecomposable groups have been completely classified by Grushko and Kurosh(see [6],[4]).

Theorem 1. *If G is a finitely generated group, then $G = G_1 * \cdots * G_n$, where each G_i is indecomposable, i.e. $G_i = A * B$ implies A or B is trivial. If also $G = G_1 * \cdots * G_n = H_1 * \cdots * H_m$ where each G_i and H_j is non-trivial and indecomposable, then $m = n$ and, by re-ordering, we have $G_i \cong H_i$ for each i. Furthermore, for each i with G_i not infinite cyclic we have G_i conjugate to H_i.*

Thus, the splitting of G into indecomposable groups is unique up to automorphisms of G. The above theorem also states that any finitely generated group G acts on a tree with trivial edge stabilizers, or equivalently, that G is the fundamental group of a graph of groups with trivial edge groups.

2 The JSJ Decomposition

A natural question at this point is to ask when a finitely generated group G can be viewed as the fundamental group of a graph of groups with non-trivial edge groups, or in other words, how does G "split".

Definition. *By a splitting of a group G, we understand a triple $\mathcal{S} = (\mathcal{G}(V,E), T, \phi)$ where $\mathcal{G}(V,E)$ is a graph of groups with underlying vertex set V and edge set E, T is a maximal subtree of (V,E), $\pi(\mathcal{G}(V,E);T)$ is the fundamental group of $\mathcal{G}(V,E)$ with respect to T and $\phi : \pi(\mathcal{G}(V,E);T) \to G$ is an isomorphism.*

The JSJ decomposition as defined by Rips and Sela [5], Dunwoody and Sageev [2], and Fujiwara and Papasoglu [3] gives a description of the possible splittings of G over certain classes of subgroups. For example, Rips and Sela's "generalized" JSJ decomposition "covers" any splitting of a finitely presented group with either trivial, finite or cyclic edge groups.

Dunwoody and Sageev have enlarged the class of groups over which a finitely presented group can split. Their JSJ decomposition describes all the splittings over *slender* groups, with the assumption that the group doesn't split over groups "smaller" than those considered. A group is *slender* if all its subgroups are finitely generated, thus all the nilpotent and even polycyclic groups are examples of slender groups. If H and K are subgroups of G, then H is *smaller* than K if $H \cap K$ has finite index in H and infinite index in K. Fujiwara and Papasoglu have removed the restriction that the group not split over *smaller* groups.

Each of the three above mentioned JSJ decompositions has been obtained using different approaches. I will describe the different methods and state the results, but I will give details only on the construction of the JSJ decomposition as given by Rips and Sela.

3 The Dunwoody Sageev Approach

In order to understand Dunwoody and Sageev's Main Theorem a few terms need to be clarified. Let us denote by K the class of slender groups and by LK the class of extensions of groups in K by 2-ended groups. The set of *ends* of a locally compact space X is the *limsup* over compact subsets C of the number of connected components of $X - C$. The set of ends of a finitely generated group G is the set of ends of its Cayley graph. For example, $\mathbb{Z}$ has two ends, $\mathbb{Z}^n$ for $n \geq 2$ has one end, and any free group on more than one generator has infinitely many ends.

The generic name $orbifold$ is used in the paper to denote a compact 2-orbifold. An informal definition describes an $orbifold$ as the quotient of a smooth, properly discontinuous action of a discrete group on a smooth manifold.

The Main Theorem claims that if we consider a finitely presented group G that does not split over any subgroup smaller than an element of LK, then either G is an extension of a K-group by a closed orbifold group, or there exists a decomposition of G as a graph of groups. The proof of the theorem uses the theory of simplicial trees, in particular patterns and track zipping.

4 The Fujiwara Papasoglu Approach

In their paper Fujiwara and Papasoglu describe an inductive procedure which produces the JSJ decomposition of a finitely presented group G over all its slender subgroups. All the intermediary steps of the algorithm give rise to graph of groups decompositions of G with the edge stabilizers being slender groups.

The first step in the procedure is proving the existence of a *minimal* splitting, i.e. a splitting that is not *hyperbolic* $-$ *elliptic* (see next section for definition) with respect to any other splitting of G over slender groups. Then this minimal splitting is being refined into graph decompositions that contain more and more minimal splittings. There exists an upper bound on the complexity of the graph decompositions, and therefore the process must terminate with a graph decomposition that "contains" all minimal splittings. This decomposition describes, in a sense, all splittings along slender groups that are not necessarily minimal.

Fujiwara and Papasoglu's approach to JSJ decompositions uses Haefliger's theory of complexes of groups and actions on products of trees.

5 The Rips and Sela Approach

Rips and Sela's papers try to understand groups from the perspective of low dimensional topology. For example, a separating simple closed curve on a surface corresponds, by van Kampen's theorem, to a free product with infinite cyclic amalgamation, $G = A *_C B$, where $C \cong \mathbb{Z}$.

Their approach is heavily based on the analysis of the most basic splittings, the *elementary $\mathbb{Z}$-splittings.* A splitting is called a *$\mathbb{Z}$-splitting* if all the edge groups are infinite cyclic, and an *elementary $\mathbb{Z}$-splitting* is a $\mathbb{Z}$-splitting for which the graph of groups contains only one edge. Consider two distinct elementary $\mathbb{Z}$-splittings of a group, $G = A_1 *_{C_1} B_1$ and $G = A_2 *_{C_2} B_2$, where $C_1 =< c_1 >$ and $C_2 =< c_2 >$. The element c_2 and the second splitting are called *elliptic* with respect to the first splitting if c_2 it is contained in a conjugate of A_1 or B_1, and *hyperbolic* otherwise; similarly for c_1 with respect to the second splitting. For example, the topological analog of a pair of hyperbolic-hyperbolic elementary $\mathbb{Z}$-splittings is a pair of intersecting simple closed curves. It turns out that the elementary $\mathbb{Z}$-splittings of a freely indecomposable group are either simultaneously elliptic or simultaneously hyperbolic.

The study of hyperbolic-hyperbolic splittings generates a "machinery" that produces *quadratically hanging (QH) subgroups.* A subgroup H of G is a *QH subgroup* if it is free, and if there exists a $\mathbb{Z}$-splitting of G such that H is the vertex group of a *QH vertex* v in this splitting. Geometrically, a *QH vertex* is a surface type vertex, where the surface has a finite number of punctures, and H is correspondingly the fundamental group of the surface, $\pi_1(S)$. The usefulness of the QH vertices resides in the fact that every simple closed curve on a surface S gives an elementary $\mathbb{Z}$-splitting of $\pi_1(S)$, and any elementary splitting of a vertex group can be extended to an elementary splitting of G by introducing a new edge e and its corresponding infinite cyclic group, and collapsing all other edges.

With the help of the "machinery" mentioned above, Rips and Sela construct certain decompositions for a single-ended finitely generated group G, called canonical quadratic decompositions. These decompositions describe all the hyperbolic-hyperbolic elementary $\mathbb{Z}$-splittings as splittings obtained from simple closed curves on one of the maximal QH subgroups of G.

The tool needed for studying elliptic-elliptic elementary $\mathbb{Z}$-splittings is the *unfolding* of elementary splittings. If $H = A *_C B$ and $C \subseteq C_1 \subseteq A$, then the splitting $H = A *_{C_1} B_1$ with $B_1 = C_1 *_C B$ is called a *folding* of H along C_1. An *unfolding* is the inverse of a folding. An important result concerning unfoldings states that every sequence of successive unfoldings of a finitely generated freely indecomposable group eventually terminates. Thus there exist elementary $\mathbb{Z}$-splittings which do not admit unfoldings. Such splittings are called *unfolded* elementary $\mathbb{Z}$-splittings, and more generally, $\mathbb{Z}$-splittings are called *unfolded* if all elementary $\mathbb{Z}$-splittings corresponding to their edges are unfolded.

Finally, a *JSJ decomposition* of a finitely presented single-ended group H will be defined as a reduced, unfolded $\mathbb{Z}$-splitting of H. In order to extend the JSJ decomposition from a single-ended finitely presented group to a finitely presented group G one starts by applying Grushko's theorem and decomposing G into noncyclic freely indecomposable factors and a free factor. Each of the noncyclic freely indecomposable factors can be replaced by its so called Dunwoody decomposition(see [1]), and then every nonfinite single-ended vertex group in this Dunwoody decomposition can be replaced by its JSJ decomposition. Thus G has a reduced, unfolded splitting with trivial, finite and cyclic edge groups which will be called the "generalized" JSJ decomposition.

The generalized JSJ decomposition "covers"any splitting of G with either trivial, finite or cyclic edge groups in a number of ways. Every canonical maximal QH subgroup of G is conjugate to a vertex group in the generalized JSJ decomposition. For a splitting Λ of G with trivial, free and cyclic edge groups there exists a splitting Λ_1 obtained from the generalized JSJ decomposition by splitting the canonical maximal subgroups along weakly essential simple closed cuves so that there exists a G-equivariant map between a subdivision of the Bass-Serre tree T_{Λ_1} and T_Λ.

References

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