The Surface Group Conjecture: Cyclically Pinched and Conjugacy Pinched One-Relator Groups

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Abstract

The general **surface group conjecture** asks whether a one-relator group where every subgroup of finite index is again one-relator and every subgroup of infinite index is free (property IF) is a surface group. We resolve several related conjectures given in [FKMRR]. First we obtain the Surface Group Conjecture B for cyclically pinched and conjugacy pinched one-relator groups. That is: if G is a cyclically pinched one-relator group or conjugacy pinched one-relator group satisfying property IF then G is free, a surface group or a solvable Baumslag-Solitar Group. Further combining results in [FKMRR] on Property IF with a theorem of H. Wilton [W] and results of Stallings [St] and Kharlampovich and Myasnikov [KhM4] we show that Surface Group Conjecture C proposed in [FKMRR] is true, namely: If G is a finitely generated nonfree freely indecomposable fully residually free group with property IF, then G is a surface group.

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1 Introduction

The **surface group conjecture** as originally proposed in the Kourovka notebook by Melnikov was the following problem.

Surface Group Conjecture. *Suppose that G is a residually finite non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group. Then G is a surface group.*

In this form the conjecture is false. Recall that the Baumslag-Solitar groups $BS(m,n)$ are the groups

$$BS(m,n) = < a, b; a^{-1}b^m a = b^n > .$$

If $|m| = |n|$ or either $|m| = 1$ or $|n| = 1$ these groups are residually finite. In all other cases these groups are not residually finite. Further $BS(m,n)$ is Hopfian if and only if $m = \pm 1$ or $n = \pm 1$ or m, n have the same set of primes ([CoLe]). If either $|m| = 1$ or $|n| = 1$ every subgroup of finite index is again a Baumslag-Solitar group and therefore a one-relator group, and every subgroup of infinite index is infinite cyclic and therefore a free group of rank one ([GKhM]). It follows that besides the surface groups, the groups $BS(1, m)$, also satisfy Melnikov's question. We then have the following modified conjecture.

Surface Group Conjecture A. *Suppose that G is a residually finite non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group. Then G is either a surface group or a Baumslag-Solitar group $BS(1, m)$ for some integer m.*

We note that the groups $BS(1, 1)$ and $BS(1, -1)$ are surface groups. In surface groups, subgroups of infinite index must be free groups. To avoid the Baumslag-Solitar groups, $BS(1, m), |m| \geq 2$, Surface Group Conjecture A, was modified to:

Surface Group Conjecture B. *Suppose that G is a non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group, there exists a noncyclic subgroup of infinite index and every subgroup of infinite index is a free group . Then G is a surface group of genus $g \geq 2$.*

In [FKMRR] the surface group conjecture was considered for fully residually free groups using the JSJ decomposition. A group G has **Property IF** if every subgroup of infinite index in G is free. In [FKMRR] it was proved that a fully residually free group satisfying property IF is either a cyclically pinched one-relator group or a conjugacy pinched one-relator group. This led to the following:

Surface Group Conjecture C. *Suppose that G is a finitely generated nonfree freely indecomposable fully residually free group with property IF. Then G is a surface group.*

In Theorem 3.2 below, we settle Surface Group Conjecture C by combining results in [FKMRR] with a recent theorem of Wilton [W] and results of Kharlampovich and Myasnikov [KhM4] and Stallings [St]. This result also appears in Wilton [W] for one-ended limit groups.

Theorem (3.2) *Suppose that G is a finitely generated nonfree freely indecomposable fully residually free group with property IF. Then G is a surface group. That is, Surface Group Conjecture C is true.*

We then improve on a result of Wilton (see Theorem 2.4) by dropping the conditions of one-ended hyperbolic from the hypothesis. Our main result concerns cyclically pinched and conjugacy pinched one-relator groups.

Theorem (3.1) *(1) Let G be a cyclically pinched one-relator group satisfying property IF. Then G is a free group or a surface group.*

(2) Let G be a conjugacy pinched one-relator group satisfying property IF Then G is a free group, a surface group or a solvable Baumslag-Solitar group.

2 Background Material and Necessary Results

Let G be the fundamental group of a compact surface of genus g. Then G has a one-relator presentation

$$< a_1, b_1, ..., a_g, b_g; [a_1, b_1]....[a_g, b_g] >$$

in the orientable case and

$$< a_1, ...a_g; a_1^2...a_g^2 >$$

in the non-orientable case. From covering space theory it follows that any subgroup of finite index is again a surface group of the same or higher genus while any subgroup of infinite index must be a free group. These results, although known since the early 1900's, were proved purely algebraically using Reidemeister-Schreier rewriting by Hoare, Karrass and Solitar in 1971 [HKS 1,2]. It is well known (see [FR]) that an orientable surface group can be faithfully represented as a discrete subgroup of $PSL_2(\mathbb{C})$ and hence each such group is linear. It follows that surface groups are residually finite. G.Baumslag [GB] showed that any orientable surface group of genus ≥ 2 must be residually free and 2-free from which it can be deduced using results of Remeslennikov [Re] and Gaglione and Spellman [GS] that they are fully residually free (see section 2). The article [AFR] surveys most of the properties of surface groups and shows how they are the primary motivating examples for much of combinatorial group theory.

A **cyclically pinched one-relator group** is a one-relator group of the following form

$$G = \langle a_1, ..., a_p, a_{p+1}, ..., a_n; U = V \rangle$$

where $1 \neq U = U(a_1, ..., a_p)$ is a cyclically reduced word in the free group F_1 on $a_1, ..., a_p$ and $1 \neq V = V(a_{p+1}, ..., a_n)$ is a cyclically reduced in the free group F_2 on $a_{p+1}, ..., a_n$.

Clearly such a group is the free product of the free groups on $a_1, ..., a_p$ and $a_{p+1}, ..., a_n$, respectively, amalgamated over the cyclic subgroups generated by U and V. From the standard one-relator presentation for an orientable surface group of genus $g \geq 2$ it follows that they are cyclically pinched one-relator groups. There is a similar decompositon in the nonorientable case.

The HNN analogs of cyclically pinched one-relator groups are called **conjugacy pinched one-relator groups** and are also motivated by the structure of orientable surface groups. In particular suppose

$$S_g = \langle a_1, b_1, ..., a_g, b_g; [a_1, b_1]...[a_g, b_g] = 1 \rangle.$$

If $b_g = t$ then S_g is an HNN group of the form

$$S_g = \langle a_1, b_1, ..., a_g, t; tUt^{-1} = V \rangle$$

where $U = a_g$ and $V = [a_1, b_1]...[a_{g-1}, b_{g-1}]a_g$. Generalizing this we say that a **conjugacy pinched one-relator group** is a one-relator group of the form

$$G = \langle a_1, ..., a_n, t; tUt^{-1} = V \rangle$$

where $1 \neq U = U(a_1, ..., a_n)$ and $1 \neq V = V(a_1, ..., a_n)$ are cyclically reduced in the free group F on $a_1, ..., a_n$.

The one-relator presentation for a surface group allows for a decomposition as a cyclically pinched one-relator group in both the orientable and non-orientable cases and as a conjugacy pinched one relator group in the orientable case (see [FRS]). In general cyclically pinched one-relator groups and conjugacy pinched one-relator groups have been shown to be extremely similar to surface groups. We refer to [FR] or [FRS], where a great deal more information about both cyclically pinched one-relator groups and conjugacy pinched one-relator groups is available.

In [FKMRR] several results were proved about the surface group conjectures. Recall that a group G is *fully residually free* if given finitely many nontrivial elements $g_1, ..., g_n$ in G there is a homomorphism $\phi : G \to F$, where F is a free group, such that $\phi(g_i) \neq 1$ for all $i = 1, ..., n$. Fully residually free groups have played a crucial role in the study of equations and first order formulas over free groups and in particular the solution of the Tarski problem (see [KhM 1-5] and [Se 1-2]). Finitely generated fully residually free groups are also known as **limit groups**. In this guise they were studied by Sela (see [Se 1-2] and [BeF 2]) in terms of studying homomorphisms of general groups into free groups.

In [FKMRR] several results concerning the Surface Group Conjectures were obtained:

Theorem 2.1 (Thm 3.1, [FKMRR])**.** *Suppose that G is a finitely generated fully residually free group with property IF. Then G is either a free group or a cyclically pinched one relator group or a conjugacy pinched one relator group.*

Corollary 2.1 (Cor 3.2, [FKMRR])**.** *Suppose that G is a finitely generated fully residually free group with property IF. Then G is either free or every subgroup of finite index is freely indecomposable and hence a one-relator group.*

Furthermore we have:

Theorem 2.2 (Thm 3.2, [FKMRR])**.** *Let G be a finitely generated fully residually free group with property IF. Then G is either hyperbolic or free abelian of rank 2.*

These results depend on the fact that the fully residually free groups have JSJ-decompositions. Roughly a JSJ-decomposition of a group G is a splitting of G as a graph of groups with abelian edges which is canonical in that it encodes all other such abelian splittings. If each edge is cyclic it is called a *cyclic JSJ-decomposition* (see [FKMRR]).

Being fully residually free provides a graph of groups decomposition. Then property IF will imply the finite index property, as follows:

Theorem 2.3 (Thm 3.3, [FKMRR])**.** *Let G be a nonfree cyclically pinched or conjugacy pinched one-relator group with property IF. Then each subgroup of finite index is again a cyclically pinched or conjugacy pinched one-relator group.*

The proof of Theorem 2.3 used the subgroup theorems for free products with amalgamation and HNN groups as described by Karrass and Solitar [KS 1,2].

Our proof of the Surface Group Conjecture C in Theorem 3.2 combines the results in [FKMRR] with the following statement, which is a rewording of a recent result of H.Wilton [W].

Theorem 2.4 (see Cor 4, [W])**.** *Let G be a hyperbolic one-ended cyclically pinched one-relator group or a hyperbolic one-ended conjugacy pinched one-relator group. Then either G is a surface group, or G has a finitely generated non-free subgroup of infinite index.*

3 Main Results

Our main result shows that Surface Group Conjecture B is true if G is a cyclically pinched or conjugacy pinched one-relator.

Theorem 3.1. *(1) Let G be a cyclically pinched one-relator group satisfying property IF. Then G is a free group or a surface group.*

(2) Let G be a conjugacy pinched one-relator group satisfying property IF. Then G is a free group, a surface group or a solvable Baumslag-Solitar group.

Before proving Theorem 3.1, we settle the Surface Group Conjecture C in the following theorem. This result appears in [Cor. 5, W] for one-ended limit groups.

Theorem 3.2. *Suppose that G is a finitely generated nonfree freely indecomposable fully residually free group with property IF. Then G is a surface group. That is, Surface Group Conjecture C is true.*

Proof. Suppose that G is a finitely generated freely indecomposable fully residually free group with property IF. If G is free abelian we are done since the free abelian group of rank 2 is a surface group, and of higher rank does not posses IF. If it is not free abelian then from Theorem 1.2 from [FKMRR] it follows that G is hyperbolic.

Since G is assumed to be a finitely generated freely indecomposable fully residually group and satisfies Property IF, from Theorem 2.1 in [FKMRR] it follows that

G must be either a cyclically pinched one relator group or a conjugacy pinched one relator group.

To apply Wilton's Theorem we show that it must be one-ended. Let $e(H)$ denote the number of ends of a group H. From a theorem of Stallings [St] $e(H) = 0$ if and only if H is finite and $e(H) > 1$ if and only if H has a non-trivial decomposition as a free product with amalgamation with finite amalgamated subgroup or a non-trivial decomposition as a HNN-group with finite associated subgroup.

Let G be the group as in the statement of the theorem. Certainly G is not finite so $e(G) > 0$. Since G satisfies Property IF it follows that if $e(G) > 1$ then the amalgamated subgroup (or the associated subgroup) in G has to be trivial. That implies that if our G is not one-ended then there is a free infinite cyclic factor and G is free or has a non-free subgroup of infinite index. Since G is freely indecomposable it follows that it must be one-ended and Wilton's result applies.

Therefore G is either a surface group or has a nonfree subgroup of infinite index. Again from property IF G must be a surface group settling Surface Group Conjecture C.

□

We now give the proof for cyclically pinched and conjugacy pinched one-relator groups.

Proof. (Theorem 3.1) (1) Let G be a cyclically pinched one-relator group amalgamated via $U = V$ and suppose that G is nonfree. Suppose that not both U and V are proper powers. Then by results of Juhasz and Rosenberger [JR], Bestvina and Feighn [BeF 1] and Kharlampovich and Myasnikov [KhM], G must be hyperbolic. Since G satisfies Property IF, as in the proof of Theorem 3.2, G must be one-ended and hence Wilton's theorem applies to give that G must be a surface group.

Suppose now that both U and V are proper powers. Let $U = g^n, n > 1$ and $V = h^m, m > 1$. If $G = \langle g, h : g^2 = h^2 \rangle$ then G is a nonhyperbolic, nonorientable surface group of genus 2.

Now assume that G is not isomorphic to a group $\langle a, b; a^2 = b^2 \rangle$. Then consider the subgroup $H = \langle g^n, gh \rangle$. H is free abelian of rank 2 and further H has infinite index in G. To see this introduce the relations $g^n = h^m = 1$. Then the image of G is a nontrivial free product, not isomorphic to the infinite dihedral group, and the image of H is infinite cyclic. However G is assumed to have Property IF and hence this case is impossible.

Thus all cyclically pinched one-relator groups with Property IF must be either free or a surface groups.

(2) Now let G be a conjugacy pinched one-relator group satisfying Property IF and assume G is nonfree. Suppose first that U and V are not both proper powers. Assume that U and V are conjugately separated in F, where F is the group generated by $a_1, ..., a_n$. This means that $\langle U \rangle \cap x \langle V \rangle x^{-1}$ is finite for all $x \in F$. Since G is torsion-free this intersection must be trivial. By a result of Kharlampovich

and Myasnikov [KhM4], G is then hyperbolic. Since G satisfies Property IF then, as in the proof of Theorem 3.2, G is one-ended and hence Wilton's theorem applies to give that G is a surface group.

Thus $\langle U\rangle \cap x\langle V\rangle x^{-1}$ is infinite, in fact infinite cyclic, for some $x \in F$. After a suitable conjugation and a possible interchange of U and V we may assume that U is not a proper power in F and $V = U^k$ for some $k \neq 0$. Let K be the subgroup generated by U and t. By normal form arguments K has a presentation $K = \langle U, t : tUt^{-1} = U^k\rangle$ (see [FRS]). If F is free of rank > 1, that is if $n > 1$, then K is a nonfree subgroup of infinite index in G. This can be seen as follows. In G introduce the relations $t = U = 1$. Then the image of G is a one-relator group $\langle a_1, ..., a_n; U = 1\rangle$ which is infinite by the Freiheitssatz. Therefore this case cannot occur since G satisfies Property IF.

Now let F be cyclic and let $a_1 = a$, so $G = \langle a, t; tat^{-1} = a^p\rangle$ with $p = \pm k$. Then G is a solvable Baumslag-Solitar group.

Finally let both U and V be proper powers so suppose that $U = g^n, n > 1$ and $V = h^m, m > 1$. By normal form arguments the subgroup N generated by $w = tgt^{-1}$ and h has a presentation $N = \langle w, h; w^n = h^m\rangle$ (see [FRS]). Note that the subgroup H generated by w^n and wh is free abelian of rank 2 and has infinite index in G. Since G is assumed to have Property IF this does not occur.

Altogether, if G is a conjugacy pinched one-relator group with Property IF then G must be either free, a surface group or a solvable Baumslag-Solitar group. □

4 Some Observations on the General Conjecture

Here we make some straightforward observations based on the proofs of Theorems 3.1 and 3.2 that might have a bearing on the general Surface Group Conjecture. From the proofs in [FKMRR] and the proofs of Theorems 3.1 and 3.2 we have the following.

Lemma 4.1. *Let G be nonfree and have a graph of groups decomposition with Property IF. Then each factor must be free and G is one-ended.*

Lemma 4.2. *Let G be nonfree and have a graph of groups decomposition with cyclic edge groups and with Property IF. Then G is a cyclically pinched or conjugacy pinched one-relator group, G is one-ended and hence either a surface group or a solvable Baumslag-Solitar group.*

We would like to note how strong Property IF is. A rewording of Lemma 4.2 implies that if G is not a free, not a surface, and not a Baumslag-Solitar group, then it cannot have a graph of groups decomposition with cyclic edge groups.

Now we consider noncyclic one-relator groups. Certainly if G has property IF it must be torsion-free. Let G be a torsion-free one-relator group with property IF. Using the standard Magnus breakdown G can be taken to be an HNN group. Hence by Property IF the base F is a free group and as in the proofs in section 3 it must be one-ended. Altogether then if G is a torsion-free finitely generated one-relator group with property IF then G has a presentation as an HNN extension with base group F, a free group. Then by rewriting we can get a presentation of the form

$$G =< F, t; tU_1t^{-1} = U_2, tU_2t^{-1} = U_3, ..., tU_{k-1}t^{-1} = U_k >$$

for some free group words $U_1, U_2, ..., U_k$. If $k = 2$ then it is a conjugacy pinched one-relator group and hence G is either free, a surface group or a solvable Baumslag-Solitar group. The general surface group conjecture would then be true if the following one is true.

Conjecture. *Suppose that*

$$G =< F, t; tU_1t^{-1} = U_2, tU_2t^{-1} = U_3, ..., tU_{k-1}t^{-1} = U_k >$$

with F a finitely generated nonabelian free group and $U_1, U_2, ..., U_k$ some nontrivial free group words. If $k > 2$ there must be a finite index subgroup that is not a one-relator group.

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References

[AFR] P.Ackermann, B.Fine and G.Rosenberger, On Surface Groups: Motivating Examples in Combinatorial Group Theory, **Groups St. Andrews 2005**. Cambridge University Press, 2007, 96–129

[BeF 1] M. Bestvina and M. Feighn, A combination theorem for negatively curved groups, **J. Diff. Geom.**, 35, 1992, 85-101

[BeF 2] M. Bestvina and M. Feighn, Notes on Sela's Work: Limit Groups and Makanin-Razborov Diagrams, preprint

[CoLe] D.Collins and F.Levin, Automorphisms and Hopficity of Certain Baumslag-Solitar Groups, **Archiv. Math.**, 40, 1983, 385–400

[FKMRR] B. Fine, O.Kharlampovich, A.Myasnikov,V.Remeslennikov and G. Rosenberger, On the Surface Group Conjecture, **Scienta: Math Series A** , 1, 2008, 1-15

[FR] B. Fine and G. Rosenberger, **Algebraic Generalizations of Discrete Groups**, Marcel-Dekker, 1999

[FRS] B.Fine, G.Rosenberger and M. Stille, Conjugacy Pinched and Cyclically Pinched One-Relator Groups, **Revista Math. Madrid** , 10, 1997, 207–227

[GS] A. Gaglione and D. Spellman, Some Model Theory of Free Groups and Free Algebras, **Houston J. Math** , 19, 1993, 327-356

[GKM] D.Gildenhuys, O. Kharlampovich and A. Myasnikov, CSA Groups and Separated Free Constructions, **Trans. Amer. Math. Soc.**, 350, 1998, 571-613

[HKS 1] A. Hoare, A. Karrass and D. Solitar, Subgroups of finite index of Fuchsian groups, **Math. Z.** , 120, 1971, 289–298

[HKS 2] A. Hoare, A. Karrass and D. Solitar, Subgroups of infinite index in Fuchsian groups, **Math. Z.** , 125, 1972, 59–69

[JR] A. Juhasz and G. Rosenberger, On the Combinatorial Curvature of Groups of F-type and Other One-Relator Products of Cyclics, **Cont. Math.**, 169, 1994, 373-384

[KS1] A. Karrass and D.Solitar, The Subgroups of a Free Product of Two Groups with an Amalgamated Subgroup, **Trans. Amer. Math. Soc.**, **Vol 150**, (1970), 227-255

[KS2] A. Karrass and D.Solitar, Subgroups of HNN Groups and Groups with One Defining Relation, **Can. J. Math. Soc.**, **Vol 23**, (1971), 627-643

[KhM] O. Kharlampovich and A. Myasnikov, Hyperbolic groups and free constructions; **Trans. Amer. Math. Soc.** 350 (1998), 571-613.

[KhM 1] O. Kharlamapovich and A.Myasnikov, Irreducible affine varieties over a free group: I. Irreducibility of quadratic equations and Nullstellensatz, **J. of Algebra**, 200, 1998, 472-516

[KhM 2] O. Kharlamapovich and A.Myasnikov, Irreducible affine varieties over a free group: II. Systems in triangular quasi-quadratic form and a description of residually free groups, **J. of Algebra**, 200, 1998, 517-569

[KhM 3] O. Kharlamapovich and A.Myasnikov, Description of fully residually free groups and Irreducible affine varieties over free groups, **Summer school in Group Theory in Banff, 1996, CRM Proceedings and Lecture notes** , 17, 1999, 71-81

[KhM 4] O.Kharlamapovich and A.Myasnikov, Hyperbolic Groups and Free Constructions, **Trans. Amer. Math. Soc.** 350, 2, 1998, 571-613

[KhM 5] O. Kharlamapovich and A.Myasnikov, Description of fully residually free groups and irreducible affine varieties over a free group, **CRM Proceeding and**

Lecture Notes: Summer School in Group Theory in Banff 1996, 17, 1998, 71-80

[Ko] Y.I. Merzlyakov, **Kourovka Notebook - Unsolved Problems in Group Theory**

[Re] V.N. Remeslennikov, ∃-free groups, **Siberian Mat. J.**, 30, 1989, 998–1001

[St] J. Stallings, On torsion-free groups with infinitely many ends, **Annals of Math.** (2) 88 (1968), 312-334

[Se 1] Z. Sela, Diophantine Geometry over Groups I: Makanin-Razborov Diagrams, **Publ. Math. de IHES**, 93, 2001, 31-105

[Se2] Z. Sela, Diophantine Geometry over Groups V: Quantifier Elimination, **Israel Jour. of Math.**, 150, 2005, 1-97

[W] H. Wilton, One ended subgroups of graphs of free groups with cyclic edge groups **Geom. Topol.** (to appear)

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