Fourier series - solution of the wave equation

We would like to justify the solution of the wave equation in a bounded domain we found by using the separation of variable technique. Let us consider the following problem

\[
\begin{align*}
    u_{tt} - Du_{xx} &= 0 & \text{in } [0, L] \times (0, \infty), \\
    u(x, 0) &= g(x) & \text{in } [0, L], \\
    u_t(x, 0) &= h(x) & \text{in } [0, L], \\
    u(0, t) &= u(L, t) = 0 & \text{for } t > 0.
\end{align*}
\]

The solution we were able to find was

\[
u(x, t) := \sum_{n=1}^{\infty} \left[ g_n \cos \left( \frac{n\pi}{L} ct \right) + \frac{L}{n\pi c} h_n \sin \left( \frac{n\pi}{L} ct \right) \right] \sin \left( \frac{n\pi}{L} x \right),
\]

by assuming the following sine Fourier series expansion of the initial data \(g\) and \(h\):

\[
\sum_{n=1}^{\infty} g_n \sin \left( \frac{n\pi}{L} x \right), \quad \sum_{n=1}^{\infty} h_n \sin \left( \frac{n\pi}{L} x \right).
\]

In order to prove that the function \(u\) above is the solution of our problem, we cannot differentiate term-by-term the series defining \(u\). We will instead use the reflection method: we consider the odd \(2L\)-periodic extensions of \(g\) and \(h\), namely we first extend \(g\) and \(h\) in \([-L, L]\) in an odd way \((g(x) = -g(x))\) for \(x \in [-L, 0]\) and same for \(h\), and then we take the periodic extensions \(\tilde{g}\) and \(\tilde{h}\) of these functions. Let us now consider the wave equation in the whole space

\[
\begin{align*}
    v_{tt} - Dv_{xx} &= 0 & \text{in } \mathbb{R} \times (0, \infty), \\
    v(x, 0) &= \tilde{g}(x) & \text{in } \mathbb{R}, \\
    v_t(x, 0) &= \tilde{h}(x) & \text{in } \mathbb{R}.
\end{align*}
\]

It is easy to see that \(v(0) = v(L) = 0\). Thus, we have that \(v\) restricted to the interval \([0, L]\) solves problem (1). But we have an explicit formula for \(v\):

\[v(x, t) = \frac{1}{2} \left[ \tilde{g}(x - ct) + \tilde{g}(x + ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \tilde{h}(y) \, dy.
\]

Since we are using the sine Fourier series for \(g\) and \(h\), these are the Fourier series of \(\tilde{g}\) and \(\tilde{h}\) (since they are odd in \([-L, L]\)). By inserting these two expansion in the above formula, we get

\[v(x, t) = \frac{1}{2} \left[ \sum_{n=1}^{\infty} h_n \sin \left( \frac{n\pi}{L} (x - ct) \right) + \sum_{n=1}^{\infty} h_n \sin \left( \frac{n\pi}{L} (x + ct) \right) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \sum_{n=1}^{\infty} h_n \sin \left( \frac{n\pi}{L} y \right).\]

We now use the identities

\[\sin \alpha + \sin \beta = 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right),\]

and

\[\cos \alpha - \cos \beta = -2 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right)\]

combined with the fact that we can integrate term-by-term the Fourier series, to obtain

\[v(x, t) = \sum_{n=1}^{\infty} h_n \sin \left( \frac{n\pi}{L} x \right) \cos \left( \frac{n\pi}{L} ct \right) + \frac{1}{2c} \sum_{n=1}^{\infty} \frac{L}{n\pi} h_n \sin \left( \frac{n\pi}{L} x \right) \sin \left( \frac{n\pi}{L} ct \right),\]

that is the expression defining \(u\) in (2). Thus, the function we found by using the separation of variables technique is a solution of the problem (1).