Quantitative Algebraic Reasoning

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Abstract

We develop a quantitative analogue of equational reasoning which we call quantitative algebra. We define an equality relation indexed by rationals: $a =_\varepsilon b$ which we think of as saying that "a is approximately equal to b up to an error of ε". We have 4 interesting examples where we have a quantitative equational theory whose free algebras correspond to well known structures. In each case we have finitary and continuous versions. The four cases are: Hausdorff metrics from quantitive semilattices; p-Wasserstein metrics (hence also the Kantorovich metric) from barycentric algebras and also from pointed barycentric algebras and the total variation metric from a variant of barycentric algebras.

1 Introduction

One of the exciting themes in research in programming language theory is the algebraic study of computational phenomena initiated by Moggi [Mog88, Mog91] where he showed how one can view notions of computation as monads. This allowed the incorporation of computational effects into a functional core in a compositional way. This became enormously influential and even led to monads being directly incorporated into programming languages like Haskell. It was a decade later that Plotkin and Power [PP01, PP02] began the study of computational effects from the point of view of equations and operations. From a categorical perspective one is moving from monads to Lawvere theories; see the excellent historical survey by Hyland and Power for more details [HP07].

One aspect of computational effects that has attracted significant attention is probabilistic computation [SD78, SD80, Koz81, Koz85, JP89]. This is, in fact,

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growing significantly with recent work spurred by interest from the machine learning community; see for example [BGG+11, GGR+14] among many other research efforts on the theory and practice of probabilistic programming as it applies to machine learning applications and [FKM+15] for a recently developed probabilistic programming language for network applications. Early work on lambda-calculi for probabilistic programming is due to Saheb-Djahromi [SD80]. Claire Jones [Jon90] developed a probabilistic λ-calculus in her thesis, gave an operational semantics and proved adequacy results. The fundamental work on probability monads is due to Lawvere [Law64] (before monads were invented!) and Gıry [Gir81]. One can develop a probabilistic λ-calculus using this monad [RP02].

In the present paper we develop an equational approach to reasoning about quantitative phenomena. The key new idea is to introduce equations annotated with rational numbers written $=_\varepsilon$ to capture the notion of *approximate* equality. One should think of $s =_\varepsilon t$ as saying that s and t are "within ε of each other." Essentially we are working with enriched Lawvere theories; see [Rob02] for an expository account of this subject. We do not emphasize the category-theoretic underpinnings here; instead we concentrate on presenting the notion of quantitative equations as concretely as possible. The bulk of the paper is spent on some very pleasing examples and on the general notions developed in the spirit of traditional universal algebra. In later work we will carefully spell out the categorical picture.

The examples are all of the following form: we give a simple set of equations and define the algebras of the resulting theory. We then induce metrics on the free algebra and identify them with commonly defined metrics. Thus, for example, we show that the Hausdorff metric arises from a quantitative version of semilattices. We show that the total variation metric arises from an axiomatization of convexity in terms of barycentric axioms. We show that the famous Kantorovich[1] metric [Vil08, vBW01, Pan09] arises from a variation of the same axioms. In fact, already the p-Wasserstein metric, which is a generalization of the Kantorovich metric arises from a variation of the same axioms. These metrics (especially Kantorovich) play a fundamental role in the study of probabilistic bisimulation [Pan09] and transport theory [Vil08]. We present both finitary and infinitary versions of these constructions.

Metric ideas have been important in denotational semantics from the beginning especially in Jaco de Bakker's school; see [vB01] for a survey. It may seem that for probabilistic reasoning one needs to work with measure theory. This is, of course, true but measure theory works best when there is metric structure; as witnessed, for example, by the ubiquity of Polish spaces in discussions of measure theory. The algebraic approach to effects [PP01, PP02, PP03, PP04, HPP06, HLPP07] has not, until now, been considered in a metric context. Owing to the increasing importance of probability in computer science it seems worthwhile to investigate this now. The first order of business then is to see

[1] This metric goes by many names: Hutchinson, Wasserstein (with numerous variations in spelling) and Kantorovich-Rubinstein. Perhaps the most commonly used name is Wasserstein.

how some familiar and important monads fit into this approach. In this paper, we only consider monads related to probabilistic and nondeterministic systems. However the well-known basic examples (exceptions, states, I/O) also fit into the framework of this paper, albeit with some inessential limitations arising from our working with operations with finite discrete arities.

2 Quantitative Equational Theories

An algebraic similarity type consists of a *sort name*[2] and a finite set of function symbols each with fixed *finite* arity. Consider an algebraic similarity type Ω and algebras of this type. Given a set X of variables, let $\mathbb{T}X$ be the set of terms constructed over Ω from X, this is the *term algebra* of Ω over X.

A *substitution* is a function $\sigma : X \longrightarrow \mathbb{T}X$. It can be canonically extended to $\sigma\colon \mathbb{T}X \longrightarrow \mathbb{T}X$ by:

- for any $f : n \in \Omega$, $\sigma(f(t_1, ..t_n)) = f(\sigma(t_1), ..\sigma(t_n))$.

In what follows $\Sigma(X)$ denotes the set of substitutions on $\mathbb{T}X$.

If $\Gamma \subseteq \mathbb{T}X$ is a set of terms and $\sigma \in \Sigma(X)$, let $\sigma(\Gamma) = \{\sigma(t) \mid t \in \Gamma\}$.

Let $\mathcal{V}(X)$ denote the set of *indexed equalities* of the form $x =_\epsilon y$ for $x, y \in X$ and $\epsilon \in \mathbb{Q}_+$; similarly, let $\mathcal{V}(\mathbb{T}X)$ denote the set of indexed equalities of the form $t =_\epsilon s$ for $t, s \in \mathbb{T}X$, $\epsilon \in \mathbb{Q}_+$. We call them *quantitative equations.*

Definition 2.1 (Deducibility Relation) *Given an algebraic similarity type Ω and a set X of variables, a* deducibility relation of type Ω over X *is a relation $\vdash \subseteq 2^{\mathcal{V}(\mathbb{T}X)} \times \mathcal{V}(\mathbb{T}X)$ closed under the following rules stated for arbitrary $t, s, u, t_1, \cdots t_n \in \mathbb{T}X$, $\epsilon, \epsilon' \in \mathbb{Q}_+$, $\Gamma, \Gamma' \subseteq \mathcal{V}(\mathbb{T}X)$ and $\phi, \psi \in \mathcal{V}(\mathbb{T}X)$; where $(\Gamma, \phi) \in \vdash$ is denoted by $\Gamma \vdash \phi$:*

(Refl) $\emptyset \vdash t =_0 t$

(Symm) $\{t =_\epsilon s\} \vdash s =_\epsilon t$.

(Triang) $\{t =_\epsilon s, s =_{\epsilon'} u\} \vdash t =_{\epsilon+\epsilon'} u$.

(Max) *For* $\epsilon' > 0$, $\{t =_\epsilon s\} \vdash t =_{\epsilon+\epsilon'} s$.

(Arch) *For* $\epsilon \geq 0$, $\{t =_{\epsilon'} s \mid \epsilon' > \epsilon\} \vdash t =_\epsilon s$.

(NExp) *For* $f : n \in \Omega$, $\{t_1 =_\epsilon s_1, \ldots, t_n =_\epsilon s_n\} \vdash f(t_1, ..t_i, ..t_n) =_\epsilon f(s_1, ..s_i, ..s_n)$

(Subst) *If* $\sigma \in \Sigma(X)$, $\Gamma \vdash t =_\epsilon s$ *implies* $\sigma(\Gamma) \vdash \sigma(t) =_\epsilon \sigma(s)$.

(Cut) *If* $\Gamma \vdash \phi$ *for all* $\phi \in \Gamma'$ *and* $\Gamma' \vdash \psi$, *then* $\Gamma \vdash \psi$.

(Assumpt) *If* $\phi \in \Gamma$, *then* $\Gamma \vdash \phi$.

[2]In later work we will develop and use many-sorted algebras.

Let $\mathcal{E}(\mathbb{T}X) = \mathcal{P}_f(\mathcal{V}(\mathbb{T}X)) \times \mathcal{V}(\mathbb{T}X)$, where $\mathcal{P}_f(A)$ is the finite powerset of A; we call its elements *quantitative inferences* on $\mathbb{T}X$. If $(V, \phi) \in \mathcal{E}(\mathbb{T}X)$, we refer to the elements of V as the *hypotheses* of the inference. An *unconditional quantitative inference* is a quantitative inference with an empty set of hypotheses.

Of particular interest for us is the subclass $\mathcal{E}(X) = \mathcal{P}_f(\mathcal{V}(X)) \times \mathcal{V}(\mathbb{T}X)$ of quantitative inferences, hereafter called *basic quantitative inferences*, where the hypotheses are finite sets of quantitative equations between variables. The basic quantitative inferences are the ones that we will use as axioms for theories.

Definition 2.2 (Quantitative Equational Theory) *Given a set $S \subseteq \mathcal{E}(\mathbb{T}X)$ of quantitative inferences on $\mathbb{T}X$, denote by $\vdash_S$ the smallest deducibility relation that contains S. The* quantitative equational theory induced by S *is the set*

$$\mathcal{U} \stackrel{\text{def}}{=} (\vdash_S) \cap \mathcal{E}(\mathbb{T}X).$$

Note that a quantitative equational theory does not contain any conditional inference with infinitely many hypotheses, nor indeed does the set S. However, in constructing $\mathcal{U}$ from S, we can use the infinitary archimedean rule in derivations. This restriction can be relaxed to allow conditional inferences with countable sets of hypothesis without really changing much from the theory developed hereafter.

If $\mathcal{U}$ is a quantitative equational theory and $\emptyset \vdash s =_e t \in \mathcal{U}$, we will abuse the notation and also write $\mathcal{U} \vdash s =_e t$.

Definition 2.3 (Consistent theories) *A quantitative equational theory $\mathcal{U}$ over $\mathbb{T}X$ is* inconsistent *if $\mathcal{U} \vdash x =_0 y$, where $x, y \in X$ are two distinct variables. $\mathcal{U}$ is* consistent *if it is not inconsistent.*

3 Quantitative Algebras

Definition 3.1 (Quantitative Algebra) *A* quantitative algebra *is a tuple $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$, where $(A, \Omega^{\mathcal{A}})$ is an algebra of type Ω and*

$$d^{\mathcal{A}} : A \times A \longrightarrow \mathbb{R}_+ \cup \{\infty\}$$

is a metric on A (possibly taking infinite values) such that all the operators in the signature are non-expansive. i.e., for any $f : n \in \Omega^{\mathcal{A}}$, any $a_i, b_i \in A$, $i = 1, ..n$ and any $\epsilon \geq 0$,

$$d^{\mathcal{A}}(a_i, b_i) \leq \epsilon \;\; \textit{for all} \;\; i = 1, ..n \quad \textit{implies} \quad d^{\mathcal{A}}(f(a_1, .., a_n), f(b_1, .., b_n)) \leq \epsilon.$$

A quantitative algebra is *degenerate* if its support is empty or a singleton.

Definition 3.2 (Homomorphism of Quantitative Algebras) *Given two quantitative algebras* $\mathcal{A}_i = (A_i, \Omega, d^{\mathcal{A}_i})$*,* $i = 1, 2$*, of type* Ω*, a* homomorphism of quantitative algebras *is a* non-expansive Ω*-homomorphism (of* Ω*-algebras); i.e., it is a map* $h : A_1 \longrightarrow A_2$ *that, for arbitrary* $a, b \in A_1$*,*

$$d^{\mathcal{A}_1}(a, b) \geq d^{\mathcal{A}_2}(h(a), h(b)) .$$

Notice that identity maps are homomorphisms and that homomorphisms are closed under composition, hence quantitative algebras of type Ω and their homomorphisms form a category, denoted Ω-**QA**.

Definition 3.3 (Subalgebras) *The quantitative algebra* $\mathcal{B} = (B, \Omega, d^{\mathcal{B}})$ *is a subalgebra of the quantitative algebra* $\mathcal{A} = (A, \Omega, d^{\mathcal{A}})$*, denoted by* $\mathcal{B} \leq \mathcal{A}$*, if* $\mathcal{B}$ *is a subalgebra of* $\mathcal{A}$ *as* Ω*-algebra and, in addition, for any* $a, b \in B$*,* $d^{\mathcal{B}}(a, b) = d^{\mathcal{A}}(a, b)$*.*

Definition 3.4 (Initiality) *Let* $\mathbb{K}$ *be a subcategory of* Ω-**QA***, hence its objects are quantitative algebras of type* Ω*. A quantitative algebra* $\mathcal{A}$ *is* initial in $\mathbb{K}$ *if* $\mathcal{A} \in \mathbb{K}$ *and, for all* $\mathcal{B} \in \mathbb{K}$*, there exists a unique homomorphism of quantitative algebras* $\alpha \colon \mathcal{A} \longrightarrow \mathcal{B}$*.*

Note that if two quantitative algebras are both initial in some subcategory $\mathbb{K}$, then they are isomorphic.

Definition 3.5 (Universal mapping property) *Let* $\mathbb{K}$ *be a subcategory of quantitative algebras of type* Ω*,* $\mathbb{C}$ *an arbitrary category,* $G \colon \mathbb{K} \longrightarrow \mathbb{C}$ *a functor and* C *an object in* $\mathbb{C}$*. A* universal morphism from C to G *is a pair* $(\mathcal{A}, \alpha)$ *consisting of a quantitative algebra* $\mathcal{A} \in \mathbb{K}$ *and a morphism* $\alpha \colon C \longrightarrow G\mathcal{A}$ *in* $\mathbb{C}$*, such that for every pair* $(\mathcal{B}, \beta)$ *with* $\mathcal{B} \in \mathbb{K}$ *and* $\beta \colon C \longrightarrow G\mathcal{B}$ *a morphism in* $\mathbb{C}$*, there exists a unique homomorphism of quantitative algebras* $h \colon \mathcal{A} \longrightarrow \mathcal{B}$ *such that* $Gh \circ \alpha = \beta$*. Diagrammatically*

$$\begin{array}{ccc} & \text{in } \mathbb{C} & \qquad \text{in } \mathbb{K} \\ C & \xrightarrow{\alpha} G\mathcal{A} & \qquad \mathcal{A} \\ & \searrow^{\beta} \quad \downarrow Gh & \qquad \downarrow h \\ & G\mathcal{B} & \qquad \mathcal{B} \end{array}$$

A quantitative algebra $\mathcal{A}$ *has the* universal mapping property for C to G *if there exists a universal morphism* $(\mathcal{A}, \alpha)$ *from* C *to* G*.*

4 Algebraic Semantics for Quantitative Equations

Definition 4.1 (Assignment) *Given a quantitative algebra* $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$ *of type* Ω *and a set* X *of variables, an* assignment *on* $\mathcal{A}$ *is a function* $\iota : X \longrightarrow A$ *that is canonically extended to* $\iota : \mathbb{T}X \longrightarrow A$ *over* Ω*-terms as follows*

- *for any* $f : n \in \Omega$, $\iota(f(t_1, \ldots, t_n)) = f^{\mathcal{A}}(\iota(t_1), \ldots \iota(t_n))$.

Note that the requirements above guarantee that ι is a homomorphism of Ω-algebras, where the operations in Ω have the canonical interpretation in $\mathbb{T}X$. We denote by $\mathbb{T}(X|\mathcal{A})$ the set of assignments on $\mathcal{A}$.

Definition 4.2 (Satisfiability) *Consider a quantitative algebra* $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}})$ *and a set* X *of variables.* $\mathcal{A}$ satisfies *a quantitative inference* $\Gamma \vdash s =_\epsilon t \in \mathcal{E}(\mathbb{T}X)$ *over* $\mathbb{T}X$*, written*

$$\Gamma \models_{\mathcal{A}} s =_\epsilon t,$$

if for all assignments $\iota \in \mathbb{T}(X|\mathcal{A})$*,*

$$[\ d^{\mathcal{A}}(\iota(t'), \iota(s')) \leq \epsilon' \ \ \textit{for all} \ \ s' =_{\epsilon'} t' \in \Gamma\ ] \ \ \textit{implies} \ \ d^{\mathcal{A}}(\iota(s), \iota(t)) \leq \epsilon.$$

In these cases we say that $\mathcal{A}$ is a *model* of the inference. Similarly, for a quantitative equational theory (or set of quantitative inferences) Γ, we say that $\mathcal{A}$ is a model of Γ if $\mathcal{A}$ satisfies each element of Γ. A quantitative inference (a quantitative equational theory) is *satisfiable* if it has a model.

For the case of unconditioned quantitative inferences of type $\emptyset \vdash s =_\epsilon t$, observe that the left-hand side of the implication in the previous definition is vacuously satisfied. For these inferences, to further simplify the notation, instead of $\emptyset \models_{\mathcal{A}} s =_\epsilon t$ we also write $\mathcal{A} \models s =_\epsilon t$.

Definition 4.3 (Equational Class of Quantitative Algebras) *For a signature* Ω *and a quantitative equational theory* $\mathcal{U}$ *over the* Ω*-terms* $\mathbb{T}X$*, the* equational class induced by $\mathcal{U}$ *is the class of quantitative algebras of signature* Ω *satisfying* $\mathcal{U}$*.*

We denote this class as well as the full subcategory of Ω-quantitative algebras satisfying $\mathcal{U}$ by $\mathbb{K}(\Omega,\mathcal{U})$. We say that a class of algebras that is an equational class is *equationally definable.*

$\mathbb{K}(\Omega,\mathcal{U})$ is obviously closed under taking isomorphic images. We prove below that it is also closed under subalgebras.

Lemma 4.4 *If* $\mathcal{A} \in \mathbb{K}(\Omega,\mathcal{U})$ *and* $\mathcal{B} \leq \mathcal{A}$*, then* $\mathcal{B} \in \mathbb{K}(\Omega,\mathcal{U})$*.*

Proof. Since $\mathcal{B} \leq \mathcal{A}$, $id_{\mathcal{B}} : B \longrightarrow A$ defined by $id_{\mathcal{B}}(b) = b$ is a morphism of quantitative algebras.

Suppose that $\{s_i =_{\epsilon_i} t_i \mid i \in I\} \vdash s =_e t \in \mathcal{U}$. Hence,

$$\{s_i =_{\epsilon_i} t_i \mid i \in I\} \models_{\mathcal{A}} s =_e t \in \mathcal{U},$$

meaning that for any $\iota \in \mathbb{T}(X|\mathcal{A})$,

$$[d^{\mathcal{A}}(\iota(s_i), \iota(t_i)) \leq \epsilon_i \ \ \text{for all} \ \ i \in I] \quad \text{implies} \quad d^{\mathcal{A}}(\iota(s), \iota(t) \leq e.$$

Consider an arbitrary $\iota \in \mathbb{T}(X|\mathcal{B})$ and observe that $\iota \in \mathbb{T}(X|\mathcal{A})$ as well.

Suppose that $[d^{\mathcal{B}}(\iota(s_i), \iota(t_i)) \leq \epsilon_i$ for all $i \in I]$. This is equivalent to

$$[d^{\mathcal{A}}(\iota(s_i), \iota(t_i)) \leq \epsilon_i \ \ \text{for all} \ \ i \in I].$$

But then, we also have $d^{\mathcal{A}}(\iota(s), \iota(t)) \leq e$. Hence, $d^{\mathcal{B}}(\iota(s), \iota(t)) \leq e$. ∎

5 The Induced Pseudometric

Given a quantitative equational theory $\mathcal{U}$ over the set $\mathbb{T}X$ for a given signature Ω, we intend to define a pseudometric over the set $\mathbb{T}X$ of terms somehow induced by $\mathcal{U}$.

There are three possible candidates for this definition:

$$d^{\mathcal{U}}(s,t) = \inf\{\epsilon \mid \emptyset \vdash s =_\epsilon t \in \mathcal{U}\}\,,$$

$$\gamma^{\mathcal{U}}(s,t) = \inf\{\epsilon \mid \forall V \in \mathcal{P}_f(\mathcal{V}(X)), V \vdash s =_\epsilon t \in \mathcal{U}\}\,,$$

and

$$\delta^{\mathcal{U}}(s,t) = \inf\{\epsilon \mid \exists V \in \mathcal{P}_f(\mathcal{V}(X)), V \vdash s =_\epsilon t \in \mathcal{U}\}\,,$$

In what follows we prove that $\delta^{\mathcal{U}} = 0$ and hence provides no useful information; and that $d^{\mathcal{U}} = \gamma^{\mathcal{U}}$, which essentially guarantees that it is sufficient to use the unconditioned quantitative inferences of a theory to derive a pseudometric between the terms.

Proposition 5.1 *For arbitrary $s,t \in \mathbb{T}(X)$, $\delta^{\mathcal{U}}(s,t) = 0$.*

Proof. Consider two variables $x, y \in X$ and let $\epsilon \geq 0$ be arbitrarily chosen.

Since $x =_\epsilon y \vdash x =_\epsilon y$, we obtain that

$$\delta^{\mathcal{U}}(x,y) \leq \epsilon \quad \text{for all} \quad \epsilon \geq 0.$$

Hence, $\delta^{\mathcal{U}}(x,y) = 0$ for any variables $x, y \in X$, implying further that for arbitrary terms $t, s \in \mathbb{T}(X)$, $\delta^{\mathcal{U}}(s,t) = 0$. ∎

Proposition 5.2 *For arbitrary $s,t \in \mathbb{T}(X)$, $\gamma^{\mathcal{U}}(s,t) = d^{\mathcal{U}}(s,t)$.*

Proof. We prove that

$$\emptyset \vdash s =_\epsilon t \in \mathcal{U} \ \text{ iff [for any } \ V \in \mathcal{P}_f(\mathcal{V}(X)), V \vdash s =_\epsilon t \in \mathcal{U}].$$

($\Leftarrow$): Since $\emptyset \in \mathcal{P}_f(\mathcal{V}(X))$, this direction is trivial.

($\Rightarrow$): Suppose that $\emptyset \vdash s =_\epsilon t \in \mathcal{U}$. Applying (Cut) instantiated with $\Gamma = V$ and $\Gamma' = \emptyset$ we get that for any $V \in \mathcal{P}_f(\mathcal{V}(X))$, $V \vdash s =_\epsilon t \in \mathcal{U}$.

The equivalence mentioned above guarantees that:

$$\inf\{\epsilon \mid \emptyset \vdash s =_\epsilon t \in \mathcal{U}\} = \inf\{\epsilon \mid \forall V \in \mathcal{P}_f(\mathcal{V}(X)), V \vdash s =_\epsilon t \in \mathcal{U}\}.$$

Hence $d^{\mathcal{U}} = \gamma^{\mathcal{U}}$. ∎

6 Free Quantitative Algebras

Fix a signature Ω and a quantitative equational theory $\mathcal{U}$ over Ω-terms in $\mathbb{T}X$ with variables in X.

Term Quantitative Algebra. Define $d_{\mathcal{U}} : \mathbb{T}X \times \mathbb{T}X \to \mathbb{R}_+ \cup \{\infty\}$ (possibly taking infinite values) as

$$d_{\mathcal{U}}(t, s) = \inf\{\epsilon \mid \mathcal{U} \vdash t =_\epsilon s\}\,.$$

By (Refl), (Symm), and (Triang), $d^{\mathcal{U}}$ is a well-defined pseudometric on $\mathbb{T}X$. Observe that due to (Arch) we can characterize the distance 0 as follows

$$d_{\mathcal{U}}(s, t) = 0 \quad \text{iff} \quad \mathcal{U} \vdash s =_0 t.$$

This pseudometric induces on $\mathbb{T}X$ the following equivalence relation

$$\sim = \{(s, t) \in \mathbb{T}X^2 \mid d_{\mathcal{U}}(s, t) = 0\}.$$

Lemma 6.1 *$\sim$ is a congruence relation w.r.t. Ω.*

Proof. Let $f : n \in \Omega$ and let $x_i, y_i \in \mathbb{T}X$, $i = 1, ..n$ be such that $x_i \sim y_i$ for all $i = 1, ..n$. We need to prove that $f(x_1, ..x_n) \sim f(y_1, ..y_n)$.

Since $x_i \sim y_i$, $\mathcal{U} \vdash x_i =_0 y_i$. Using (NExp) and (Cut) we further derive that $\mathcal{U} \vdash f(x_1, .., x_n) =_0 f(y_1, .., y_n)$. Hence, $f(x_1, ..x_n) \sim f(y_1, ..y_n)$. ∎

Since $\sim$ is a congruence, the quotient $\mathbb{T}X^\sim$ of $\mathbb{T}X$ w.r.t. $\sim$ can be organized as an Ω-algebra by lifting arbitrary $f : n \in \Omega$ to the quotient set and defining the operator $f^\sim$:

$$f^\sim(x_1^\sim, .., x_n^\sim) = (f(x_1, .., x_n))^\sim,$$

where $x^\sim$ denotes the $\sim$-equivalence class of $x \in \mathbb{T}X$.

Similarly, $d_{\mathcal{U}}$ became the metric $d^\sim$ on the quotient set characterized by

$$d^\sim(x^\sim, y^\sim) = d_{\mathcal{U}}(x, y).$$

Under these interpretations $\mathbb{T}X^\sim$ became a quantitative algebra and a model for $\mathcal{U}$.

Lemma 6.2 *$\mathbb{T}X^\sim = (\mathbb{T}X^\sim, \Omega, d^\sim)$ is an Ω-quantitative algebra and a model of $\mathcal{U}$, i. e., $\mathbb{T}X^\sim \in \mathbb{K}(\Omega, \mathcal{U})$.*

Proof. The construction, together with the fact that $\sim$ is a congruence, guarantees that $\mathbb{T}X$ is an Ω-algebra and that $(\mathbb{T}X^\sim, d^\sim)$ is a metric space.

We prove now that all the operators $f^\sim : n$ are non-expansive w.r.t. $d^\sim$.

Suppose that for each $i = 1, ..n$, $d^\sim(x_i^\sim, y_i^\sim) \leq \epsilon$, where $x_i, y_i \in \mathbb{T}X$. Hence, for any $e \in \mathbb{Q}_+$ s.t. $e \geq \epsilon$, $\mathcal{U} \vdash x_i =_e y_i$. (NExp) and (Cut) gives us further that

$$\mathcal{U} \vdash f(x_1, .., x_n) =_e f(y_1, .., y_n).$$

Hence, for any $e \geq \epsilon$, $d_{\mathcal{U}}(f(x_1, .., x_n), f(y_1, .., y_n)) \leq e$, i.e.,

$$d_{\mathcal{U}}(f(x_1, .., x_n), f(y_1, .., y_n)) \leq \epsilon.$$

But then,

$$d^{\sim}(f^{\sim}(x_1^{\sim}, .., x_n^{\sim}), f^{\sim}(y_1^{\sim}, .., y_n^{\sim})) \leq \epsilon$$

meaning that, indeed, $f^{\sim}$ is non-expansive w.r.t. $d^{\sim}$. This proves that $\mathbb{T}X^{\sim}$ is an Ω-quantitative algebra.

It remains now to show that $\mathbb{T}X^{\sim}$ is a model of $\mathcal{U}$.

Suppose that $\{s_i =_{\epsilon_i} t_i \mid i \in I\} \vdash s =_e t \in \mathcal{U}$. Then, using (Subst), for any substitution $\sigma \in \Sigma(X)$, $\{\sigma(s_i) =_{\epsilon_i} \sigma(t_i) \mid i \in I\} \vdash \sigma(s) =_e \sigma(t) \in \mathcal{U}$.

Consider an assignment $\iota \in \mathbb{T}(X|\mathbb{T}X^{\sim})$ and let $\sigma_\iota \in \Sigma(X)$ be such that for any $x \in X$, $\sigma_\iota(x) \in \mathbb{T}X$ is an element of the equivalence class of $\iota(x) \in \mathbb{T}X^{\sim}$. Hence, our hypothesis guarantees that

$$\{\sigma_\iota(s_i) =_{\epsilon_i} \sigma_\iota(t_i) \mid i \in I\} \vdash \sigma_\iota(s) =_e \sigma_\iota(t) \in \mathcal{U}.$$

In order to prove that $\{s_i =_{\epsilon_i} t_i \mid i \in I\} \models_{\mathbb{T}X^{\sim}} s =_e t \in \mathcal{U}$, we need to prove that for any $\iota \in \mathbb{T}(X|\mathbb{T}X^{\sim})$,

$$[d^{\sim}(\iota(s_i), \iota(t_i)) \leq \epsilon_i \text{ for any } i \in I] \quad \text{implies} \quad d^{\sim}(\iota(s), \iota(t) \leq e.$$

Since ϵ_i are indexes of quantitative equations, they must be rationals, hence $[d^{\sim}(\iota(s_i)\iota(t_i)) \leq \epsilon_i \text{ for any } i \in I]$ is equivalent to

$$[\emptyset \vdash \sigma_\iota(s_i) =_{\epsilon_i} \sigma_\iota(t_i) \in \mathcal{U} \text{ for any } i \in I].$$

Because $\{\sigma_\iota(s_i) =_{\epsilon_i} \sigma_\iota(t_i) \mid i \in I\} \vdash \sigma_\iota(s) =_e \sigma_\iota(t) \in \mathcal{U}$, we obtain that

$$\emptyset \vdash \sigma_\iota(s) =_e \sigma_\iota(t) \in \mathcal{U},$$

which is equivalent to $d^{\sim}(\iota(s), \iota(t)) \leq e$ – and this concludes our proof.

∎

Freely Generated Algebra. In what follows, for a set M of generators, we construct a quantitative algebra $\mathbb{T}[M]$ with support a quotient of $\mathbb{T}M$. The equivalence relation by which we quotient is

$$s \sim t \text{ iff } \mathcal{U} \vdash s =_0 t$$

which we call 0-provability for $\mathcal{U}$.

We endow $\mathbb{T}M$ with a pseudometric in order to eventually get a quantitative algebra. This pseudometric is defined following the model theoretic intuition: it is generated by the quantitative equational theory $\mathcal{U}$ such that $\mathbb{T}M$ is a model for $\mathcal{U}$.

$$d(m, n) = \inf\{\epsilon \mid \exists \iota \in \mathbb{T}(X|\mathbb{T}M), \exists s, t \in \mathbb{T}X, m = \iota(s), n = \iota(t) \text{ and } \mathcal{U} \vdash s =_\epsilon t\}.$$

Notice that due to the density of rationals and due to (Arch), if the infimum is a rational number, then it belongs to the set, hence it is a minimum.

Lemma 6.3 *$d : \mathbb{T}M^2 \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ is a pseudometric.*

Proof. The symmetry and the fact that $d(m, m) = 0$ for any $m \in \mathbb{T}M$ derive trivially from the definition. In what follows we prove the triangle inequality.

Let $m_1, m, m_2 \in \mathbb{T}M$. We aim to prove that

$$d(m_1, m_2) \leq d(m_1, m) + d(m, m_2).$$

For this, we prove that for each $\epsilon_1, \epsilon_2 \in \mathbb{Q}_+$, $d(m_1, m) \leq \epsilon_1$ and $d(m, m_2) \leq \epsilon_2$ implies that $d(m_1, m_2) \leq \epsilon_1 + \epsilon_2$.

Suppose that $d(m_1, m) \leq \epsilon_1$ and $d(m, m_2) \leq \epsilon_2$. Since $\epsilon_1, \epsilon_2 \in \mathbb{Q}_+$, there exist $\iota_1, \iota_2 \in \mathbb{T}(X|\mathbb{T}M)$, $t_1, t_2, s_1, s_2 \in \mathbb{T}X$, such that $\iota_i(t_i) = m_i$, $\iota_i(s_i) = m$, $\mathcal{U} \vdash t_1 =_{\epsilon_1} s_1$ and $\mathcal{U} \vdash s_2 =_{\epsilon_2} t_2$.

For each $i = 1, 2$, since $s_i, t_i \in \mathbb{T}X$, there exist $y^i_1, .., y^i_{k_i} \in X$ and two functions f^i, h^i that can be constructed from the operators of the signature Ω, such that $s_i = f^i(y^i_1, .., y^i_{k_i})$ and $t_i = h^i(y^i_1, .., y^i_{k_i})$ – some of these functions might not depend of all these parameters, but possibly by a subset of them.

For each $i = 1, 2$, since $\iota_i : \mathbb{T}X \longrightarrow \mathbb{T}M$ is a morphism, there exist $m^i_1, .., m^i_{k_i} \in \mathbb{T}M$ such that $m = f^i(m^i_1, .., m^i_{k_i})$ and $m_i = h^i(m^i_1, .., m^i_{k_i})$; and moreover, $\iota_i(y^i_j) = m^i_j$ for each $j = 1, .., k_i$.

However, it is not necessarily that $m^i_j \in M$. For this reason, there must exist $n_1, .., n_l \in M$ and some functions g^i_j definable from the signature such that $m^i_j = g^i_j(n_1, .., n_l)$ for each $i = 1, 2$ and each $j = 1, .., k$ – again, some of these functions might not depend of all these l parameters, but possibly by a subset of them. Obviously, $n_1, .., n_l$ are (distinct by choice and) unique and moreover,

$$f^1(g^1_1(n_1, .., n_l), .., g^1_{k_1}(n_1, .., n_l)) = f^2(g^2_1(n_1, .., n_l), .., g^2_{k_2}(n_1, .., n_l)) = m, \quad (1)$$

and

$$h^i(g^i_1(n_1, .., n_l), .., g^i_{k_i}(n_1, .., n_l)) = m_i, \quad i = 1, 2.$$

Since $n_1, .., n_l \in M$ and $m \in \mathbb{T}M$, from the universality of the term algebra (as an Ω-algebra) we get that equation (1) also prove that for arbitrary $x_1, .., x_l \in X$,

$$f^1(g^1_1(x_1, .., x_l), .., g^1_{k_1}(x_1, .., x_l)) = f^2(g^2_1(x_1, .., x_l), .., g^2_{k_2}(x_1, .., x_l)). \quad (2)$$

Consider now a set $\{x_1, .., x_l\} \subseteq X$ of distinct variables in X and two substitutions $\sigma^1, \sigma^2 \in \Sigma(X)$ such that $\sigma^i(y^i_j) = g^i_j(x_1, .., x_l)$ for each $i = 1, 2$ and each $j = 1, .., k_i$. Observe that such substitutions can always be defined, since y^i_j are variables in X.

Observe now that

$$\sigma^i(s_i) = \sigma^i(f^i(y^i_1, .., y^i_{k_i})) = f^i(\sigma^i(y^i_1), .., \sigma^i(y^i_{k_i}))$$

$$= f^i(g^i_1(x_1, .., x_l), .., g^i_{k_i}(x_1, .., x_l)),$$

and similarly,

$$\sigma^i(t_i) = \sigma^i(h^i(y_1^i, .., y_{k_i}^i)) = h^i(\sigma^i(y_1^i), .., \sigma^i(y_{k_i}^i))$$

$$= h^i(g_1^i(x_1, .., x_l), .., g_{k_i}^i(x_1, .., x_l)).$$

Applying equation (2) and (Refl), we obtain that

$$\sigma^1(s_1) = \sigma^2(s_2),\ i.e.,\ \ \mathcal{U} \vdash \sigma^1(s_1) =_0 \sigma^2(s_2) \tag{3}$$

We already know that $\mathcal{U} \vdash t_1 =_{\epsilon_1} s_1$ and applying (Subst) we further obtain

$$\mathcal{U} \vdash \sigma^1(t_1) =_{\epsilon_1} \sigma^1(s_1). \tag{4}$$

Similarly, from $\mathcal{U} \vdash s_2 =_{\epsilon_2} t_2$ we obtain

$$\mathcal{U} \vdash \sigma^2(s_2) =_{\epsilon_2} \sigma^2(t_2). \tag{5}$$

Applying (Triang) to equations (3), (4) and (5) we obtain that

$$\mathcal{U} \vdash \sigma^1(t_1) =_{\epsilon_1+\epsilon_2} \sigma^2(t_2). \tag{6}$$

Consider an assignment $\iota \in \mathbb{T}(X|\mathbb{T}M)$ such that $\iota(x_j) = n_j$ for $j = 1, .., l$. Such an assignment can always be defined since x_j are variables in X. Then,

$$\iota(\sigma^i(t_i)) = \iota(h^i(g_1^i(x_1, .., x_l), .., g_{k_i}^i(x_1, .., x_l)))$$

$$= h^i(g_1^i(\iota(x_1), .., \iota(x_l)), .., g_{k_i}^i(\iota(x_1), .., \iota(x_l)))$$

$$= h^i(g_1^i(n_1, .., n_l), .., g_{k_i}^i(n_1, .., n_l)) = h^i(m_1^i, .., m_{k_i}^i) = m_i.$$

Consequently, we have proven that there exist $t_1' = \sigma^1(t_1), t_2' = \sigma^2(t_2) \in \mathbb{T}X$ and $\iota \in \mathbb{T}(X|\mathbb{T}M)$ such that $\iota(t_1') = m_1$, $\iota(t_2') = m_2$ and $\mathcal{U} \vdash t_1' =_{\epsilon_1+\epsilon_2} t_2'$.

This, by definition, guarantees that $d(m_1, m_2) \leq \epsilon_1 + \epsilon_2$ and it concludes our proof.

∎

Let $(\mathbb{T}[M], d^{\cong})$ be the metric space induced by the pseudometric d by quotienting $\mathbb{T}M$ w.r.t. the equivalence relation $\cong\ = \{(p, q) \mid d(p, q) = 0\}$ on $\mathbb{T}M$. Explicitly, $\mathbb{T}[M]$ is the set of $\cong$-equivalence classes on $\mathbb{T}M$ and

$$d^{\cong}(p^{\cong}, q^{\cong}) = d(p, q)$$

where $p^{\cong}, q^{\cong}$ are the $\cong$-equivalence classes of $p, q \in \mathbb{T}M$, respectively.

Lemma 6.4 $\cong$ *is a congruence w.r.t. the operators in* Ω*, i.e., for arbitrary* $f : n \in \Omega$ *and* $p_i, q_i \in \mathbb{T}M$*,* $i = 1, \ldots, n$*,*

$$p_i \cong q_i \ \ \textit{implies} \ \ f(p_1, \ldots, p_n) \cong f(q_1, \ldots, q_n).$$

Proof. $p_i \cong q_i$ is equivalent with the statement that for any rational $\epsilon > 0$ there exists $\iota_i \in \mathbb{T}(X|\mathbb{T}M)$ such that $p_i = \iota_i(s_i)$, $q_i = \iota_i(t_i)$ and $\mathcal{U} \vdash s_i =_\epsilon t_i$. Thus, when we characterize $p_i \cong q_i$ for all $i = 1,..n$ we initially get a set

$$\{\iota_i \mid i = 1,..n\} \subseteq \mathbb{T}(X|\mathbb{T}M).$$

Suppose that s_i and t_i depend on the variables $x^i_1,..,x^i_k$. Then for any other set $\{y_1,..,y_k\}$ of variables in X there exists a bijective substitution $\sigma \in \Sigma(X)$ such that $\sigma(x^i_j) = y_j$, $\sigma(y_j) = x^i_j$ and $\sigma(x) = x$ for the rest of variables. Applying (Subst) we will further get that $\mathcal{U} \vdash \sigma(s_i) =_\epsilon \sigma(t_i)$ and $\iota_i \circ \sigma^{-1} \in \mathbb{T}(X|TM)$ is such that $(\iota_i \circ \sigma^{-1})(\sigma(s_i)) = p_i$ and $(\iota_i \circ \sigma^{-1})(\sigma(t_i)) = q_i$.

This observation allows us to assume that when we chose ι_i to characterize $p_i \cong q_i$, for $i \neq j$ the set of variables on which s_i and t_i depend is disjoint of the set of variables of s_j and t_j.

Since the active role of ι_i in the characterization of $p_i \cong q_i$ has to do only with the variables on which s_i and t_i depend and because these variables are distinct for $i \neq j$, there exists $\iota \in \mathbb{T}(X|\mathbb{T}M)$ such that for each $i = 1,..n$ ι coincides with ι_i on the set of variables on which s_i and t_i depend.

Hence, for any $\epsilon > 0$, there exists $\iota \in \mathbb{T}(X|\mathbb{T}M)$ such that for any $i = 1,..n$, $p_i = \iota(s_i)$, $q_i = \iota(t_i)$ and $\mathcal{U} \vdash s_i =_\epsilon t_i$. Applying (NExp) and (Cut) we get further that $\mathcal{U} \vdash f(s_1,..,s_n) =_\epsilon f(t_1,..,t_n)$.

Observe now that because

$$f(p_1,..,p_n) = f(\iota(s_1),..,\iota(s_n)) = \iota(f(s_1,..,s_n))$$

and

$$f(q_1,..,q_n) = f(\iota(t_1),..,\iota(t_n)) = \iota(f(t_1,..,t_n)),$$

we obtain that there exist $s = f(s_1,..s_n), t = f(t_1,..,t_n) \in \mathbb{T}X$ and $\iota \in \mathbb{T}(X|\mathbb{T}M)$ such that $\iota(f(p_1,..,p_n)) = s$, $\iota(f(q_1,..,q_n)) = t$ and $\mathcal{U} \vdash s =_\epsilon t$. Hence, for any $\epsilon > 0$, $d(f(p_1,..,p_n), f(q_1,..,q_n)) \leq \epsilon$ implying

$$d(f(p_1,..,p_n), f(q_1,..,q_n)) = 0.$$

This means that $f(p_1,..,p_n) \cong f(q_1,..,q_n)$. ∎

Due to the fact that $\cong$ is a congruence, we can endow $\mathbb{T}[M]$ with the structure of an Ω-algebra by interpreting $f : n \in \Omega$, for arbitrary $p_1,\ldots,p_n \in \mathbb{T}M$ as follows:

$$f(p_1^{\cong},\ldots,p_n^{\cong}) = (f(p_1,\ldots p_n))^{\cong}.$$

Notice that, by this construction, what we actually get is a quantitative algebra.

Lemma 6.5 $\mathbb{T}[M] = (\mathbb{T}[M], \Omega, d^{\cong})$ *is an* Ω*-quantitative algebra.*

Proof. That check that $d^{\cong}$ is a metric on $\mathbb{T}[M]$ derives from the fact that d is a pseudometric and from the fact that $d^{\cong}(p^{\cong}, q^{\cong}) = 0$ iff $p^{\cong} = q^{\cong}$.

It remains to prove that every $f : n \in \Omega$ is non-expansive w.r.t. $d^{\cong}$, i.e., if $d^{\cong}(p_i^{\cong}, q_i^{\cong}) \leq \epsilon$ for all $i = 1, ..n$, then $d^{\cong}(f(p_1^{\cong}, .., p_n^{\cong}), f(q_1^{\cong}, .., q_n^{\cong})) \leq \epsilon$.

But since $\cong$ is a congruence, $d^{\cong}(p_i^{\cong}, q_i^{\cong}) \leq \epsilon$ is equivalent to $d(p_i, q_i) \leq \epsilon$ and $d^{\cong}(f(p_1^{\cong}, .., p_n^{\cong}), f(q_1^{\cong}, .., q_n^{\cong})) \leq \epsilon$ is equivalent to $d(f(p_1, .., p_n), f(q_1, .., q_n)) \leq \epsilon$. Hence, the aforementioned implication can be proven by reusing part of the proof of Lemma 6.4. ∎

Now returning to the term quantitative algebra $\mathbb{T}X^{\sim}$, observe that the previous construction can also be done for the case when $M = X$ and we obtain the quantitative algebra $\mathbb{T}[X] = (\mathbb{T}[X], \Omega, d^{\cong})$ of terms modulo 0-provability.

Lemma 6.6 *$\mathbb{T}[X] = (\mathbb{T}[X], \Omega, d^{\cong})$ and $\mathbb{T}X^{\sim} = (\mathbb{T}X^{\sim}, \Omega, d^{\sim})$ are isomorphic quantitative algebras. Moreover, $d = d_{\mathcal{U}}$, hence $d^{\cong} = d^{\sim}$.*

Proof. If we prove that $d = d_{\mathcal{U}}$, we get firstly that $\sim = \cong$, and being the way the two algebras have been constructed, we eventually prove that they are isomorphic, where the isomorphism $h : \mathbb{T}X^{\sim} \longrightarrow \mathbb{T}X^{\cong}$ is the coincidence of the congruence classes: for any $t \in \mathbb{T}X$, $h(t^{\sim}) = t^{\cong}$.

Note that any assignment $\iota \in \mathbb{T}(X|\mathbb{T}X)$ is also a substitution $\iota \in \Sigma(X)$, since they are both just homomorphisms from $\mathbb{T}X$ to $\mathbb{T}X$. Consequently, for any $\iota \in \mathbb{T}(X|\mathbb{T}X)$, any $s, t \in \mathbb{T}X$ and any $\epsilon \in \mathbb{Q}_+$,

$$\mathcal{U} \vdash s =_{\epsilon} t \quad \text{iff} \quad \mathcal{U} \vdash \iota(s) =_{\epsilon} \iota(t).$$

But then,

$$\begin{aligned}
d(s_1, s_2) &= \inf\{\epsilon \mid \exists t_1, t_2 \in \mathbb{T}X, \exists \iota \in \mathbb{T}(X|\mathbb{T}X) \text{ s.t. } \iota(t_i) = s_i \text{ and } \mathcal{U} \vdash t_1 =_{\epsilon} t_2\} \\
&= \inf\{\epsilon \mid \exists t_1, t_2 \in \mathbb{T}X, \exists \iota \in \Sigma(X) \text{ s.t. } \iota(t_i) = s_i \text{ and } \mathcal{U} \vdash \iota(t_1) =_{\epsilon} \iota(t_2)\} \\
&= \inf\{\epsilon \mid \exists t_1, t_2 \in \mathbb{T}X, \exists \iota \in \Sigma(X) \text{ s.t. } \iota(t_i) = s_i \text{ and } \mathcal{U} \vdash s_1 =_{\epsilon} s_2\} \\
&= \inf\{\epsilon \mid \mathcal{U} \vdash s_1 =_{\epsilon} s_2\} \\
&= d^{\mathcal{U}}(s_1, s_2).
\end{aligned}$$

∎

The previous lemma allows us to prove that $\mathbb{T}[M]$ indeed satisfies the quantitative equational theory $\mathcal{U}$ used for its construction.

Theorem 6.7 *For an arbitrary set M, $\mathbb{T}[M] = (\mathbb{T}[M], \Omega, d_{\mathcal{U}}^{\cong}) \in \mathbb{K}(\Omega, \mathcal{U})$.*

Proof. We know already that $\mathbb{T}[X] \in \mathbb{K}(\Omega, \mathcal{U})$, since we have proven that $\mathbb{T}X^{\sim} \in \mathbb{K}(\Omega, \mathcal{U})$.

We split the cases according to whether the cardinality of M is less than or greater than that of X.

I. If $|M| \leq |X|$, let $Y \subset X$ be such that $|Y| = |M|$. Obviously, $\mathbb{T}[Y] \leq \mathbb{T}[X]$ and since $\mathbb{T}[X] \in \mathbb{K}(\Omega,\mathcal{U})$, applying Lemma 4.4, we obtain that $\mathbb{T}[Y] \in \mathbb{K}(\Omega,\mathcal{U})$.

Since $|M| = |Y|$, $\mathbb{T}[M]$ and $\mathbb{T}[Y]$ are isomorphic (freely generated) quantitative algebras with the isomorphism generated by any bijective map between M and Y. So, from $\mathbb{T}[Y] \in \mathbb{K}(\Omega,\mathcal{U})$, we get $\mathbb{T}[M] \in \mathbb{K}(\Omega,\mathcal{U})$.

II. If $|M| > |X|$, let $Z \supseteq X$ be such that $|M| = |Z|$. As before, $\mathbb{T}[M]$ and $\mathbb{T}[Z]$ are isomorphic. Observe now that because $X \subseteq Z$, $\mathcal{U}$ generates a quantitative equational theory $\mathcal{U}'$ on Z, which contains $\mathcal{U}$ and in addition some other inferences that might be obtain, for instance, from inferences of $\mathcal{U}$ by applying the (Subst) rule that involve some variables from $Z \setminus X$. What is essential here is that $\mathcal{U} \subseteq \mathcal{U}'$.

Lemma 6.2 guarantees that for any $\Gamma \vdash \phi \in \mathcal{U}'$, $\Gamma \models_{\mathbb{T}[Z]} \phi$. In particular, for $\Gamma \vdash \phi \in \mathcal{U}$ we have $\Gamma \models_{\mathbb{T}[Z]} \phi$. And since $\mathbb{T}[Z]$ and $\mathbb{T}[M]$ are isomorphic, we also obtain $\Gamma \models_{\mathbb{T}[M]} \phi$. ∎

Completeness. Having in hand all these results, we are ready now to prove the following strong completeness theorem.

For a quantitative inference $\Gamma \vdash \phi$, we write $\Gamma \models_{\mathbb{K}(\Omega,\mathcal{U})} \phi$ whenever

$$\Gamma \models_{\mathcal{A}} \phi \ \text{ for all } \ \mathcal{A} \in \mathbb{K}(\Omega,\mathcal{U}).$$

Theorem 6.8 (Completeness) *Given a quantitative equational theory $\mathcal{U}$ over the set X of variables and signature Ω,*

$$\Gamma \models_{\mathbb{K}(\Omega,\mathcal{U})} \phi \quad \textit{iff} \quad \Gamma \vdash \phi \in \mathcal{U}.$$

Proof. The right-to-left implication (soundness) is a direct consequence of the definition of $\mathbb{K}(\Omega,\mathcal{U})$.

It remains for us to prove the left-to-right implication:

$$[\Gamma \models_{\mathcal{A}} \phi \ \text{ for any } \ \mathcal{A} \in \mathbb{K}(\Omega,\mathcal{U})] \quad \text{implies } \Gamma \vdash \phi \in \mathcal{U}.$$

Suppose that the left-hand side is satisfied. Assume that ϕ is the quantitative equation $s =_e t$.

Let $\overline{\mathcal{U} \cup \Gamma}$ be the quantitative equational theory induced by $\mathcal{U} \cup \{\emptyset \vdash \psi \mid \psi \in \Gamma\}$. Obviously, $\overline{\mathcal{U} \cup \Gamma}$ is a theory over $\mathbb{T}X$. Applying Lemma 6.2, we obtain that $(\mathbb{T}[X], \Omega, d^{\cong}_{\overline{\mathcal{U}\cup\Gamma}})$ is a model for $\overline{\mathcal{U} \cup \Gamma}$, hence both for $\mathcal{U}$ and for $\{\emptyset \vdash \psi \mid \psi \in \Gamma\}$.

Because $(\mathbb{T}[X], \Omega, d^{\cong}_{\overline{\mathcal{U}\cup\Gamma}}) \in \mathbb{K}(\Omega,\mathcal{U})$, $(\mathbb{T}[X], \Omega, d^{\cong}_{\overline{\mathcal{U}\cup\Gamma}})$ satisfies $\Gamma \vdash \phi$. And because $(\mathbb{T}[X], \Omega, d^{\cong}_{\overline{\mathcal{U}\cup\Gamma}})$ is a model of Γ, (Cut) proves that $(\mathbb{T}[X], \Omega, d^{\cong}_{\overline{\mathcal{U}\cup\Gamma}})$ is also a model for $s =_e t$. Consequently, $\inf\{\epsilon \mid \overline{\mathcal{U} \cup \Gamma} \vdash s =_\epsilon t\} \leq e$, i.e.,

$$d_{\overline{\mathcal{U}\cup\Gamma}}(s,t) \leq e.$$

Suppose now that $\Gamma \vdash s =_e t \notin \mathcal{U}$.
If $\emptyset \vdash s =_e t \in \mathcal{U}$, applying (Cut) we get that $\Gamma \vdash s =_e t \in \mathcal{U}$ - contradiction.

Also, $\emptyset \vdash s =_e t \notin \overline{\{\emptyset \vdash \psi \mid \psi \in \Gamma\}}$, because otherwise $s =_e t$ is derived from the hypothesis in Γ and the use of some of the closure conditions in Definition 2.2, i.e., $\Gamma \vdash s =_e t$ is guaranteed by the closure rules in Definition 2.2. But then, we also have $\Gamma \vdash s =_e t \in \mathcal{U}$ - contradiction.

Since $\emptyset \vdash s =_e t \notin \mathcal{U} \cup \overline{\Gamma}$, if $\emptyset \vdash s =_e t \in \overline{\mathcal{U} \cup \Gamma}$, then there exists $\Gamma' \subseteq \Gamma$ and $\Delta \in \mathcal{U}$ such that $\Gamma' \cup \Delta \vdash s =_e t \in \mathcal{U}$. Then, using (Assumpt) and (Cut), we must also have $\Gamma \cup \Delta \vdash s =_e t \in \mathcal{U}$.

Because $\emptyset \vdash \rho \in \mathcal{U}$ for all $\rho \in \Delta$, we also have $\Gamma \vdash \rho \in \mathcal{U}$ for all $\rho \in \Delta$ and applying (Assumpt) we get further $\Gamma \vdash \rho \in \mathcal{U}$ for all $\rho \in \Gamma \cup \Delta$. Since $\Gamma \cup \Delta \vdash s =_e t \in \mathcal{U}$, applying (Cut) we get $\Gamma \vdash s =_e t \in \mathcal{U}$ - contradiction.

Hence, $\emptyset \vdash s =_e t \notin \overline{\Gamma \cup \mathcal{U}}$.

Let $i = \inf\{\epsilon \mid \overline{\Gamma \cup \mathcal{U}} \vdash s =_\epsilon t\} = d_{\overline{\Gamma \cup \mathcal{U}}}(s,t)$.

If $i \in \mathbb{Q}$, then using (Arch) we can prove that $\overline{\Gamma \cup \mathcal{U}} \vdash s =_i t$ and further (Max) guarantees that $i > e$, since $\emptyset \vdash s =_e t \notin \overline{\Gamma \cup \mathcal{U}}$.

If $i \notin \mathbb{Q}$, from $\emptyset \vdash s =_e t \notin \overline{\Gamma \cup \mathcal{U}}$ we derive that $i \geq e$. But since $e \in \mathbb{Q}$, this means that $i > e$.

Hence, $d_{\overline{\Gamma \cup \mathcal{U}}}(s,t) > e$. Thus, we derive a contradiction, since we have already proved that $d_{\overline{\mathcal{U} \cup \Gamma}}(s,t) \leq e$. ∎

Universality. The next theorem proves that the construction of $\mathbb{T}[M]$ is universal (in the categorical sense) with respect to all the quantitative algebras satisfying the quantitative equational theory $\mathcal{U}$. Specifically, $\mathbb{T}[M]$ has the universal mapping property for M to the (obvious) forgetful functor

$$U_{\mathbf{Set}} \colon \mathbb{K}(\Omega, \mathcal{U}) \longrightarrow \mathbf{Set}.$$

This situation is described by the following commutative diagram (*cf.* Definition 3.5):

$$\begin{array}{ccc} \text{in } \mathbf{Set} & & \text{in } \mathbb{K}(\Omega,\mathcal{U}) \\ \begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[dr, "\alpha"'] & \mathbb{T}[M] \arrow[d, "h"] \\ & A \end{tikzcd} & & \begin{tikzcd} \mathbb{T}[M] \arrow[d, dashed, "h"] \\ \mathcal{A} \end{tikzcd} \end{array}$$

where $\eta_M \colon M \longrightarrow \mathbb{T}[M]$ is the map given by $\eta_M(m) = m^{\cong}$.

Theorem 6.9 $(\mathbb{T}[M], \eta_M)$ *is a universal arrow from* $M \in \mathbf{Set}$ *to* $U_{\mathbf{Set}}$.

Proof. We have already proven that $\mathbb{T}[M] \in \mathbb{K}(\Omega,\mathcal{U})$. Consider an arbitrary quantitative algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathbb{K}(\Omega,\mathcal{U})$ and a set-map $\alpha \colon M \longrightarrow A$. This map can be canonically extended to a morphism $\hat{\alpha} : \mathbb{T}M \longrightarrow \mathcal{A}$ of Ω-universal algebras by defining, for arbitrary $f : k \in \Omega$ and $t_1, .., t_k \in \mathbb{T}M$,

$$\hat{\alpha}(f^M(t_1, .., t_k)) = f^{\mathcal{A}}(\hat{\alpha}(t_1), .., \hat{\alpha}(t_k)).$$

To start with, we prove that $\hat{\alpha}$ is nonexpansive. Since the metric $d^{\cong}$ on $\mathbb{T}[M]$ is just the pseudo-metric d on $\mathbb{T}M$ quotiented by the equivalence relation, what we need to prove is that for arbitrary $s, t \in \mathbb{T}M$,

$$d^{\mathcal{A}}(\hat{\alpha}(s), \hat{\alpha}(t)) \leq d(s, t).$$

To do that, we show that for two arbitrary elements $s, t \in \mathbb{T}M$ such that $d(s, t) \leq e_0 \in \mathbb{Q}_+$, we also have $d^{\mathcal{A}}(\hat{\alpha}(s), \hat{\alpha}(t)) \leq e_0$.

Because $d(s, t) \leq e_0$, for any rational $e > 0$ there exists $\iota_e \in \mathbb{T}(X|\mathbb{T}M)$ and there exist $s^e, t^e \in \mathbb{T}X$ such that $\mathcal{U} \vdash s^e =_{e+e_0} t^e$ and $\iota_e(s^e) = s$, $\iota_e(t^e) = t$.

Because $\mathcal{U} \vdash s^e =_{e+e_0} t^e$ and $\mathcal{A} \in \mathbb{K}(\Omega, \mathcal{U})$, we have that $\mathcal{A} \models s^e =_{e+e_0} t^e$, i.e., for any $\iota \in \mathbb{T}(X|\mathcal{A})$, $d^{\mathcal{A}}(\iota(s^e), \iota(t^e)) \leq e + e_0$. Note that $\hat{\alpha} \circ \iota_\epsilon \in \mathbb{T}(X|\mathcal{A})$. Hence,

$$d^{\mathcal{A}}((\hat{\alpha} \circ \iota_e)(s^e), (\hat{\alpha} \circ \iota_e)(t^e)) \leq e + e_0$$

or equivalently,

$$d^{\mathcal{A}}(\hat{\alpha}(s), \hat{\alpha}(t)) \leq (e + e_0).$$

Since this is true for any $e > 0$, we obtain that necessarily

$$d^{\mathcal{A}}(\hat{\alpha}(s), \hat{\alpha}(t)) \leq e_0,$$

hence, $\hat{\alpha}$ is nonexpansive.

If we consider now two arbitrary elements $s, t \in \mathbb{T}M$ such that $d(s, t) = 0$, the previous argument also proves that

$$d^{\mathcal{A}}(\hat{\alpha}(s), \hat{\alpha}(t)) = 0.$$

Since $d^{\mathcal{A}}$ is a metric, not just a pseudo-metric, we get further that $\hat{\alpha}(s) = \hat{\alpha}(t)$.

$$\begin{tikzcd}
\mathbb{T}X \arrow[r, "\iota_\epsilon"] \arrow[dr, "\hat{\alpha}\circ\iota_\epsilon"] & \mathbb{T}M \arrow[r, "\tau"] \arrow[d, "\hat{\alpha}"] & \mathbb{T}[M] \arrow[dl, "h"] \\
& A &
\end{tikzcd}$$

Let τ be the map that sends an element in $\mathbb{T}M$ to its equivalence class in $\mathbb{T}[M]$. What we have shown is that elements in the same equivalence class in $\mathbb{T}M$ are sent by $\hat{\alpha}$ to the same element of A. Thus we can define $h : \mathbb{T}[M] \longrightarrow A$ by mapping $[t]$ to $\hat{\alpha}(t)$; i.e., by $h([t]^{\cong}) = \hat{\alpha}(t)$ for any $t \in \mathbb{T}M$. It is evident that $h \circ \tau = \hat{\alpha}$.

By definition of the interpretation of the operators $f : n \in \Omega$ in $\mathbb{T}[M]$, it immediately follows that h is a homomorphism of Ω-algebras. Moreover, by the definition of h and of τ, the map h is also the unique Ω-algebra homomorphism from $(\mathbb{T}[M], \Omega)$ to (A, Ω^A) satisfying $h \circ \tau = \hat{\alpha}$.

It remains to prove that h is non-expansive, i.e.,

$$d^{\mathcal{A}}(h(p^{\cong}), h(q^{\cong})) \leq d^{\cong}(p^{\cong}, q^{\cong}).$$

However, $d(p, q) = d^{\cong}(p^{\cong}, q^{\cong})$ and $h(p^{\cong}) = \hat{\alpha}(p)$, $h(q^{\cong}) = \hat{\alpha}(q)$. We have shown above that $d^{\mathcal{A}}(\hat{\alpha}(p), \hat{\alpha}(q)) \leq d(p, q)$ hence,

$$d^{\mathcal{A}}(h(p^{\cong}), h(q^{\cong})) = d^{\mathcal{A}}(\hat{\alpha}(p), \hat{\alpha}(q)) \leq d(p, q) = d^{\cong}(p^{\cong}, q^{\cong}).$$

This completes the proof. ∎

Since X and $\mathcal{U}$ are arbitrarily chosen, Theorem 6.9 justifies calling $\mathbb{T}[X]$ the *free* Ω*-quantitative algebra* generated over X in $\mathbb{K}(\Omega, \mathcal{U})$.

In standard presentation of universal algebra, the set of terms gives rise to a monad, the *term monad*. As one would expect, this is the case also for quantitative algebras, with the only difference that now terms are quotiented w.r.t. 0-provability in $\mathcal{U}$. (Note that $\mathcal{U}$ can be chosen arbitrarily).

Indeed the free-construction above gives rise to a functor

$$\mathbb{T}_{\mathcal{U}} \colon \mathbf{Set} \longrightarrow \mathbf{Set}$$

that maps objects $M \in \mathbf{Set}$ to the set $\mathbb{T}[M]$ of Ω-terms constructed over M and quotiented w.r.t. 0-provability in $\mathcal{U}$.

Moreover, $\mathbb{T}_{\mathcal{U}}$ is monadic, with unit and multiplication given, respectively, by the natural transformations $\eta \colon Id \Rightarrow \mathbb{T}_{\mathcal{U}}$ and $\mu \colon \mathbb{T}_{\mathcal{U}}\mathbb{T}_{\mathcal{U}} \Rightarrow \mathbb{T}_{\mathcal{U}}$, defined, for arbitrary $m \in M$, $t \in \mathbb{T}[M]$, $f : n \in \Omega$, and $C_1, .., C_n \in \mathbb{T}[\mathbb{T}[M]]$, as

$$\eta_M(m) = m^{\cong}\,, \qquad \begin{aligned} \mu_M(t) &= t\,, \\ \mu_M(f(C_1, .., C_n)^{\cong}) &= f\big(\mu_M(C_1), .., \mu_M(C_n)\big)^{\cong}\,. \end{aligned}$$

Note that this monad corresponds to the standard equational term monad, that can be constructed from the equational algebras.

In Section 7, we will show that quantitative equational theories are actually stronger then their non-quantitative counterparts, by allowing the construction of *metric term monads*.

7 Free Quantitative Algebras over Metric Spaces

Fix a set of variables X and a quantitative equational theory $\mathcal{U}$ of type Ω over X. In this section we focus on the equational class $\mathbb{K}(\Omega, \mathcal{U})$.

There is an obvious forgetful functor $U_{\mathbf{Met}} \colon \mathbb{K}(\Omega, \mathcal{U}) \longrightarrow \mathbf{Met}$ to the category of metric spaces (possibly taking infinite values) and non-expansive maps. Similarly to Theorem 6.9, we aim to show that any metric space (M, d) admits a free quantitative algebra $\mathbb{T}^d[M]$ generated over (M, d) in $\mathbb{K}(\Omega, \mathcal{U})$.

Define $\Omega_M = \Omega \cup \{m : 0 \mid m \in M\}$ as the extension of Ω with additional constant symbols taken from M (for this reason we assume that $\Omega \cap M = \emptyset$); and let $\mathcal{U}_M$ be the smallest quantitative equational theory of type Ω_M over X, containing $\mathcal{U}$ and such that, for all $m, n \in M$, $\epsilon \in \mathbb{Q}_+$,

$$\emptyset \vdash m =_\epsilon n \in \mathcal{U}_M, \qquad \text{whenever } d(m, n) \leq \epsilon\,.$$

The construction of $\mathcal{U}_M$ guarantees that any algebra in $\mathbb{K}(\Omega_M, \mathcal{U}_M)$ can be turned into an algebra in $\mathbb{K}(\Omega, \mathcal{U})$ simply by forgetting the interpretations of the constants in M. Conversely, given a non-expansive map $\alpha \colon M \longrightarrow A$, any algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathbb{K}(\Omega,\mathcal{U})$ can be turned into an algebra in $\mathbb{K}(\Omega_M,\mathcal{U}_M)$ just by interpreting each constant symbol $m : 0 \in M$ as $\alpha(m) \in A$.

Lemma 7.1 (Conversion Lemma) *The following two statements hold:*

1. *If $\mathcal{A} = (A, \Omega^{\mathcal{A}} \cup M^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathbb{K}(\Omega_M,\mathcal{U}_M)$, then $\mathcal{A}' = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathbb{K}(\Omega,\mathcal{U})$;*
2. *If $\alpha\colon M \longrightarrow A$ is a non-expansive map and $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathbb{K}(\Omega,\mathcal{U})$, then $\mathcal{A}' = (A, \Omega^{\mathcal{A}} \cup \{m^{\mathcal{A}'} \mid m \in M \text{ and } m^{\mathcal{A}'} = \alpha(m)\}, d^{\mathcal{A}}) \in \mathbb{K}(\Omega_M,\mathcal{U}_M)$.*

Proof. (1) We need to check that $\mathcal{A}' \models \mathcal{U}$. This follows immediately by $\mathbb{T}(X|\mathcal{A}') \subseteq \mathbb{T}(X|\mathcal{A})$ and $\mathcal{U} \subseteq \mathcal{U}_M$ from the hypothesis $\mathcal{A} \models \mathcal{U}_M$.

(2) We need to check that $\mathcal{A}' \models \mathcal{U}_M$. Since $\mathcal{A} \models \mathcal{U}$ and $\mathcal{U}_M$ is the smallest quantitative equational theory that contains $\mathcal{U}$ and satisfies the inferences of type $\emptyset \vdash m =_\epsilon n$, for all $m, n \in M$ with $d(m, n) \leq \epsilon \in \mathbb{Q}_+$, we only need to prove that $\mathcal{A}' \models m =_\epsilon n$ for all $m, n \in M$ with $d(m, n) \leq \epsilon \in \mathbb{Q}_+$. For any assignment ι and $m, n \in M$, we have that

$$\begin{aligned} d^{\mathcal{A}}(\iota(m), \iota(n)) &= d^{\mathcal{A}}(m^{\mathcal{A}}, n^{\mathcal{A}}) && (\iota \text{ homomorphism}) \\ &= d^{\mathcal{A}}(\alpha(m), \alpha(n)) && (\text{def. } m^{\mathcal{A}}) \\ &\leq d(m, n) \leq \epsilon\,. && (\alpha \text{ non-expansive}) \end{aligned}$$

Hence, $\mathcal{A}' \models m =_\epsilon n$. ∎

The definition of the conversion from an algebra in $\mathbb{K}(\Omega_M,\mathcal{U}_M)$ to an algebra in $\mathbb{K}(\Omega,\mathcal{U})$ in Lemma 7.1(1) is functorial, and it gives the (forgetful) functor

$$U\colon \mathbb{K}(\Omega_M,\mathcal{U}_M) \longrightarrow \mathbb{K}(\Omega,\mathcal{U})\,.$$

Consider $\mathbb{T}[\emptyset] \in \mathbb{K}(\Omega_M,\mathcal{U}_M)$, the free Ω_M-quantitative algebra generated over the empty set, given by the construction in Section 6, and define

$$\mathbb{T}^d[M] = U(\mathbb{T}[\emptyset]) \in \mathbb{K}(\Omega,\mathcal{U})$$

which is the quantitative Ω-algebra obtained from $\mathbb{T}[\emptyset]$ by forgetting the interpretations of the constants in M. Denote its quantitative algebraic structure by $\mathbb{T}^d[M] = (\mathbb{T}^d[M], \Omega, d^{\cong}_M)$.

The following theorem states that $\mathbb{T}^d[M]$ is the quantitative algebra in $\mathbb{K}(\Omega,\mathcal{U})$ freely generated from the metric space (M, d). Specifically, $\mathbb{T}^d[M]$ has the universal mapping property for $(M, d) \in \mathbf{Met}$ to the forgetful functor

$$U_{\mathbf{Met}}\colon \mathbb{K}(\Omega,\mathcal{U}) \longrightarrow \mathbf{Met}\,.$$

This situation is described by the commutative diagram below (*cf.* Definition 3.5):

$$\begin{array}{cc} \text{in } \mathbf{Met} & \text{in } \mathbb{K}(\Omega,\mathcal{U}) \\ \begin{tikzcd} (M,d) \arrow[r, "\eta_M"] \arrow[dr, "\alpha"'] & (\mathbb{T}^d[M], d^{\cong}_M) \arrow[d, "h"] \\ & (A, d^{\mathcal{A}}) \end{tikzcd} & \begin{tikzcd} \mathbb{T}^d[M] \arrow[d, dashed, "h"] \\ \mathcal{A} \end{tikzcd} \end{array}$$

where $\eta_M \colon M \longrightarrow \mathbb{T}^d[M]$ is the map given by $\eta_M(m) = m^{\cong}$. Observe that η_M is non-expansive since for arbitrary $m, n \in M$,

$$d^{\cong}_M(\eta(m), \eta(n)) = d^{\cong}_M(m^{\cong}, n^{\cong}) = \inf\{\epsilon \mid \mathcal{U}_M \vdash m =_\epsilon n\} \leq d(m,n)\,.$$

Theorem 7.2 $(\mathbb{T}^d[M], \eta_M)$ *is a universal arrow from* $(M,d) \in \mathbf{Met}$ *to* $U_{\mathbf{Met}}$.

Proof. Consider an arbitrary quantitative algebra $\mathcal{A} = (A, \Omega^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathbb{K}(\Omega,\mathcal{U})$ and a non-expansive map $\alpha \colon M \longrightarrow A$. Let $\mathcal{A}'$ be the algebra constructed from $\mathcal{A}$ by interpreting the constants $m \in M$ as $\alpha(m)$. By Lemma 7.1(2), $\mathcal{A}' \in \mathbb{K}(\Omega_M,\mathcal{U}_M)$.

Let $\mathbb{T}[\emptyset] \in \mathbb{K}(\Omega_M,\mathcal{U}_M)$ be the free algebra generated over $\emptyset$, given as in Section 6. Recall that $\emptyset$ is an initial object in **Set**. Therefore, by Theorem 6.9, $\mathbb{T}[\emptyset]$ is initial in $\mathbb{K}(\Omega_M,\mathcal{U}_M)$. Let $h \colon \mathbb{T}[\emptyset] \longrightarrow \mathcal{A}'$ be the *unique* Ω_M-homomorphism from the initial algebra $\mathbb{T}[\emptyset]$ in $\mathbb{K}(\Omega_M,\mathcal{U}_M)$. Since h is a homomorphism of Ω_M-algebras, for all $m \in M$, the following hold:

$$\begin{aligned} h(\eta_M(m)) &= h(m^{\cong}) && \text{(def. } \eta_M) \\ &= \alpha(m)\,, && (h \text{ homomorphism}) \end{aligned}$$

hence $h \circ \eta = \alpha$. Recall that $\mathbb{T}^d[M] = U(\mathbb{T}[\emptyset])$. Moreover $\mathcal{A} = U(\mathcal{A}')$ and $U(h) = h$. Hence $U(h)$ is the unique Ω-homomorphism from $\mathbb{T}^d[M]$ to $\mathcal{A}$ such that $U(h) \circ \eta_M = \alpha$. This concludes the proof. ∎

The free-construction described above gives rise to what we will call *the metric term monad*. Indeed, given an arbitrary quantitative equational theory $\mathcal{U}$, one can define the functor

$$\mathbb{T}_{\mathcal{U}} \colon \mathbf{Met} \to \mathbf{Met}$$

that map an object $(M,d) \in \mathbf{Met}$ (hence a metric space) to the metric space $(\mathbb{T}^d[M], d^{\cong}_M)$ of Ω-terms constructed over M and quotiented w.r.t. 0-provability in $\mathcal{U}_M$, with metric $d^{\cong}_M$ induced by the equational theory $\mathcal{U}_M$.

Moreover, $\mathbb{T}_{\mathcal{U}}$ is monadic, with unit and multiplication given, respectively, by the natural transformations $\eta \colon Id \Rightarrow \mathbb{T}_{\mathcal{U}}$ and $\mu \colon \mathbb{T}_{\mathcal{U}}\mathbb{T}_{\mathcal{U}} \Rightarrow \mathbb{T}_{\mathcal{U}}$, given, for arbitrary $m \in M$, $t \in \mathbb{T}^d[M]$, $f : n \in \Omega$, and $C_1, .., C_n \in \mathbb{T}^d[\mathbb{T}^d[M]]$, by

$$\eta_M(m) = m^{\cong}\,, \qquad \begin{aligned} \mu_M(t) &= t\,, \\ \mu_M(f(C_1,..,C_n)^{\cong}) &= f\big(\mu_M(C_1),..,\mu_M(C_n)\big)^{\cong}\,. \end{aligned}$$

Unlike the situation in Section 6, this monad lives in **Met** and the metrics associated with the set of terms are uniquely induced by the quantitative equational theories $\mathcal{U}$.

We conclude this section with a characterization of the consistency of $\mathcal{U}_M$ from a metric perspective.

We say that a metric space is *degenerate* if its support is empty or a singleton. Otherwise, they are non-degenerate. Notice that the support of a degenerate quantitative algebra is a degenerate metric space and that any quantitative algebra supported by a degenerate metric space is degenerate.

Theorem 7.3 *If (M, d) is a non-degenerate metric space, then $\mathcal{U}_M$ is consistent iff the map $\eta_M : (M, d) \longrightarrow (\mathbb{T}^d[M], d^{\cong}_M)$ is an isometry*[3]*.*

Proof. ($\Rightarrow$) Suppose that η_M is an isometry. Let $m, n \in M$ two distinct points.

$$d^{\cong}_M(\eta_M(m), \eta_M(n)) = \inf\{\epsilon \mid \mathcal{U}_M \vdash m =_\epsilon n\} = d(m, n) > 0\,.$$

Suppose that $\mathcal{U}_M$ is inconsistent. Then, there exist two variables $x, y \in X$ such that $\mathcal{U}_M \vdash x =_0 y$. Applying (Subst) for any substitution σ such that $\sigma(x) = m$, $\sigma(y) = n$, we get $\mathcal{U}_M \vdash m =_0 n$. Hence, $d^{\cong}_M(\eta_M(m), \eta_M(n)) = \inf\{\epsilon \mid \mathcal{U}_M \vdash m =_\epsilon n\} = 0$. But $0 < d(m, n)$, since m and n are distinct points in a metric space – contradiction.

($\Leftarrow$) Suppose that $\mathcal{U}_M$ is consistent and η_M is not an isometry. Then, there exist two distinct elements $m, n \in M$ such that

$$d^{\cong}_M(\eta_M(m), \eta_M(n)) = \inf\{\epsilon \mid \mathcal{U}_M \vdash m =_\epsilon n\} < d(m, n)\,.$$

However, this strict inequality can be true only if $\mathcal{U}_M$ proves an equation of type $m =_\delta n$ for some $\delta < d(m, n)$. Suppose this is the case, then since $m, n \in M$, we have that

$$d(m, n) \leq diam(M) = \sup\{\epsilon \mid d(u, v) \leq \epsilon, u, v \in M\}\,.$$

Because m and n are constants in Ω_M, the equation $m =_\delta n$ cannot be derived from the axioms induced by the metric space (M, d) only. On the other hand, because $M \cap \Omega = \emptyset$ and $\mathcal{U}$ is a quantitative equational theory of type Ω, the elements of M cannot be distinguished by the theory $\mathcal{U}$ alone. Hence, the proof of $\mathcal{U}_M \vdash m =_\delta n$ must be derived using (Subst) from a provable quantitative statement of type $\mathcal{U}_M \vdash x =_\delta y$ for some variables $x, y \in X$. Substituting these variables with all pairs of constants in M, we obtain that $diam(M) \leq \delta$ – contradiction. ∎

[3] By isometry in this context we mean a distance-preserving map, since η is obviously not a bijection.

Corollary 7.4 *If (M, d) is non-degenerate, then $\mathcal{U}_M$ is inconsistent iff $\mathcal{U}$ is inconsistent.*

Proof. ($\Leftarrow$) It follows by $\mathcal{U} \subseteq \mathcal{U}_M$.

($\Rightarrow$) Assume $\mathcal{U}_M$ to be inconsistent. By construction, $\mathcal{U}_M$ is the smallest quantitative equational theory that contains $\mathcal{U}$ and satisfies the inferences $\emptyset \vdash m =_\epsilon n$ whenever $d(m,n) \leq \epsilon$, for all $m, n \in M$. If $\mathcal{U}$ is consistent, then the algebra $\mathbb{T}^d[M]$ is non-degenerate and we know that $\mathbb{T}^d[M] \models \mathcal{U}_M$ – contradiction. Hence, $\mathcal{U}$ must be inconsistent. ∎

8 Free models over complete metric spaces

A basic result that we will show in this section is the following: if one takes a quantitative theory and forms its free algebra in the category of metric spaces (possibly with infinite values) and then takes its metric completion (suitably extending the operations) then that is the free algebra in the category of complete metric spaces (possibly with infinite values). This gives a general characterization of the monad on complete metric spaces; though, of course, for specific examples one can give much better characterizations. The corresponding result fails for dcpos.

Recall that we could have metrics that take values in the extended reals $[0, \infty]$. This means that notions like completion are a little different from the usual situation. In particular, we have to deal with the components of the (extended) metric space. Let **FMet** be the category of what are usually called metric spaces, "F" signifies that the the metric takes finite values in $[0, \infty)$, and non-expansive maps. We write **Met** for the category where the objects are metric spaces with the metric taking values in $[0, \infty]$; the maps are non-expansive. Clearly **FMet** is a full subcategory of **Met**.

In the category **FMet** one has the familiar notion of Cauchy completion where one adds points corresponding to (equivalence classes of) Cauchy sequences. Given a space X in **FMet** one gets a complete metric space $\overline{X}$ in **FMet**. A non-expansive function[4] f from X to a complete metric space Y can be extended to a function $\overline{f}$ from $\overline{X}$ to Y by the standard formula

$$\overline{f}(\lim x_i) = \lim f(x_i)$$

for any Cauchy sequence (x_i). Cauchy completion defines a functor $\mathbb{C}$ from **FMet** to **CFMet**, the category of complete metric spaces with finite metrics. This functor is left adjoint to the inclusion functor $\mathbb{I} : \mathbf{CFMet} \to \mathbf{FMet}$.

More precisely, there exists an adjunction $\mathbb{C} \dashv \mathbb{I}$ defined by the following natural transformations:

- $\eta : Id_{\mathbf{FMet}} \Longrightarrow \mathbb{IC}$ defined for a metric space $X \in \mathbf{FMet}$ by $\eta_X : X \to \overline{X}$.

[4] Recall, such functions are automatically continuous.

- $\epsilon : \mathbb{CI} \Longrightarrow Id_{\mathbf{CFMet}}$ defined for a complete metric space K by $\epsilon_K : \overline{K} \longrightarrow K$.

Stated another way, $(X, e : X \longrightarrow \mathbb{I}(\overline{X}))$ is a universal arrow from $X \in \mathbf{FMet}$ to $\mathbb{I}$. This means that for any morphism $f : X \longrightarrow \mathbb{I}(Y)$, where Y is an object of **CFMet**, there is a morphism $\overline{f} : \overline{X} \longrightarrow Y$ of **CFMet** such that

$$\mathbb{I}(\overline{f}) \circ e = f.$$

Thus, the Cauchy completion is the *universal* completion of X. This is illustrated in the diagram below where we have dropped explict mention of the inclusion functor $\mathbb{I}$.

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "f"] & \overline{X} \arrow[d, "\overline{f}"] \\ & Y \end{tikzcd}$$

We have to mimic this in the case of **Met** where the metrics are not finite.

Coproducts in **Met** are easy to describe[5]. Let $\{(X_i, d_i) | i \in I\}$ be an indexed family of objects. The underlying set of the coproduct $\coprod_{i \in I} X_i$ is the disjoint union $\biguplus_{i \in I} X_i$. We write (x, i) for a typical element of this set, where $x \in X_i$. The metric d on $\coprod_{i \in I} X_i$ is given by

$$d((x,i),(y,j)) = \begin{cases} \infty & \text{if } i \neq j, \\ d_i(x,y) & \text{if } i = j. \end{cases}$$

The verification that this is indeed the coproduct is straightforward.

Products in **Met** are obtained by taking the cartesian product of the underlying sets and using the supremum of the distances in the base spaces. More precisely, let $\{(X_i, d_i) | i \in I\}$ be a family in **Met**. The distance in $\prod_{i \in I} X_i$ is

$$d((x_i),(y_i)) = \sup_{i \in I} d_i(x_i, y_i)$$

where we have written (x_i) for an element of the product. The usual category of metric spaces only has finite products and this is what we need for our purposes.

Let (X, d) be an object in **Met**. We define an equivalence relation $\sim$ on X by $x \sim y$ if $d(x, y) < \infty$. We call the equivalence classes the *components*[6] of X; each component is an ordinary metric space. It is easy to see that X is (isomorphic to) the coproduct of its components.

[5] The category of ordinary metric spaces does not have coproducts.

[6] Do not confuse this with the connected components of the underlying topological space.

Suppose (X,d) has components $\{(X_i,d_i)|i\in I\}$. We claim that the Cauchy completion of X, written $\overline{X}$, is isomorphic to $\coprod\limits_{i\in I}\overline{X_i}$ where by $\overline{X_i}$ we mean the usual Cauchy completion.

$$\begin{tikzcd}
\overline{X_i} \arrow[r, "\iota_i"] \arrow[dr, "\alpha_i"'] & \coprod_{i\in}\overline{X_i} \arrow[d, "\alpha"] \\
& \overline{X}
\end{tikzcd}$$

The diagram shows that the coproduct of the completions of the components naturally embeds in $\overline{X}$. To see that this is an isomorphism note that any Cauchy sequence (x_n) in X will, apart from at most finitely many elements, be in one component. If we drop the elements that are not in the same component as the majority of the elements we still have a Cauchy sequence and it will be in some component, say X_i. Thus when we complete the space X, the new point added associated with the Cauchy sequence (x_n) can be identified with a point in the Cauchy completion of X_i corresponding to the Cauchy sequence obtained from (x_n) by dropping the finitely many points that are not in the component X_i.

It is easy to see that in fact $\coprod X_i$ is the universal completion in the same sense that we had for ordinary metric spaces. If we define **CMet** as the category of complete metric spaces then, as before, we have Cauchy completion defining a functor $\mathbb{C}$ from **Met** to **CMet** which is left adjoint to the functor $\mathbb{I}$ embedding **CMet** in **Met**. Thus, if we have a space (X,d) in **Met** with components $\{(X_i,d_i)\}$ then $\coprod\overline{X_i}$ is the universal completion of X. To see this, consider the diagram below.

$$\begin{tikzcd}
X_i \arrow[r, "\iota_i"] \arrow[d, "e_i"] \arrow[rr, bend left, "f_i"] & \coprod_i X_i \simeq X \arrow[r, "f"] \arrow[d, "e"] & Y \\
\overline{X_i} \arrow[r, "\kappa_i"] \arrow[rru, bend right, "\overline{f_i}"'] & \coprod_i \overline{X_i} \simeq \overline{X} \arrow[ru, "\overline{f}"'] &
\end{tikzcd}$$

The ι_i and κ_i are the canonical injections into the coproduct. The e_i are the embeddings of spaces into their completions and e is the embedding of X in $\overline{X}$. Let $f:X\longrightarrow Y$ be a morphism to the complete metric space Y. The maps $f_i:X_i\longrightarrow Y$ are given by $f_i=f\circ\iota_i$. Since the universal completions of X_i is the $\overline{X_i}$, we have maps $\overline{f_i}:\overline{X_i}\longrightarrow Y$ such that $f_i=\overline{f_i}\circ e_i$. Finally, by couniversality there is an induced morphism $\overline{f}$ from $\overline{X}$ to Y. It is clear that everything commutes as required and is unique. Thus, $\coprod\overline{X_i}$ is the universal completion of X.

In the usual category of metric spaces, Cauchy completion of a finite product of spaces is isomorphic to the product of the Cauchy completions of the individual spaces. Thus the finite product of the universal completions is the universal completion of the product. One can combine this with the above discussion of components to see that the completion of a finite product[7] of spaces in our category **Met** is the product of the completions of the individual spaces. Now we can extend n-ary functions to their completions and discuss the completions of quantitative algebras.

Definition 8.1 *Given a quantitative algebra $\mathcal{A} = (A, \Omega, d)$, its completion is the quantitative algebra $\overline{\mathcal{A}} = (\overline{A}, \Omega, \overline{d})$, where $(\overline{A}, \overline{d})$ denotes the completion of the metric space (A, d); and for arbitrary $f : n \in \Omega$, $a_1, .., a_n \in A$, $y_1, .., y_n \in \overline{A}$, $(x_i)_i \subseteq A$ a Cauchy sequence converging to x in $\overline{A}$, the following hold*

- $f^{\overline{\mathcal{A}}}(a_1, .., a_n) = f^{\mathcal{A}}(a_1, .., a_n)$;
- $\lim_i f^{\overline{\mathcal{A}}}(y_1, .., y_k, x_i, y_{k+2}, .., y_n) = f^{\overline{\mathcal{A}}}(y_1, .., y_k, x, y_{k+2}, .., y_n)$

Notice that the previous definition is indeed correct and $\overline{\mathcal{A}}$ is a quantitative algebra. What we need to verify for this is that $f^{\overline{\mathcal{A}}}$ are non-expansive w.r.t. $\overline{d}$.

Let $x^i, y^i \in \overline{A}$ for $i = 1, .., n$. We need to prove that that

$$\text{for each } i = 1, .., n, \quad \overline{d}(x^i, y^i) \geq \overline{d}(f^{\overline{\mathcal{A}}}(x^1, .., x^n), f^{\overline{\mathcal{A}}}(y^1, .., y^n)).$$

For each $i = 1, .., n$, let $(x^i_j)_j \subseteq A$ be a Cauchy sequence in A that converges to x^i in $\overline{A}$.

Because $f^{\mathcal{A}}$ is non-expansive in $\mathcal{A}$ we have that for arbitrary i, j,

$$d(x^i_j, y^i_j) \geq d(f^{\mathcal{A}}(x^1_j, ..., x^n_j), f^{\mathcal{A}}(y^1_j, .., y^n_j)),$$

and since x^i_j are all in $\mathcal{A}$, this is equivalent to

$$\overline{d}(x^i_j, y^i_j) \geq \overline{d}(f^{\overline{\mathcal{A}}}(x^1_j, ..., x^n_j), f^{\overline{\mathcal{A}}}(y^1_j, .., y^n_j)).$$

Now taking this inequality repeatedly to the limit, we will eventually get that

$$\text{for each } i = 1, .., n, \quad \overline{d}(x^i, y^i) \geq \overline{d}(f^{\overline{\mathcal{A}}}(x^1, .., x^n), f^{\overline{\mathcal{A}}}(y^1, .., y^n)).$$

If we consider a category **K** of quantitative algebras and the category **CK** of their completions, we can easily observe that, as for metric spaces, we can define two functors

- $\mathbb{C} : \mathbf{K} \longrightarrow \mathbf{CK}$ that maps a quantitative algebra to its completion and any morphism $f : \mathcal{A} \longrightarrow \mathcal{A}'$ to the morphism $\overline{f} : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}'$, where $\overline{X} \in \mathbf{CK}$ denotes the completion of $X \in \mathbf{K}$; and $\overline{f}$ is defined by $\overline{f}(x) = f(x)$ for $x \in A$ and such that for any Cauchy sequence $(x_i)_i$ in A we have $\overline{f}(\lim_i x_i) = \lim_i f(x_i)$.

[7] In fact this works for any products.

- $\mathbb{I} : \mathbf{CK} \to \mathbf{K}$ is the embedding of $\mathbf{CK}$ into $\mathbf{K}$.

$$\begin{tikzcd} \mathbf{K} \arrow[r, bend left, "\mathbb{C}"] & \mathbf{CK} \arrow[l, bend left, "\mathbb{I}"] \end{tikzcd}$$

As for the case of metric spaces, there exists an adjunction $\mathbb{C} \dashv \mathbb{I}$ defined by the following natural transformations:

- $\eta : Id_{\mathbf{K}} \Longrightarrow \mathbb{I}\mathbb{C}$ defined for an arbitrary quantitative algebra $\mathcal{A} \in \mathbf{K}$ by $\eta_{\mathcal{A}} : \mathcal{A} \to \overline{\mathcal{A}}$.
- $\epsilon : \mathbb{C}\mathbb{I} \Longrightarrow Id_{\mathbf{CK}}$ defined for an arbitrary complete quantitative algebra $K \in \mathbf{CK}$ by $\epsilon_K : \overline{K} \to K$ – notice that in this case $\overline{K} = K$.

Definition 8.2 (Continuous equation scheme) *Let Ω be an algebraic similarity type. A set*

$$\{\{x_1 =_{e_1} y_1, .., x_n =_{e_n} y_n\} \vdash s =_{f(e_1,..,e_n)} t \mid e_1, .., e_n \in \mathbb{R}_+\}$$

of basic quantitative inference over $\mathbb{T}X$ such that f is a continuous function in all variables is called a continuous equation scheme on $\mathbb{T}X$.

We say that an algebra satisfies a continuous equation scheme if it satisfies all the elements of the continuous equation scheme.

Proposition 8.3 *If a quantitative algebra $\mathcal{A}$ satisfies a continuous equation scheme, so does its completion $\overline{\mathcal{A}}$.*

Proof. Assume that $\mathcal{A}$ is a model for the continuous equation scheme

$$\{\{x^1 =_{e_1} y^1, .., x^n =_{e_n} y^n\} \vdash s =_{f(e_1,..,e_n)} t \mid e_1, .., e_n \in \mathbb{R}_+\}.$$

This means that for any $e_1, ..e_n \in \mathbb{R}_+$ and any assignment $\iota \in \mathbb{T}(X|\mathcal{A})$,

$$[d^{\mathcal{A}}(\iota(x^i), \iota(y^i)) \leq e_i \text{ for all } i = 1, .., n] \quad \text{implies} \quad d^{\mathcal{A}}(\iota(s), \iota(t)) \leq f(e_1, .., e_n).$$

Consider arbitrary $e_1, .., e_n \in \mathbb{R}_+$ and let $\iota' \in \mathbb{T}(X|\overline{\mathcal{A}})$ be such that

$$d^{\overline{\mathcal{A}}}(\iota'(x^i), \iota'(y^i)) \leq e_i \text{ for all } i = 1, .., n.$$

We need to prove that $d^{\overline{\mathcal{A}}}(\iota'(s), \iota'(t)) \leq f(e_1, .., e_n)$.

For each $i = 1, .., n$, let $(a^i_j)_j \subseteq A$ be a Cauchy sequence in A that converges to $\iota(x^i)$ in $\overline{A}$ and similarly $(b^i_j)_j \subseteq A$ be a Cauchy sequence in A that converges to $\iota(y^i)$ in $\overline{A}$ - in case some of these variables coincide, we chose the same Cauchy sequence.

For each j, let $\iota_j \in \mathbb{T}(X|\overline{\mathcal{A}})$ be such that $\iota_j(x^i) = a^i_j$, $\iota_j(y^i) = b^i_j$ and the interpretation of all variables are in $\mathcal{A}$ - we can always define such assignments.

This construction guarantees that for each $i = 1, .., n$ there exists a convergent sequence $(\alpha_j^i)_j \subseteq \mathbb{R}_+$ such that $\lim_j \alpha_j^i = 0$ and

$$d^{\overline{\mathcal{A}}}(\iota_j(x^i), \iota_j(y^i)) \leq e_i + \alpha_j^i.$$

Observe that, due to the way they were defined, we also have $\iota_j \in \mathbb{T}(X|\mathcal{A})$. Hence,

$$d^{\mathcal{A}}(\iota_j(x^i), \iota_j(y^i)) \leq e_i + \alpha_j^i.$$

Applying now the hypothesis, we get

$$d^{\mathcal{A}}(\iota_j(s), \iota_j(t)) \leq f(e_1 + \alpha_j^1, .., e_n + \alpha_j^n).$$

Hence,

$$d^{\overline{\mathcal{A}}}(\iota_j(s), \iota_j(t)) \leq f(e_1 + \alpha_j^1, .., e_n + \alpha_j^n)$$

and taking this to the limit after j and using the continuity of f in all variables, we obtain further

$$d^{\overline{\mathcal{A}}}(\iota'(s), \iota'(t)) \leq f(e_1, .., e_n).$$

∎

From the above, one concludes the following.

Theorem 8.4 *Consider a quantitative equational theory $\mathcal{U}$ axiomatized by continuous equation schemes and a metric space (M, d). The freely generated quantitative algebra $\mathbb{T}^{\overline{d}}[\overline{M}]$ over the completion $(\overline{M}, \overline{d})$ of (M, d) is isomorphic to the completion $\overline{\mathbb{T}^d[M]}$ of the quantitative algebra $\mathbb{T}^d[M]$.*

Proof. We already know that $\mathbb{T}^{\overline{d}}[\overline{M}]$ is the freely generated algebra from $(\overline{M}, \overline{d})$.

If we consider the adjunction $\mathbb{C} \dashv \mathbb{I}$ where $\mathbb{C} : \mathbf{K} \longrightarrow \mathbf{CK}$ and $\mathbb{I} : \mathbf{CK} \longrightarrow \mathbf{K}$, it is sufficient to prove that there exists a universal morphism from $\mathbb{T}^d[M]$ to $\mathbb{I}$ given by the tuple $(\overline{\mathbb{T}^d[M]}, \alpha)$ where $\alpha : \mathbb{T}^d[M] \longrightarrow \mathbb{I}\overline{\mathbb{T}^d[M]}$ is the embedding in $\mathbf{K}$. This is sufficient in the light of Theorem 7.2.

$$\begin{tikzcd}
 & \text{in } \mathbf{K} & \text{in } \mathbf{CK} \\
\mathbb{T}^d[M] \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathbb{I}\overline{\mathbb{T}^d[M]} \arrow[d, "\mathbb{I}h"] & \overline{\mathbb{T}^d[M]} \arrow[d, dashed, "h"] \\
 & \mathbb{I}\mathcal{A} & \mathcal{A}
\end{tikzcd}$$

This is not difficult to verify. Indeed, we can uniquely extend the morphism β of quantitative algebras to the morphism $\overline{\beta}$ between their completions, as defined at the beginning of this section. Now note that $\mathcal{A}$ coincides with its completion, hence we can take $h = \overline{\beta}$. ∎

It is pleasing that separability is also preserved as complete separable metric spaces play a major role in probability theory.

Corollary 8.5 *Consider a quantitative equational theory axiomatized by continuous equation schemes, over a signature with countably many operation symbols. Then the free model over a complete separable metric space M is separable, with countable set of generators being the least subalgebra containing any countable set of generators of M.*

9 Left-Invariant Barycentric Algebras

In this section we present a first example of quantitative algebra, the *left-invariant barycentric algebra*, and demonstrate that the freely generated one is, in this case, the algebra of probability distributions with finite support over the set of generators and the metric space is induced by the total-variation distance between distributions.

Consider the algebraic similarity type

$$\mathcal{B} = \{+_e : 2 \mid e \in [0,1]\}$$

containing, for each $e \in [0,1]$, a binary operator $+_e$. We call it the *barycentric signature.*

Definition 9.1 (Left-Invariant Barycentric Equational Theory) *This theory is given by the following axiom schemata, where $x, x', x'' \in X$ (X is the countable set of variables) and $e, e' \in [0,1]$:*

(B1) $\vdash x +_1 x' =_0 x$

(B2) $\vdash x +_e x =_0 x$

(SC) $\vdash x +_e x' =_0 x' +_{1-e} x$

(SA) $\vdash (x +_e x') +_{e'} x'' =_0 x +_{ee'} (x' +_{\frac{e'-ee'}{1-ee'}} x'')$ *provided that $e, e' \in (0,1)$*

(LI) $\vdash x' +_e x =_\epsilon x'' +_e x$ *where $e \leq \epsilon \in \mathbb{Q}_+$*

(SC) stands for *skew commutativity* and (SA) for *skew associativity.* We call (LI) the *left-invariance axiom schema.* Observe that if $e \in \mathbb{Q}$, (LI) takes the simpler form:

$$\vdash x' +_e x =_e x'' +_e x.$$

The algebras satisfying left-invariant barycentric equational theories are called *left-invariant barycentric algebras* or LIB algebras for short.

Hereafter we focus on the the class $\mathbb{K}(\mathcal{B},\mathcal{U}^{LI})$ defined by the left-invariant barycentric equational theory $\mathcal{U}^{LI}$.

Total-Variation Duality

Let (M, Σ) be a measurable space and $\Delta[M, \Sigma]$ the class of probability measures over (M, Σ). The *total variation distance* between probability measures is defined, for arbitrary $\mu, \nu \in \Delta[M, \Sigma]$ by

$$T(\mu, \nu) = \sup_{E \in \Sigma} |\mu(E) - \nu(E)| .$$

This is a metric that has a well known dual characterization, based on the notion of *coupling*.

For $\mu, \nu \in \Delta[M, \Sigma]$, a coupling for the pair (μ, ν) is a probability measure ω on the product space $(M \times M, \Sigma \otimes \Sigma)$, such that, for all $E \in \Sigma$

$$\omega(E \times M) = \mu(E) \qquad \text{and} \qquad \omega(M \times E) = \nu(E) .$$

We denote by $\mathcal{C}(\mu, \nu)$ the set of couplings for (μ, ν).

Let $\equiv_\Sigma = \bigcap\{E \times E \mid E \in \Sigma\}$, called the *inseparability relation of* Σ.

Lemma 9.2 (Total variation duality [Lin02, Th.5.2]) *Let μ, ν be probability measures on (X, Σ). Then, provided that $\equiv_\Sigma$ is measurable in $\Sigma \otimes \Sigma$,*

$$T(\mu, \nu) = \min\{\omega(\not\equiv_\Sigma) \mid \omega \in \mathcal{C}(\mu, \nu)\} .$$

Next we state two technical lemmas that will be useful in what follows.

Lemma 9.3 (Convex Combination of Couplings) *Let $\mu_i, \nu_i \in \Delta[M, \Sigma]$ and $\omega_i \in \mathcal{C}(\mu_i, \nu_i)$, for $i \in \{1, 2\}$. Then, for all $e \in [0, 1]$*

$$(e\omega_1 + (1-e)\omega_2) \in \mathcal{C}\Big((e\mu_1 + (1-e)\mu_2), (e\nu_1 + (1-e)\nu_2)\Big) .$$

Proof. We show only the left marginal, the other is similar. Let $E \in \Sigma$, then

$$\begin{aligned}
(e\omega_1 + (1-e)\omega_2)(E \times M) &= e\omega_1(E \times M) + (1-e)\omega_2(E \times M) && \text{(by def.)} \\
&= e\mu_1(E) + (1-e)\mu_2(E) && \text{(by } \omega_i \in \mathcal{C}(\mu_i, \nu_i)\text{)} \\
&= (e\mu_1 + (1-e)\mu_2)(E) . && \text{(by def.)}
\end{aligned}$$

∎

The result above states that the set of all couplings between arbitrary measures in $\Delta[M, \Sigma]$ is a convex set (note that $\Delta[M, \Sigma]$ is a convex set too).

Lemma 9.4 (Splitting Lemma) *Let $\mu, \nu \in \Delta[M, \Sigma]$ and $e = T(\mu, \nu)$. Then, there exist $\mu', \nu', \rho \in \Delta[M, \Sigma]$ such that*

$$\mu = e\mu' + (1-e)\rho \qquad \textit{and} \qquad \nu = e\nu' + (1-e)\rho .$$

Proof. If $e \in 1$, choose $\mu' = \mu$ and $\nu' = \nu$. If $e = 0$ (hence $\mu = \nu$) choose $\rho = \mu = \nu$. Otherwise, let $\omega \in \mathcal{C}(\mu,\nu)$ minimal in Lemma 9.2 for $T(\mu,\nu)$. By hypothesis $T(\mu,\nu) = \omega(\not\equiv_\Sigma) \neq \{0,1\}$. Define $\mu_i, \nu_i \in \Delta[M,\Sigma]$, for arbitrary $E \in M$, as follows:

$$\mu_1(E) = \frac{\omega((E \times M) \cap \not\equiv_\Sigma)}{\omega(\not\equiv_\Sigma)}\,, \qquad \mu_2(E) = \frac{\omega((E \times M) \cap \equiv_\Sigma)}{1 - \omega(\not\equiv_\Sigma)}\,,$$
$$\nu_1(E) = \frac{\omega((M \times E) \cap \not\equiv_\Sigma)}{\omega(\not\equiv_\Sigma)}\,, \qquad \nu_2(E) = \frac{\omega((M \times E) \cap \equiv_\Sigma)}{1 - \omega(\not\equiv_\Sigma)}\,.$$

We show $\mu = \omega(\not\equiv_\Sigma)\mu_1 + (1 - \omega(\not\equiv_\Sigma))\mu_2$. Let $E \in \Sigma$, then

$$\begin{aligned}
&\omega(\not\equiv_\Sigma)\mu_1(E) + (1 - \omega(\not\equiv_\Sigma))\mu_2 = \\
&\qquad = \omega((E \times M) \cap \not\equiv_\Sigma) + \omega((E \times M) \cap \equiv_\Sigma) && \text{(def. } \mu_1, \mu_2) \\
&\qquad = \omega(E \times M) && \text{(additivity of } \omega) \\
&\qquad = \mu(E)\,. && (\omega \in \mathcal{C}(\mu,\nu))
\end{aligned}$$

Similarly, $\nu = \omega(\not\equiv_\Sigma)\nu_1 + (1 - \omega(\not\equiv_\Sigma))\nu_2$.

By definition of $\equiv_\Sigma$, for all $E \in \Sigma$, $(E \times M) \cap \equiv_\Sigma = (M \times E) \cap \equiv_\Sigma$. Thus it follows that $\mu_2 = \nu_2$. This concludes the proof. ∎

9.1 The Freely-Generated Algebra

Fix a set M. Let $\mathbb{T}[M]$ be the variation barycentric algebra in $\mathbb{K}(\mathcal{B},\mathcal{U}^{LI})$ freely generated from M, as constructed in Section 6. By Theorem 6.9, $\mathbb{T}[M]$ has the universal mapping property for M to

$$U_{\mathbf{Set}} \colon \mathbb{K}(\mathcal{B},\mathcal{U}^{LI}) \to \mathbf{Set}.$$

Denote by $\Pi[M]$ the set of *finitely-supported* discrete probability distributions on M (i.e., defined on the discrete σ-algebra 2^M). Next we will show that $\Pi[M]$ endowed with the total-variation distance can be organized as a LIB algebra in $\mathbb{K}(\mathcal{B},\mathcal{U}^{LI})$ having the universal mapping property for M to $U_{\mathbf{Set}}$. By uniqueness (up to isomorphism) of universal arrows of the same type, this proves that $\Pi[M]$ and $\mathbb{T}[M]$ are isomorphic.

We can regard $\Pi[M]$ as an algebra of type $\mathcal{B}$ by interpreting each operator $+_e : 2 \in \mathcal{B}$, for arbitrary $\mu, \nu \in \Pi[M]$, as follows

$$\mu +_e \nu = e\mu + (1 - e)\nu\,,$$

We can further regard $\Pi[M]$ as a quantitative algebra by taking the total-variation distance as a metric on $\Pi[M]$.

Theorem 9.5 $\Pi[M] = (\Pi[M], \mathcal{B}, T) \in \mathbb{K}(\mathcal{B},\mathcal{U}^{LI})$, *i.e.*, $\Pi[M] \models \mathcal{U}^{LI}$.

Proof. Note that each assignment $\iota\colon X \to \Pi[M]$ maps variables to distributions in $\Pi[M]$, hence we only need to check the axioms on $\mathcal{B}$-terms constructed over $\Pi[M]$. (Refl), (Symm), (Triang), (Max), and (Arch) follow from the fact that T is a metric. As for (NExp) we need to prove that, for arbitrary $e \in [0,1]$ and $\mu, \mu', \nu, \nu' \in \Pi[M]$,

1. $T(\mu +_e \nu, \mu +_e \nu') \le T(\nu, \nu')$;
2. $T(\mu +_e \nu, \mu' +_e \nu') \le T(\mu, \mu')$.

We show only (1); the other follows similarly.

Let $\omega \in \mathcal{C}(\nu, \nu')$ and $\rho \in \mathcal{C}(\mu, \mu)$ be two couplings that attain the minimum in Lemma 9.2 for $T(\nu, \nu')$ and $T(\mu, \mu)$, respectively. (Note that the inseparability relation is simply the equality relation in M, obviously measurable in a discrete σ-algebra). Then

$$\begin{aligned}
T(\nu, \nu') &= T(\mu, \mu) + T(\nu, \nu') && (T \text{ metric})\\
&\ge eT(\mu, \mu) + (1-e)T(\nu, \nu') && (e \in [0,1])\\
&= e\rho(\neq) + (1-e)\omega(\neq) && (\text{hp. on } \rho, \omega)\\
&\ge T(e\mu + (1-e)\nu, e\mu + (1-e)\nu') && (\text{Lemmas 9.3 \& 9.2})\\
&= T(\mu +_e \nu, \mu +_e \nu')\,. && (\text{def. } +_e)
\end{aligned}$$

(B1), (B2), (SA), and (SC) follow by the definition of the interpretation for $+_e : 2 \in \mathcal{B}$ and the fact that T is a metric. As for (LI), assume $\omega \in \mathcal{C}(\mu, \nu)$ and $\omega \in \mathcal{C}(\rho, \rho)$ be two couplings that attains the minimum in Lemma 9.2 for $T(\mu, \nu)$ and $T(\rho, \rho)$, respectively. Then, for $e \le \epsilon \in \mathbb{Q}_+$

$$\begin{aligned}
\epsilon &\ge e\,T(\mu, \nu) && (e \le \epsilon \;\&\; T \text{ 1-bounded})\\
&= e\,T(\mu, \nu) + (1-e)T(\rho, \rho) && (T \text{ metric})\\
&= e\omega(\neq) + (1-e)\omega'(\neq) && (\text{hp. on } \omega, \omega')\\
&\ge T(e\mu + (1-e)\rho, e\nu + (1-e)\rho) && (\text{Lemmas 9.3 \& 9.2})\\
&= T(\mu +_e \rho, \nu +_e \rho)\,. && (\text{def. } +_e)
\end{aligned}$$

This concludes the proof. ∎

The next theorem shows that $\Pi[M]$ has the universal mapping property for M to $U_{\mathbf{Set}}$, with universal arrow $(\Pi[M], \delta_M)$, where $\delta_M\colon M \to \Pi[M]$ maps $m \in M$ to $\delta_m \in \Pi[M]$ —the Dirac measure with probability mass concentrated at $m \in M$.

Theorem 9.6 $(\Pi[M], \delta_M)$ *is an universal arrow from* $M \in \mathbf{Set}$ *to* $U_{\mathbf{Set}}$.

Proof. Let $\mathcal{A} = (A, \mathcal{B}^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathbb{K}(\mathcal{B}, \mathcal{U}^{LI})$ and $\alpha\colon M \to A$ a set-map.

Theorem 2 in [Sto49] proves that any barycentric algebra $(A, \mathcal{B}^{\mathcal{A}})$ has a one-to-one embedding into a convex subset of a suitable vector space. By this result, if $a_1, \ldots, a_n \in A$, then also $\sum_{i=1}^{n} e_i a_i \in A$, provided that $e_i \in [0,1]$ and $\sum_{i=1}^{n} e_i = 1$.

For $\Pi[M]$, we additionally note that $\Pi[M]$ is a simplex, *i.e.* any $\mu \in \Pi[M]$, can be canonically represented as a *finite* convex combination of the form $\mu = \sum_{i=1}^{k} c_i \delta_{m_i}$, where $supp(\mu) = \{m_1, \ldots, m_k\}$ and $c_i \in (0,1]$ are such that $\sum_{i=1}^{k} c_i = 1$.

By using these facts we define the map $h \colon \Pi[M] \to A$ as follows:

$$h(\sum_{i=1}^{k} c_i \delta_{m_i}) = \sum_{i=1}^{k} c_i \alpha(m_i)\,.$$

Clearly $h \circ \delta_M = \alpha$: for any $m \in M$, $h(\delta_M(m)) = h(\delta_m) = \alpha(m)$. Now we show that h is a homomorphism. Let $\mu = \sum_{i=1}^{k} c_i \delta_{m_i}, \nu = \sum_{j=1}^{n} d_j \delta_{n_j}$ and $e \in [0,1]$. Then the following holds:

$$\begin{aligned}
h(\mu +_e \nu) &= h(e\mu + (1-e)\nu) && \text{(def. } +_e\text{)}\\
&= h(e \sum_{i=1}^{k} c_i \delta_{m_i} + (1-e) \sum_{j=1}^{n} d_j \delta_{n_j}) && \text{(canonical rep.)}\\
&\overset{(*)}{=} e \sum_{i=1}^{k} c_i \alpha(m_i) + (1-e) \sum_{j=1}^{n} d_j \alpha(n_j) && \text{(def. } h \text{ \& } \Pi[M], \mathcal{A} \models \text{(B2))}\\
&= h(\mu) +^{\mathcal{A}}_e h(\nu)\,. && \text{(def. } h \text{ \& def. } +^{\mathcal{A}}_e\text{)}
\end{aligned}$$

Note that in $(*)$, the formal definition of h requires that the measure is canonically represented without repetitions of Dirac measures δ_m. The repetitions can be removed by applying (B2) in $\Pi[M]$ and, once h is applied, they can be recovered by applying $(B2)$ in $\mathcal{A}$ in the reverse direction.

As for the uniqueness, assume that there exists another homomorphism $h' \neq h$ such $h' \circ \delta_M = \alpha$. Let $\mu = \sum_{i=1}^{k} c_i \delta_{m_i}$ be a measure in $\Pi[M]$ with minimal support such that $h'(\mu) \neq h(\mu)$. If $k = 1$, we get a contradiction by $h' \circ \delta_M = \alpha = h \circ \delta_M$. Assume $k > 1$, then we get the following contradiction

$$\begin{aligned}
h'(\sum_{i=1}^{k} c_i \delta_{m_i}) &= h'(\delta_{m_1}) +^{\mathcal{A}}_{c_1} h'(\sum_{i=2}^{k} \frac{c_i}{1-c_1} \delta_{m_i}) && (h' \text{ homo.})\\
&= h(\delta_{m_1}) +^{\mathcal{A}}_{c_1} h(\sum_{i=2}^{k} \frac{c_i}{1-c_1} \delta_{m_i}) && \text{(minimal supp.)}\\
&= h(\sum_{i=1}^{k} c_i \delta_{m_i}) && (h' \text{ homo.})
\end{aligned}$$

It remains to show that h is non-expansive, i.e., that for all $\mu, \nu \in \Pi[M]$, $T(\mu, \nu) \geq d^{\mathcal{A}}(h(\mu), h(\nu))$. Let $\mu, \nu \in \Pi[M]$. By the splitting lemma (Lemma 9.4), there exist $\mu', \nu', \rho \in \Pi[M]$ such that

$$\mu = \mu' +_e \rho \qquad \text{and} \qquad \nu = \nu' +_e \rho\,.$$

where $e = T(\mu, \nu)$, then the following holds

$$
\begin{aligned}
d^{\mathcal{A}}(h(\mu), h(\nu)) &= d^{\mathcal{A}}(h(\mu' +_e \rho), h(\nu' +_e \rho)) && \text{(splitting lemma)} \\
&= d^{\mathcal{A}}(h(\mu') +^{\mathcal{A}}_e h(\rho), h(\nu') +^{\mathcal{A}}_e h(\rho)) && (h \text{ homo.}) \\
&\leq T(\mu, \nu) && (\mathcal{A} \models \text{B3})
\end{aligned}
$$

This concludes the proof. ∎

The next result follows directly by Theorem 6.9 and 9.6.

Corollary 9.7 *The left-invariant barycentric algebras* $\Pi[M]$ *and* $\mathbb{T}[M]$ *are isomorphic with bijective isometry* $h\colon \mathbb{T}^d[M] \longrightarrow \Pi[M]$ *given, for* $m \in M$ *and* $t, s \in \mathbb{T}M$ *by*

$$h(m^{\cong}) = \delta_m\,, \qquad\qquad h((t +_e s)^{\cong}) = eh(t^{\cong}) + (1-e)h(s^{\cong})\,.$$

Consequently, the metric induced by the quantitative equational theory $\mathcal{U}^{LI}$ coincides with the total variation distance on $\Pi[M]$. Thus we say that $\mathcal{U}^{LI}$ axiomatizes the total variation distance.

10 Quantitative Semilattices with Zero

In this section we provide a first example of free quantitative algebra over metric spaces. We discuss the case of the quantitative semilattices and show how their axiomatization induces Hausdorff distances both in the finitary and in the continuous case.

Consider the algebraic similarity type of (bounded join-) semilattices

$$\mathcal{S} = \{+ : 2, 0 : 0\}$$

containing one binary operator $+$ and one constant 0. We shall call it the *semilattice signature.*

Definition 10.1 (Quantitative Semilattice Equational Theory) *This theory is given by the following axiom schemata where* $x, x', x'', y, y' \in X$ *(X is the countable set of variables) and* $\epsilon, \epsilon' \in [0, 1]$*:*

(S0) $\vdash x + 0 =_0 x$

(S1) $\vdash x + x =_0 x$

(S2) $\vdash x + x' =_0 x' + x$

(S3) $\vdash (x + x') + x'' =_0 x + (x' + x'')$

(S4) $\{x =_\epsilon y, x' =_{\epsilon'} y'\} \vdash x + x' =_\delta y + y'$*, where* $\delta = \max\{\epsilon, \epsilon'\}$*.*

In this section we focus on the algebras satisfying quantitative semilattice equational theories; we call these *quantitative semilattices with zero.*

Fix a set of variables X and a quantitative semilattice equational theory $\mathcal{U}^{\mathcal{S}}$ of type $\mathcal{S}$ over X. In this section we focus on the class $\mathbb{K}_{\mathcal{S}} = \mathbb{K}(\mathcal{S}, \mathcal{U}^{\mathcal{S}})$.

Hausdorff Duality

Let (M, d) be a metric space (possibly taking infinite values). The *Hausdorff metric induced by* d on the set of all closed subsets of M (in the ope-ball topology induced by d), is defined, for arbitrary closed sets $A, B \subseteq M$ as

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\} ,$$

where, $d(m, N) = \inf_{n \in N} d(m, n)$ denotes the distance from an element $m \in M$ to a set $N \subseteq M$.

As usual, we assume that $\sup \emptyset = 0$ and $\inf \emptyset = \infty$; so, for any closed nonempty set A,

$$H_d(\emptyset, A) = H_d(A, \emptyset) = \infty \quad \text{and} \quad H_d(\emptyset, \emptyset) = 0.$$

Our definition is somehow non-standard and generalizes, for the case of metric spaces with infinite values, the standard definition which is usually given either for compact sets or for closed and bounded sets. For our purposes neither the compactness nor the boundness are needed, as we will prove hereafter.

Before proving that H_d is indeed a metric, we provide a dual characterization for H_d. For an arbitrary set $A \subseteq M$ and arbitrary $\epsilon > 0$, let

$$A_\epsilon = \{x \in M \mid \exists a \in A, d(x, a) \leq \epsilon\}.$$

Lemma 10.2 *If A and B are closed subsets of M, then*

$$H_d(A, B) = \inf\{\epsilon \mid A \subseteq B_\epsilon \textit{ and } B \subseteq A_\epsilon\}.$$

Proof. Let's observe for the beginning that if $x \in A_\epsilon$, then $d(x, A) \leq \epsilon$.

Suppose that A and B are such that $A \subseteq B_\epsilon$ and $B \subseteq A_\epsilon$.

Since $A \subseteq B_\epsilon$, for any $a \in A$, $d(a, B) \leq \epsilon$. Hence, $\sup_{a \in A} d(a, B) \leq \epsilon$. Similarly, from $B \subseteq A_\epsilon$ we get $\sup_{b \in B} d(b, A) \leq \epsilon$. From these we derive firstly that $H_d(A, B) \leq \epsilon$ and further,

$$H_d(A, B) \leq \inf\{\epsilon \mid A \subseteq B_\epsilon \text{ and } A \subseteq B_\epsilon\}.$$

Suppose that $H_d(A, B) < \inf\{\epsilon \mid A \subseteq B_\epsilon \text{ and } A \subseteq B_\epsilon\}$. Let δ be such that $H_d(A, B) < \delta < \inf\{\epsilon \mid A \subseteq B_\epsilon \text{ and } A \subseteq B_\epsilon\}$. Then, either $A \not\subseteq B_\delta$ or $B \not\subseteq A_\delta$.

If $A \not\subseteq B_\delta$, then there exists $a \in A$ s.t. $a \notin B_\delta$. Hence, for any $b \in B$, $d(a,b) > \delta$, implying further that there exists $a \in A$ such that $d(a,B) \geq \delta$. But then,

$$\sup_{a \in A} d(a,B) \geq \delta,$$

which guarantees that

$$H_d(A,B) \geq \delta$$

contradicting the assumption that $H_d(A,B) < \delta$.

Hence, $H_d(A,B) = \inf\{\epsilon \mid A \subseteq B_\epsilon \text{ and } A \subseteq B_\epsilon\}$. ∎

We show now that H_d is indeed a metric, possibly taking infinite values, over the set of closed sets of M.

To achieve this, we firstly prove a lemma.

Lemma 10.3 *Let A, B be nonempty closed sets in the open-ball topology and $a \in A$ an arbitrary point. Then, for any $\epsilon > 0$, there exists $b \in B$ such that*

$$d(a,b) \leq H_d(A,B) + \epsilon.$$

Proof. We can assume, without loosing generality, that

$$H_d(A,B) = \sup_{x \in A} d(x,B).$$

Hence,

$$H_d(A,B) \geq d(a,B) = \inf_{y \in B} d(a,y).$$

This implies $H_d(A,B) + \epsilon > \inf_{y \in B} d(a,y)$. Then, there exists $b \in B$ such that $d(a,b) \leq H_d(A,B) + \epsilon$. ∎

Theorem 10.4 *H_d is a metric on the set of closed subsets of M, possibly taking infinite values.*

Proof. 1. Assume that $H_d(A,B) = 0$. We prove that $A = B$.
If at least one of the two sets is empty, the other one must be empty too, since otherwise $H_d(A,B) = \infty$.
Assume that $A \neq \emptyset \neq B$. Because $H_d(A,B) = 0$, for any $a \in A$, $d(a,B) = 0$, i.e., $\inf_{b \in B} d(a,b) = 0$. Hence, there exists a sequence (b_i) of elements in B such that

$$\lim_{i \longrightarrow \infty} d(a,b_i) = 0.$$

But then, (b_i) converges to a and since B is closed, $a \in B$. Hence, $A \subseteq B$.
Similarly one can prove that $B \subseteq A$, hence $A = B$.

2. That $H_d(A,B) = H_d(B,A)$ derives from the symmetry of max.

3. We prove now that for arbitrary closed sets A, B, C,

$$H_d(A,C) \leq H_d(A,B) + H_d(B,C).$$

Observe that if at least one of them is empty, the inequality is trivially true since $r + \infty = \infty + \infty = \infty > r$ for any $r \in \mathbb{R}_+$.

Assume they are not empty. Let $a \in A$. Fora any $\epsilon > 0$, we can apply Lemma 10.3 and get that there exists $b \in B$ such that

$$d(a, b) \leq H_d(A, B) + \epsilon.$$

We apply again Lemma 10.3 for $b \in B$ and C and we obtain that there exists $c \in C$ such that

$$d(b, c) \leq H_d(B, C) + \epsilon.$$

Consequently, after applying the triangle inequality for d, we get

$$d(a, c) \leq d(a, b) + d(b, c) \leq H_d(A, B) + H_d(B, C) + 2\epsilon.$$

It follows that

$$d(a, C) \leq H_d(A, B) + H_d(B, C) + 2\epsilon$$

and further that

$$\sup_{a \in A} d(a, C) \leq H_d(A, B) + H_d(B, C) + 2\epsilon.$$

Symmetrically (using also the symmetry of H_d),

$$\sup_{c \in C} d(c, A) \leq H_d(A, B) + H_d(B, C) + 2\epsilon.$$

Hence for any $\epsilon > 0$,

$$H_d(A, C) \leq H_d(A, B) + H_d(B, C) + 2\epsilon$$

which further entails

$$H_d(A, C) \leq H_d(A, B) + H_d(B, C).$$

∎

Next we prove a dual characterization for H_d in terms of what we call *relational coupling*. For M a set and $A, B \subseteq M$, a relational coupling for the pair (A, B) is a relation $R \subseteq M \times M$ such that

$$\pi_1(R) = A \qquad \text{and} \qquad \pi_2(R) = B\,,$$

where π_1 and π_2 are the canonical projections of R. We denote by $\mathcal{C}(A, B)$ the set of couplings for (A, B).

Theorem 10.5 (Hausdorff Duality) *Let (M, d) be a metric space possibly taking infinite values and $A, B \subseteq M$ closed sets in the open-ball topology, then*

$$H_d(A, B) = \inf \left\{ \sup_{(m,n) \in R} d(m, n) \mid R \in \mathcal{C}(A, B) \right\},$$

Proof. If either A or B is empty while the other one is not empty, the equality is trivially true since $H_d(A,B)=\infty$ and $\mathcal{C}(A,B)=\emptyset$ constricting

$$\left\{\sup_{(m,n)\in R} d(m,n) \mid R\in\mathcal{C}(A,B)\right\}=\emptyset.$$

On the other hand, if $A=B=\emptyset$, then $H_d(A,B)=0$ and moreover

$$\left\{\sup_{(m,n)\in R} d(m,n) \mid R\in\mathcal{C}(A,B)\right\}=\{0\}.$$

Assume now that $A\neq\emptyset\neq B$.

($\leq$): It suffices to show that for any relational coupling $R\in\mathcal{C}(A,B)$,

$$H_d(A,B)\leq\sup_{(m,n)\in R} d(m,n).$$

Let $R\in\mathcal{C}(A,B)$. Note that, for any $a\in A$ and $b\in B$, since $\pi_1(R)=A$ and $\pi_2(R)=B$, the following sets are nonempty:

$$R_a=\{n\in M \mid (a,n)\in R\}\subseteq\pi_2(R)\,,\quad R^b=\{m\in M\mid (m,b)\in R\}\subseteq\pi_1(R)\,.$$

Then, the following holds

$$\begin{aligned}
\sup_{(m,n)\in R} d(m,n) &\geq \max\left\{\sup_{a\in A} d(a,R_a),\sup_{b\in B} d(b,R^b)\right\} && (\text{def. } R_a,\ R^b)\\
&\geq \max\left\{\sup_{a\in A} d(a,B),\sup_{b\in B} d(b,A)\right\} && (R\in\mathcal{C}(A,B))\\
&= H_d(A,B)\,. && (\text{def. } H_d)
\end{aligned}$$

($\geq$): Recall the equivalent characterization of H_d, given as

$$H_d(A,B)=\inf\{\epsilon\geq 0\mid A\subseteq B_\epsilon \text{ and } B\subseteq A_\epsilon\}\,.$$

If $\{\epsilon\geq 0\mid A\subseteq B_\epsilon \text{ and } B\subseteq A_\epsilon\}=\emptyset$, since we have assumed that $A\neq\emptyset\neq B$, then $H_d(A,B)=\infty$ and the inequality is trivially satisfied.

Suppose now that $\{\epsilon\geq 0\mid A\subseteq B_\epsilon \text{ and } B\subseteq A_\epsilon\}\neq\emptyset$. To prove the inequality it is sufficient to show that for any $\epsilon\geq 0$ such that $A\subseteq B_\epsilon$ and $B\subseteq A_\epsilon$, the following inequality holds:

$$\epsilon\geq\inf\left\{\sup_{(m,n)\in R} d(m,n)\mid R\in\mathcal{C}(A,B)\right\}.$$

Let $\epsilon\geq 0$ be such that $A\subseteq B_\epsilon$ and $B\subseteq A_\epsilon$. We define $R_\epsilon\subseteq M\times M$ as $R_\epsilon=\{(a,b)\in A\times B\mid d(a,b)\leq\epsilon\}$ and we show that $\mathcal{R}_\epsilon\in\mathcal{C}(A,B)$.

$$\begin{aligned}
\pi_1(R_\epsilon) &= \pi_1(\{(a,b)\in A\times B\mid d(a,b)\leq\epsilon\}) && (\text{def. } R_\epsilon)\\
&= \{a\in A\mid b\in B\,, d(a,b)\leq\epsilon\} && (\text{def. } \pi_1)\\
&= B_\epsilon\cap A && (\text{def. } B_\epsilon)\\
&= A && (A\subseteq B_\epsilon)
\end{aligned}$$

Similarly, $\pi_2(R_\epsilon) = B$. The following concludes the proof:

$$
\begin{aligned}
\epsilon &\geq \sup_{(m,n)\in R_\epsilon} d(m,n) && \text{(by def. } R_\epsilon\text{)}\\
&\geq \inf\left\{ \sup_{(m,n)\in R} d(m,n) \mid R \in \mathcal{C}(A,B) \right\}. && (R_\epsilon \in \mathcal{C}(A,B))
\end{aligned}
$$

∎

10.1 The Finitary Case

Fix a metric space (M, d). Let $\mathbb{T}^d[M]$ be the quantitative semilattice in $\mathbb{K}(\mathcal{S},\mathcal{U}^{\mathcal{S}})$ freely generated from (M, d), constructed in Section 7. By Theorem 7.2, $\mathbb{T}^d[M]$ has the universal mapping property for (M, d) to $U_{\mathbf{Met}} \colon \mathbb{K}(\mathcal{S},\mathcal{U}^{\mathcal{S}}) \longrightarrow \mathbf{Met}$.

Theorem 10.6 *If (M, d) is a non-degenerate metric space then $\mathbb{T}^d[M]$ is a non-degenerate quantitative semilattice. In particular, $\mathcal{U}^{\mathcal{S}}$ and $\mathcal{U}^{\mathcal{S}}_M$ are consistent quantitative theories.*

Denote by $\mathbb{F}[M]$ the set of all *finite* subsets of M. In what follows we show that $\mathbb{F}[M]$ can be organized as a quantitative semilattice in $\mathbb{K}(\mathcal{S},\mathcal{U}^{\mathcal{S}})$ that has the universal mapping property for (M, d) to $U_{\mathbf{Met}} \colon \mathbb{K}(\mathcal{S},\mathcal{U}^{\mathcal{S}}) \longrightarrow \mathbf{Met}$. This will prove that $\mathbb{F}[M]$ and $\mathbb{T}^d[M]$ are isomorphic $\mathcal{S}$-quantitative algebras.

We organize $\mathbb{F}[M]$ as an algebra of type $\mathcal{S}$ by defining, for arbitrary $A, B \in \mathbb{F}[M]$, $A + B = A \cup B$, $0 = \emptyset$. We can further organize $\mathbb{F}[M]$ as a quantitative algebra by taking the Hausdorff metric H_d induced by d.

Theorem 10.7 $\mathbb{F}[M] = (\mathbb{F}[M], \mathcal{S}, H_d) \in \mathbb{K}(\mathcal{S},\mathcal{U}^{\mathcal{S}})$, *i.e.*, $\mathbb{F}[M] \models \mathcal{U}^{\mathcal{S}}$.

Proof. Note that each assignment $\iota \in \mathbb{T}(X|\mathbb{F}[M])$ maps variables in X to sets in $\mathbb{F}[M]$. Hence we only need to check the axioms on $\mathcal{S}$-terms constructed over M. (Refl), (Symm), (Triang), (Max), and (Arch) follows from the fact that H_d is a metric on $\mathbb{F}[M]$ (note that finite sets are closed in any metric d). As for (Add) we need to prove that, for arbitrary $A, A', B, B' \in \mathbb{F}[M]$,

1. $H_d(A + B, A + B') \leq H_d(B, B')$;
2. $H_d(A + B, A' + B') \leq H_d(A, A')$.

We show only (1); (2) follows similarly. To this end we use the Hausdorff duality. Note that since B, B' are finite sets, there are only finitely many relations over them, hence there exists a coupling $R \in \mathcal{C}(B, B')$ that attains the minimum in

Lemma 10.5, i.e., $H_d(B, B') = \max_{(m,n)\in R} d(m, n)$. Then the following hold

$$\begin{aligned}
H_d(B, B') &= H_d(A, A) + H_d(B, B') && (H_d \text{ metric})\\
&= \max_{(m,n)\in id(A)} d(m, n) + \max_{m,n\in R} d(m, n) && (\text{Lemma 10.5})\\
&\geq \max_{(m,n)\in id(A)\cup R} d(m, n) && (\text{triangular ineq. for max})\\
&\geq H_d(A \cup A, B \cup B') && (id(A) \cup R \in \mathcal{C}(A \cup B, A \cup B'))\\
&= H_d(A + A, B + B')\,. && (\text{def. } +)
\end{aligned}$$

(S0)–(S3) follow by the interpretations for $+ : 2 \in \mathcal{S}$ and $0 : 0 \in \mathcal{S}$ and the fact that H_d is a metric on $\mathbb{F}[M]$. As for (S4), assume $H_d(A, B) \leq \epsilon$, $H_d(A', B') \leq \epsilon'$, and let $R \in \mathcal{C}(A, B)$ and $R' \in \mathcal{C}(A', B')$ optimal, i.e. such that $H_d(A, B) = \max_{(m,n)\in R} d(m, n)$ and $H_d(A', B') = \max_{(m,n)\in R'} d(m, n)$. Then

$$\begin{aligned}
\max\{\epsilon, \epsilon'\} &\geq \max\left\{H_d(A, B), H_d(A', B')\right\} && (\text{by hp.})\\
&= \max\left\{\max_{(m,n)\in R} d(m, n), \max_{(m,n)\in R'} d(m, n)\right\} && (R, R' \text{ optimal})\\
&= \max_{(m,n)\in R\cup R'} d(m, n) && (\max \text{ on } \cup)\\
&\geq H_d(A \cup B, A' \cup B') && (R \cup R' \in \mathcal{C}(A \cup B, A' \cup B'))\\
&= H_d(A + B, A' + B')\,. && (\text{def. } +)
\end{aligned}$$

This concludes the proof. ∎

The next theorem shows that $\mathbb{F}[M]$ has the universal mapping property for (M, d) to $U_{\mathbf{Met}}$, with universal arrow $(\mathbb{F}[M], \chi_M)$, where $\chi_M : M \longrightarrow \mathbb{F}[M]$ is the map that assigns to arbitrary $m \in M$, the singleton set $\chi_M(m) = \{m\}$. Note that, $H_d(\{m\}, \{n\}) = d(m, n)$, hence χ_M is non-expansive.

Theorem 10.8 $(\mathbb{F}[M], \chi_M)$ *is an universal arrow from* $(M, d) \in \mathbf{Met}$ *to* $U_{\mathbf{Met}}$.

Proof. Let $\mathcal{A} = (A, \mathcal{S}^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathbb{K}(\mathcal{S}, \mathcal{U}^{\mathcal{S}})$ and $\alpha : M \longrightarrow A$ a non-expansive map. Define $h : \mathbb{F}[M] \longrightarrow A$ as follows, by induction on the size of the sets:

- $h(\emptyset) = 0^{\mathcal{A}}$;
- for $m \in P \in \mathbb{F}[M]$, $h(P) = \alpha(m) +^{\mathcal{A}} h(P \setminus \{m\})$.

To show that h is well-defined need to prove that its definition is independent of the choice of the element $m \in P$. Assume $m, n \in P$, be two distinct elements. Then the following hold:

$$\begin{aligned}
h(P) &= \alpha(m) +^{\mathcal{A}} (\alpha(n) +^{\mathcal{A}} h(P \setminus \{m, n\})) && (\text{def. } h)\\
&= (\alpha(m) +^{\mathcal{A}} \alpha(n)) +^{\mathcal{A}} h(P \setminus \{m, n\}) && (\mathcal{A} \models (\text{S3}))\\
&= (\alpha(n) +^{\mathcal{A}} \alpha(m)) +^{\mathcal{A}} h(P \setminus \{m, n\}) && (\mathcal{A} \models (\text{S2}))\\
&= \alpha(n) +^{\mathcal{A}} (\alpha(m) +^{\mathcal{A}} h(P \setminus \{m, n\})) && (\mathcal{A} \models (\text{S3}))\\
&= \alpha(n) +^{\mathcal{A}} h(P \setminus \{n\}) && (\text{def. } h)\\
&= h(P)\,. && (\text{def. } h)
\end{aligned}$$

Clearly, by definition of h, we have that $h \circ \chi_M = \alpha$. Now we prove that h is an homomorphism. By definition $h(0) = h(\emptyset) = 0^{\mathcal{A}}$. Let $P, Q \in \mathbb{F}[M]$, by induction on the size of P we show $h(P + Q) = h(P) + h(Q)$.

(Base case) Let $P = \emptyset$

$$\begin{aligned} h(\emptyset + Q) &= h(Q) && \text{(def. +)} \\ &= 0^{\mathcal{A}} +^{\mathcal{A}} h(Q) && (\mathcal{A} \models \text{(S0),(S2)}) \\ &= h(\emptyset) + h(Q)\,. && (def.\ h) \end{aligned}$$

(Inductive step) Assume $m \in P \cap Q$. We consider two cases: $m \in Q$ and $m \notin Q$. We show only the first case; the other can be derived by avoiding the application of (S1) in what follows:

$$\begin{aligned} h(P + Q) &= h(P \cup Q) && \text{(def. +)} \\ &= \alpha(m) +^{\mathcal{A}} h((P \cup Q)\backslash\{m\}) && \text{(def. } h) \\ &= (\alpha(m) +^{\mathcal{A}} \alpha(m)) +^{\mathcal{A}} h((P\backslash\{m\}) \cup (Q\backslash\{m\})) && (\mathcal{A} \models \text{(S1)}) \\ &= (\alpha(m) +^{\mathcal{A}} \alpha(m)) +^{\mathcal{A}} (h(P \setminus \{m\}) +^{\mathcal{A}} h(Q\backslash\{m\})) && \text{(hp. ind)} \\ &= (\alpha(m) +^{\mathcal{A}} h(P\backslash\{m\})) +^{\mathcal{A}} (\alpha(m) +^{\mathcal{A}} h(Q\backslash\{m\})) && (\mathcal{A} \models \text{(S2),(S3)}) \\ &= h(P) + h(Q) && \text{(def. } h) \end{aligned}$$

As for the uniqueness of the homomorphism. Let $h' \neq h$ be another homomorphism such that $h' \circ \chi_M = \alpha$; and let P a minimal set such that $h'(P) \neq h(P)$. If $P = \emptyset$, the contradiction derives by the fact the h', h are homomorphisms and the interpretation of $\emptyset$ as the constant 0. Assume $P \neq \emptyset$, and let $m \in P$. Then

$$\begin{aligned} h(P) &= \alpha(m) +^{\mathcal{A}} h(P\backslash\{m\}) && \text{(def. } h) \\ &= \alpha(m) +^{\mathcal{A}} h'(P\backslash\{m\}) && \text{(minimality)} \\ &= h'(\{m\}) +^{\mathcal{A}} h'(P\backslash\{m\}) && (h' \circ \chi_M = \alpha) \\ &= h(\{m\} + P\backslash\{m\}) && (h' \text{ homo.}) \\ &= h(P) && \text{(def. +)} \end{aligned}$$

It remains to show that h is non-expansive, i.e., that for arbitrary $P, Q \in \mathbb{F}[M]$, $H_d(P, Q) \geq d^{\mathcal{A}}(h(P), h(Q))$. We proceed by Noetherian induction on pairs or measures (P, Q) partially ordered by $(P, Q) \sqsubseteq (P', Q')$ iff $|P| \leq |P'|$ and $|Q| \leq |Q'|$ (note that this is a well-founded partial order).

(Base case) Assume $P, Q = \emptyset$. Then, by definition of H_d, $H_d(\emptyset, \emptyset) = \infty$. Hence $H_d(\emptyset, \emptyset) \geq d^{\mathcal{A}}(h(\emptyset), h(\emptyset))$ is trivially satisfied.

(Inductive step) Without loss of generality, assume $P, Q \neq \emptyset$. Then, there exists $m \in P$ and $n \in Q$. If $P = \{m\}$, then

$$\begin{aligned} H_d(\{m\}, Q) &\geq d(m, n) && \text{(def. } H_d) \\ &\geq d^{\mathcal{A}}(\alpha(m), \alpha(n)) && (\alpha \text{ non-expansive}) \\ &= d^{\mathcal{A}}(h(\{m\}), h(\{n\}))\,. && (\alpha = h \circ \chi_M) \end{aligned}$$

Similarly for the case $Q = \{n\}$. Assume that $P \neq \{m\}$ and $Q \neq \{n\}$. Since P and Q are finite, the Hausdorff distance between them must be realised as the distance between some element in P and some element in Q. Consequently, there exist $m \in P$ and $n \in Q$ such that[8]

$$H_d(P, Q) = \max\{H_d(\{m\}, \{n\}), H_d(P\backslash\{m\}, B\backslash\{n\})\}\,. \tag{7}$$

Then, the following holds

$$\begin{aligned}
d^{\mathcal{A}}(h(P), h(Q)) &= d^{\mathcal{A}}(\alpha(m) +^{\mathcal{A}} h(P\backslash\{m\}), \alpha(m) +^{\mathcal{A}} h(P\backslash\{m\})) && \text{(def. h)} \\
&\leq \max\{d^{\mathcal{A}}(\alpha(m), \alpha(n)), d^{\mathcal{A}}(h(P\backslash\{m\}), h(B\backslash\{n\}))\} \\
& && (\mathcal{A} \models (S4)) \\
&\leq \max\{d^{\mathcal{A}}(\alpha(m), \alpha(n)), H_d(P\backslash\{m\}, B\backslash\{n\})\} && \text{(hp. ind.)} \\
&\leq \max\{d(m, n), H_d(P\backslash\{m\}, B\backslash\{n\})\} && (\alpha \text{ non-expansive}) \\
&= \max\{H_d(\{m\}, \{n\}), H_d(P\backslash\{m\}, B\backslash\{n\})\} && (\text{def. } H_d) \\
&= H_d(P, Q) && \text{(equation (7))}
\end{aligned}$$

This concludes the proof. ∎

An immediate consequence of Theorem 7.2 and 10.8 is the following.

Corollary 10.9 *The quantitative $\mathcal{S}$-algebras $\mathbb{F}[M]$ and $\mathbb{T}^d[M]$ are isomorphic with bijective isometry $h\colon \mathbb{T}^d[M] \longrightarrow \mathbb{F}[M]$ given, for $m \in M$ and $t, s \in \mathbb{T}M$ by*

$$h(m^{\cong}) = \{m\}\,, \qquad\qquad h((t+s)^{\cong}) = h(t^{\cong}) \cup h(s^{\cong})\,.$$

Hence, the distance induced by the quantitative equational theory $\mathcal{U}^{\mathcal{S}}$ extended with the axioms relative to the generator (M, d) is the Hausdorff metric induced by d. Thus we say that $\mathcal{U}^{\mathcal{S}}_M$ axiomatizes the Hausdorff distance.

10.2 The Continuous Case

We now focus on the class of the closed subsets of a complete separable metric space and prove that it can be organized as a quantitative semilattice by interpreting, as before, $+$ by $\cup$, 0 by $\emptyset$ and the constants as singletons. It turns out that this is the freely generated algebra in the category of quantitative semilattices over complete separable metric spaces. As might be expected, the proofs here are more analytic rather than the combinatorial proofs of the previous subsection.

Consider a complete separable metric space (M, d). Let $\mathbb{G}[M]$ be the set of the closed subsets of M in the open-ball topology of d. We show that by interpreting $+$ by $\cup$, 0 by $\emptyset$ and endowing $\mathbb{G}[M]$ with the Hausdorff metric H_d, we obtain a quantitative semilattice that satisfies $\mathcal{U}^{\mathcal{S}}$.

[8] Notice that this equation is reminiscent of the *splitting lemma* used, for example, with the Kantorovich metric.

As shown in the previous section, we can also construct the freely generated quantitative semilattice $\mathbb{T}^d[M] = (\mathbb{T}^d[M], \mathcal{S}, d_M^{\cong})$, which is isomorphic to $\mathbb{F}[M] = (\mathbb{F}[M], \mathcal{S}, H_d)$.

However, $(\mathbb{T}^d[M], d_M^{\cong})$ is separable (with countable dense subset given by $\mathbb{T}^d[D]$, where D is the countable dense set in M) but it is not a complete metric space.

Consider $(\overline{\mathbb{T}^d[M]}, \overline{d_M^{\cong}})$, the completion of $(\mathbb{T}^d[M], d_M^{\cong})$. Since $\mathbb{T}^d[M]$ is isomorphic to $\mathbb{F}[M]$, their completions must be isomorphic metric spaces.

Let $\overline{\mathbb{K}}_{\mathcal{S}}$ be the subcategory of quantitative semilattices with complete separable metric spaces. We prove that $\mathbb{G}[M] = (\mathbb{G}[M], \mathcal{S}, H_d)$ and $\overline{\mathbb{T}^d[M]} = (\overline{\mathbb{T}^d[M]}, \mathcal{S}, \overline{d_M^{\cong}})$ are isomorphic quantitative semilattices with zero.

Theorem 10.10 *If (M, d) is a complete separable metric space, then $\mathbb{G}[M] \in \overline{\mathbb{K}}_{\mathcal{S}}$. Moreover, $\mathbb{G}[M]$ is isomorphic to $\overline{\mathbb{T}^d[M]}$.*

Proof. Verifying the axioms of the quantitative semilattices with zero for $\mathbb{G}[M]$ is routine. What we need to prove further is that $(\mathbb{G}[M], H_d)$ is a complete separable metric space.

Let $D \subseteq M$ be a countable dense subset of M (its existence is guaranteed by the fact that (M, d) is a separable space). $\mathbb{F}[D]$ is countable and we now show that it is dense in $\mathbb{G}[M]$.

Consider an arbitrary closed set $C \in \mathbb{G}[M]$. The set $S = D \cap C$ is countable and dense in C. Suppose that $S = \{s_1, s_2, \ldots\}$. Then, the sets $S_i = \{s_1, \ldots, s_i\}$, $i \in \mathbb{N}$ are all closed, hence elements of $\mathbb{F}[M]$, and their sequence converges to C in $(\mathbb{G}[M], H_d)$. Hence, $\mathbb{F}[D]$ is dense in $\mathbb{G}[M]$.

Previously, we have shown that $d_M^{\cong} = H_d$ on $\mathbb{F}[M]$, hence also on $\mathbb{F}[D]$. Since the completion of $\mathbb{F}[D]$ is unique and it gives us $(\mathbb{G}[M], H_d)$, we obtain the isomorphism between $\overline{\mathbb{T}^d[M]}$ and $\mathbb{G}[M]$; hence also an isomorphism of metric spaces. ∎

Next we state that $\overline{\mathbb{T}^d[M]}$ is the quantitative algebra in $\overline{\mathbb{K}}_{\mathcal{S}}$ freely generated from the complete separable metric space (M, d). Specifically, $\overline{\mathbb{T}^d[M]}$ has the universal mapping property for $(M, d) \in \mathbf{CSMet}$ (the category of complete separable metric spaces with non-expansive maps) to the forgetful functor

$$U_{\mathbf{CSMet}} \colon \overline{\mathbb{K}}_{\mathcal{S}} \to \mathbf{CSMet}\,.$$

This situation is described by the commutative diagram below (*cf.* Definition 3.5):

$$\begin{array}{ccc}
 & \text{in } \mathbf{CSMet} & \text{in } \overline{\mathbb{K}}_{\mathcal{S}} \\
(M,d) \xrightarrow{\eta_M} & (\overline{\mathbb{T}^d[M]}, \overline{d_M^{\cong}}) & \overline{\mathbb{T}^d[M]} \\
\quad\searrow^{\alpha} & \downarrow h & \downarrow h \\
 & (A, d^{\mathcal{A}}) & \mathcal{A}
\end{array}$$

Theorem 10.11 *$(\overline{\mathbb{T}^d[M]}, \eta_M)$ is a universal morphism from $(M, d) \in$* **CSMet** *to $U_{\mathbf{CSMet}}$.*

Proof. Note that if **SMet** denotes the category of separable metric spaces with non-expansive maps and **CSMet** the category of complete separable metric spaces with non-expansive maps, then the functors $\mathbb{C} : \mathbf{SMet} \to \mathbf{CSMet}$ that maps a metric space to its completion and $\mathbb{I} : \mathbf{CSMet} \to \mathbf{SMet}$ that maps a complete metric space to itself define an adjunction $\mathbb{C} \dashv \mathbb{I}$ as follows.

$$\begin{tikzcd} \mathbf{SMet} \arrow[r, bend left, "\mathbb{C}"] & \mathbf{CSMet} \arrow[l, bend left, "\mathbb{I}"] \end{tikzcd}$$

$\eta : Id_{\mathbf{SMet}} \Longrightarrow \mathbb{IC}$ defined for an arbitrary object $M \in \mathbf{SMet}$ by $\eta_M : M \to \overline{M}$, where $\overline{M}$ denotes the completion of M; and $\epsilon : \mathbb{CI} \Longrightarrow Id_{\mathbf{CSMet}}$ defined for arbitrary object $K \in \mathbf{CSMet}$ by $\epsilon_K : \overline{K} \to K$, where $\overline{K}$ denotes the completion of K.

In fact this adjunction already exists between the similar functors connecting the category of metric spaces and the category of complete metric spaces. The additional separability condition specializes further this general adjunction.

The result now follows from Theorem 10.8 and from the universal property implicit in the adjunction.

∎

11 Interpolative Barycentric Algebras

In this section we study a variation of quantitative barycentric algebras, which is similar to the left-invariant barycentric algebra discussed in Section 9 but with one slightly stronger axiom than (LI). The signature remains the same but the axioms though, superficially, only slightly different give a very different metric. Instead of axiomatizing the total variation distance, we get an axiomatization of the p-Wasserstein metric for $p \geq 1$, both in the finitary and the continuous cases. For $p = 1$ this reduces to the Kantorovich metric. We call these algebras *interpolative barycentric* algebras or p-IB algebras for short. The new axiom is a kind of interpolation axiom. In this section we are always assuming the underlying metric takes values in $[0, 1]$; they are called one-bounded metrics.

Consider the barycentric signature $\mathcal{B} = \{+_e : 2 \mid e \in [0, 1]\}$ from Section 9.

Definition 11.1 (p-IB Equational Theory) *This theory is given by the axiom-schemata (B1), (B2), (SC), (SA) from Definition 9.1 and the following axiom-scheme (IB$_p$), where $\epsilon_1, \epsilon_2 \in [0, 1]$ and $\delta \in \mathbb{Q}_+ \cap [0, 1]$:*

(IB$_p$) $\{x =_{\epsilon_1} y, x' =_{\epsilon_2} y'\} \vdash x +_e x' =_\delta y +_e y'$, *where* $(e\epsilon_1^p + (1 - e)\epsilon_2^p)^{1/p} \leq \delta$.

Note that (IB$_p$) is not an unconditional quantitative inference as are the previous examples. Moreover, it is stronger than the axiom (LI) in Definition 9.1 for 1-bounded metrics, in the sense that (LI) is than just an instantiation of (IB$_p$). Hence, this new proof system can prove more basic quantitative equations. If we set $\epsilon_1 = \epsilon_2$ in this axiom we just get $\epsilon \leq \delta$; in other words it says that the $+_e$ operations are non-expansive.

If we state (IB$_1$), we get the axiom below.

(IB$_1$) $\{x =_{\epsilon_1} y, x' =_{\epsilon_2} y'\} \vdash x +_e x' =_\delta y +_e y'$, where $e\epsilon_1 + (1-e)\epsilon_2 \leq \delta$.

In this section we focus on the class $\mathbb{K}(\mathcal{B}, \mathcal{U}^{IB})$ defined by the p-IB barycentric equational theory $\mathcal{U}^{IB}$ over a countable set X of variables.

11.1 Kantorovich-Wasserstein metrics

Let (M, d) be a one-bounded complete separable metric space and let $p \geq 1$. The *p-Wasserstein metric induced by d* on the set $\Delta[M]$ of Radon[9] probability measures over M, is defined, for arbitrary $\mu, \nu \in \Delta[M]$ as

$$W_d^p(\mu,\nu)^p = \inf\left\{\int d^p \, \mathrm{d}\omega \mid \omega \in \mathcal{C}(\mu,\nu)\right\}. \tag{8}$$

The Kantorovich metric induced by d is usually defined by:

$$K_d(\mu,\nu) = \sup\left\{\left|\int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\nu\right|\right\}.$$

with supremum ranging over the set positive 1-bounded non-expansive real-valued functions over M.

We generally work with Polish spaces in this section. A Polish space is a separable topological space for which can be metrized so that it is complete. Note that a space like $(0, 1)$ is Polish even, though it is not complete with the usual metric. However, it is homeomorphic to $(0, \infty)$, hence can be given a complete metric that gives the same topology. In a Polish space all Borel measures are Radon.

Theorem 11.2 (Kantorovich Duality - Thm 5.10, [Vil08]) *Let (M, d) be a Polish metric space with the metric taking real values. Then, for arbitrary Borel probability measures $\mu, \nu \in \Delta[M]$*

$$K_d(\mu,\nu) = \min\left\{\int d \, \mathrm{d}\omega \mid \omega \in \mathcal{C}(\mu,\nu)\right\}.$$

Kantorovich duality implies that for $p = 1$ in equation (8), one gets the *Kantorovich metric induced by d on $\Delta[M]$*.

[9] Radon measures are tight. This means that for every $\epsilon > 0$ there is a compact set K_ϵ such that the measure of $M \setminus K_\epsilon$ is less than ϵ. On complete separable metric spaces all Borel measures are Radon.

In the case of Kantorovich metric, we also know that there exists an optimal coupling for W_d^1, i.e., there exists a coupling that attains the infimum (hence, it is a minimum) in equation (8). It is not clear whether this holds for the Wasserstein metric but fortunately we do not need that in the splitting lemma below.

Note that the total variation distance is just a particular case of the Wassertstein metric, namely, $T(\mu,\nu) = K_{1\neq}(\mu,\nu) = W^1_{1\neq}(\mu,\nu)$, where $1_{\neq}$ is the metric that assigns distance 1 to all distinct pairs of points.

Lemma 11.3 (Splitting Lemma) *Let $\mu,\nu \in \Delta[M]$ and $\omega \in \mathcal{C}(\mu,\nu)$. If R is a measurable set such that $\omega(R) \notin \{0,1\}$, then*

$$\mu = \omega(R)\mu_1 + (1-\omega(R))\mu_2 \qquad \textit{and} \qquad \nu = \omega(R)\nu_1 + (1-\omega(R))\nu_2\,,$$

for some $\mu_1,\mu_2,\nu_1,\nu_2 \in \Delta[M]$, such that $supp(\mu_1) \subseteq \pi_1(R)$, $supp(\mu_2) \subseteq \pi_1(R^c)$, $supp(\nu_1) \subseteq \pi_2(R)$, and $supp(\nu_2) \subseteq \pi_2(R^c)$, where $R^c = (M\times M)\setminus R$.

Moreover, for any $\varepsilon > 0$, if $\int d^p \mathrm{d}\omega \leq W_d^p(\mu,\nu) + \varepsilon$ then

$$\omega(R)W_d^p(\mu_1,\nu_1)^p + (1-\omega(R))W_d^p(\mu_2,\nu_2)^p \leq W_d^p(\mu,\nu)^p + \varepsilon.$$

Proof. They key step is to use the conditional probabilities given R and R^c to construct the splitting. Define $\mu_i,\nu_i \in \Delta[M]$, for an arbitrary Borel set E, as follows:

$$\mu_1(E) = \frac{\omega((E\times M)\cap R)}{\omega(R)}\,, \qquad \mu_2(E) = \frac{\omega((E\times M)\cap R^c)}{1-\omega(R)}\,,$$
$$\nu_1(E) = \frac{\omega((M\times E)\cap R)}{\omega(R)}\,, \qquad \nu_2(E) = \frac{\omega((M\times E)\cap R^c)}{1-\omega(R)}\,.$$

We show that $\mu = \omega(R)\mu_1 + (1-\omega(R))\mu_2$. Let E be any Borel set, then

$$\begin{aligned}
&\omega(R)\mu_1(E) + (1-\omega(R))\mu_2 = \\
&\quad = \omega((E\times M)\cap R) + \omega((E\times M)\cap R^c) && (\text{def. } \mu_1,\mu_2)\\
&\quad = \omega(E\times M) && (\text{additivity of } \omega)\\
&\quad = \mu(E)\,. && (\omega \in \mathcal{C}(\mu,\nu))
\end{aligned}$$

Similarly, $\nu = \omega(R)\nu_1 + (1-\omega(R))\nu_2$.

Now we prove that $(\omega|R) \in \mathcal{C}(\mu_1,\nu_1)$ and $(\omega|R^c) \in \mathcal{C}(\mu_2,\nu_2)$, where $(\omega|R)$ $(\omega|R^c)$ are the conditional probability measures of ω given R and R^c, respectively. We show only one membership, the other is similar. Let E be a Borel measurable set. Then, by definitions of μ_1,ν_1 and conditional probability

$$(\omega|R)(E\times M) = \frac{\omega((E\times M)\cap R)}{\omega(R)} = \mu_1(E)\,, \qquad (\text{left marginal})$$

$$(\omega|R)(M\times E) = \frac{\omega((M\times E)\cap R)}{\omega(R)} = \nu_1(E)\,. \qquad (\text{right marginal})$$

The conditions on the supports follow immediately by the definitions of μ_i, ν_i, $(i = \{1, 2\})$.

For the last assertion in the lemma we proceed as follows. Fix $\varepsilon > 0$ and suppose that $\int d^p \, \mathrm{d}\omega \leq W^p_d(\mu, \nu)^p + \varepsilon$. We compute as follows:

$$\begin{aligned}
&\omega(R) W^p_d(\mu_1, \nu_1)^p + (1 - \omega(R)) W^p_d(\mu_2, \nu_2)^p \\
&\quad \leq \omega(R) \int d^p \, \mathrm{d}(\omega|R) + (1 - \omega(R)) \int d^p \, \mathrm{d}(\omega|R^c) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } \omega|R \text{ and } \omega|R^c \text{ are couplings)} \\
&\quad = \int_R d^p \, \mathrm{d}\omega + \int_{R^c} d^p \, \mathrm{d}\omega && \text{(def. } \mu|R \text{ \& linearity of } \textstyle\int) \\
&\quad = \int d^p \, \mathrm{d}\omega && \text{(additivity of } \textstyle\int) \\
&\quad \leq W^p_d(\mu, \nu)^p + \varepsilon \, . && \text{(hyp. on } \omega)
\end{aligned}$$

∎

11.2 The Finitary Case

Fix a one-bounded metric space (M, d). Let $\mathbb{T}^d[M]$ be the p-IB algebra in $\mathbb{K}(\mathcal{B}, \mathcal{U}^{IB})$ freely generated from (M, d), as constructed in Section 7. By Theorem 7.2, $\mathbb{T}^d[M]$ has the universal mapping property for (M, d) to

$$U_{\mathbf{Met}} \colon \mathbb{K}(\mathcal{B}, \mathcal{U}^{IB}) \to \mathbf{Met}.$$

Theorem 11.4 *If (M, d) is a non-degenerate metric space then $\mathbb{T}^d[M]$ is a non-degenerate p-IB algebra. In particular, $\mathcal{U}^{IB}$ is a consistent quantitative theory.*

Denote by $\Pi[M]$ the set of finitely supported Borel probability measures on M —i.e., those that can be represented as finite convex combinations of Dirac distributions δ_m, for $m \in M$. Next we will show that $\Pi[M]$ can be organized as a p-IB algebra in $\mathbb{K}(\mathcal{B}, \mathcal{U}^{IB})$, with metric given by the p-Wasserstein metric W^p_d. Moreover, we show that this algebra enjoys the universal mapping property for (M, d) to $U_{\mathbf{Met}} \colon \mathbb{K}(\mathcal{B}, \mathcal{U}^{IB}) \to \mathbf{Met}$; consequently $\Pi[M]$ and $\mathbb{T}^d[M]$ are isomorphic $\mathcal{B}$-algebras.

Similarly to Section 9, we regard $\Pi[M]$ as an algebra of type $\mathcal{B}$ by interpreting each operator $+_e : 2 \in \mathcal{B}$, for arbitrary $\mu, \nu \in \Pi[M]$, as

$$\mu +_e \nu = e\mu + (1 - e)\nu \, ,$$

However, unlike the situation in Section 9, $\Pi[M]$ will be viewed as a quantitative algebra by taking as a metric the p-Wasserstein metric W^p_d induced by d, rather then the total variation distance. Note that finitely supported Borel probability measures are Radon, so that W^p_d is a well defined metric on $\Pi[M]$.

Theorem 11.5 $\Pi[M] = (\Pi[M], \mathcal{B}, W_d^p) \in \mathbb{K}(\mathcal{B}, \mathcal{U}^{IB})$*, i.e.,* $\Pi[M] \models \mathcal{U}^{IB}$*.*

Proof. Note that each assignment $\iota\colon X \longrightarrow \Pi[M]$ maps variables to Borel measures in $\Pi[M]$, hence we only need to check the axioms on $\mathcal{B}$-terms constructed over $\Pi[M]$. (Refl), (Symm), (Triang), (Max), and (Arch) follows from the fact that W_d^p is a metric. As for (NExp) we need to prove that, for arbitrary $e, \epsilon \in [0,1]$ and $\mu, \mu', \nu, \nu' \in \Pi[M]$,

$$W_d^p(\mu, \mu') \leq \epsilon \text{ and } W_d^p(\nu, \nu') \leq \epsilon \text{ implies } W_d^p(\mu +_e \nu, \mu' +_e \nu') \leq \epsilon.$$

Let $\delta > 0$; let $\omega \in \mathcal{C}(\nu, \nu')$ and $\rho \in \mathcal{C}(\mu, \mu)$ be two couplings such that

$$\int d^p \, \mathrm{d}\omega - \delta \leq W_d^p(\nu, \nu') \text{ and } \int d^p \, \mathrm{d}\omega' - \delta \leq W_d^p(\mu, \mu).$$

Then,

$$\begin{aligned}
\epsilon^p &= e\epsilon^p + (1-e)\epsilon^p \\
&\geq eW_d^p(\mu, \nu)^p + (1-e)W_d^p(\mu', \nu')^p && \text{(by hp.)} \\
&\geq e \int d^p \, \mathrm{d}\omega + (1-e) \int d^p \, \mathrm{d}\omega' - \delta && \text{(hp. on } \omega, \omega') \\
&= \int d^p \, \mathrm{d}(e\omega + (1-e)\omega') - \delta && \text{(linearity of } \textstyle\int) \\
&\geq W_d^p(e\mu + (1-e)\mu', e\nu + (1-e)\nu')^p - \delta && \text{(Lemma 9.3)} \\
&= W_d^p(\mu +_e \mu', \nu +_e \nu')^p - \delta \,. && \text{(def. } +_e)
\end{aligned}$$

Since this is satisfied by any $\delta > 0$, we obtain that $W_d^p(\mu +_e \nu, \mu' +_e \nu') \leq \epsilon$.

(B1), (B2), (SA), and (SC) follow by the definition of the interpretation for $+_e : 2 \in \mathcal{B}$ and the fact that W_d^p is a metric.

As for (IB$_p$), assume $W_d^p(\mu, \nu) \leq \epsilon_1$, $W_d^p(\mu', \nu') \leq \epsilon_2$. Consider an arbitrary $\delta > 0$; and let $\omega \in \mathcal{C}(\nu, \nu')$ and $\rho \in \mathcal{C}(\mu, \mu)$ be two couplings such that

$$\int d^p \, \mathrm{d}\omega - \delta \leq W_d^p(\nu, \nu') \text{ and } \int d^p \, \mathrm{d}\omega' - \delta \leq W_d^p(\mu, \mu).$$

Then,

$$\begin{aligned}
& e\epsilon_1^p + (1-e)\epsilon_2^p \\
&\geq eW_d^p(\mu, \nu)^p + (1-e)W_d^p(\mu', \nu')^p && \text{(by hp.)} \\
&\geq e \int d^p \, \mathrm{d}\omega + (1-e) \int d^p \, \mathrm{d}\omega' - \delta && \text{(hp. on } \omega, \omega') \\
&= \int d^p \, \mathrm{d}(e\omega + (1-e)\omega') - \delta && \text{(linearity of } \textstyle\int) \\
&\geq W_d^p(e\mu + (1-e)\mu', e\nu + (1-e)\nu')^p - \delta && \text{(Lemma 9.3)} \\
&= W_d^p(\mu +_e \mu', \nu +_e \nu')^p - \delta \,. && \text{(def. } +_e)
\end{aligned}$$

Since this inequality holds for any $\delta > 0$, we obtain that

$$W_d^p(\mu +_e \mu', \nu +_e \nu') \leq e\epsilon_1^p + (1-e)\epsilon_2^p.$$

∎

The next theorem shows that $\Pi[M]$ has the universal mapping property for (M, d) to $U_{\mathbf{Met}}$, with universal arrow $(\Pi[M], \delta_M)$, where $\delta_M \colon M \longrightarrow \Pi[M]$ maps $m \in M$ to $\delta_m \in \Pi[M]$ —the Dirac measure with probability mass in $m \in M$. Note that, $W_d(\delta_m, \delta_n) = d(m, n)$, hence δ_M is non-expansive.

Theorem 11.6 $(\Pi[M], \delta_M)$ *is an universal arrow from* $(M, d) \in \mathbf{Met}$ *to* $U_{\mathbf{Met}}$.

Proof. Let $\mathcal{A} = (A, \mathcal{B}^{\mathcal{A}}, d^{\mathcal{A}}) \in \mathbb{K}(\mathcal{B}, \mathcal{U}^{IB})$ and $\alpha \colon M \longrightarrow A$ a non-expansive map.

By Theorem 2 [Sto49], any barycentric algebra $(A, \mathcal{B}^{\mathcal{A}})$ has a one-to-one embedding into a convex subset of a suitable vector space. By this result, if $a_1, \dots, a_n \in A$, then also $\sum_{i=1}^{n} e_i a_i \in A$, provided that $e_i \in [0, 1]$ and $\sum_{i=1}^{n} e_i = 1$.

As for the set $\Pi[M]$ of finitely supported Borel probability measures over M, we additionally have that any $\mu \in \Pi[M]$, can be canonically represented as a *finite* convex combination of the form $\mu = \sum_{i=1}^{k} c_i \delta_{m_i}$, where $supp(\mu) = \{m_1, \dots, m_k\}$ and $c_i \in (0, 1]$ are such that $\sum_{i=1}^{k} c_i = 1$.

By using these facts we define the map $h \colon \Pi[M] \longrightarrow A$ as follows:

$$h(\sum_{i=1}^{k} c_i \delta_{m_i}) = \sum_{i=1}^{k} c_i \alpha(m_i) \,.$$

Clearly $h \circ \delta_M = \alpha$: for any $m \in M$, $h(\delta_M(m)) = h(\delta_m) = \alpha(m)$. Now we show that h is a homomorphism. Let $\mu = \sum_{i=1}^{k} c_i \delta_{m_i}, \nu = \sum_{j=1}^{n} d_j \delta_{n_j}$ and $e \in [0, 1]$. Then the following holds:

$$\begin{aligned}
h(\mu +_e \nu) &= h(e\mu + (1-e)\nu) && \text{(def. } +_e\text{)} \\
&= h(e \sum_{i=1}^{k} c_i \delta_{m_i} + (1-e) \sum_{j=1}^{n} d_j \delta_{n_j}) && \text{(canonical repr.)} \\
&\overset{(*)}{=} e \sum_{i=1}^{k} c_i \alpha(m_i) + (1-e) \sum_{j=1}^{n} d_j \alpha(n_j) && \text{(def. } h \text{ \& } \Pi[M], \mathcal{A} \models \text{(B2))} \\
&= h(\mu) +_e^{\mathcal{A}} h(\nu) \,. && \text{(def. } h \text{ \& def. } +_e^{\mathcal{A}}\text{)}
\end{aligned}$$

Note that in $(*)$, the formal definition of h requires that the measure is canonically represented without repetitions of Dirac measures δ_m. The repetitions can be removed by applying (B2) in $\Pi[M]$ and, once h is applied, they can be recovered by applying $(B2)$ in $\mathcal{A}$ in the reverse direction.

As for the uniqueness, assume that there exists another homomorphism $h' \neq h$ such $h' \circ \delta_M = \alpha$. Let $\mu = \sum_{i=1}^{k} c_i \delta_{m_i}$ be a measure in $\Pi[M]$ with minimal support such that $h'(\mu) \neq h(\mu)$. If $k = 1$, we get a contradiction by $h' \circ \delta_M = \alpha = h \circ \delta_M$. Assume $k > 1$, then we get the following contradiction

$$\begin{aligned}
h'(\sum_{i=1}^{k} c_i \delta_{m_i}) &= h'(\delta_{m_1}) +^{\mathcal{A}}_{c_1} h'(\sum_{i=2}^{k} \frac{c_i}{1-c_1} \delta_{m_i}) && (h' \text{ homo.}) \\
&= h(\delta_{m_1}) +^{\mathcal{A}}_{c_1} h(\sum_{i=2}^{k} \frac{c_i}{1-c_1} \delta_{m_i}) && (\text{minimal supp.}) \\
&= h(\sum_{i=1}^{k} c_i \delta_{m_i}) && (h' \text{ homo.})
\end{aligned}$$

It remains to show that h is non-expansive, *i.e.*, that for arbitrary $\mu, \nu \in \Pi[M]$, $W_d^p(\mu, \nu) \geq d^{\mathcal{A}}(h(\mu), h(\nu))$. We proceed by well-founded induction on pairs or measures (μ, ν) partially ordered by $(\mu, \nu) \sqsubseteq (\mu', \nu')$ iff $|\,supp(\mu)| \leq |\,supp(\mu')|$ and $|\,supp(\nu)| \leq |\,supp(\nu')|$; the $|\cdot|$ notation means cardinality of the set. Note that it is indeed well-founded as we are dealing with finite sets here.

(Base case) Assume $\mu = \delta_m$ and $\nu = \delta_n$, for some $m, n \in M$.

$$\begin{aligned}
W_d^p(\delta_m, \delta_n) &= d(m, n) && (\text{def. } W_d) \\
&\geq d^{\mathcal{A}}(\alpha(m), \alpha(n)) && (\alpha \text{ non-exp.}) \\
&= d^{\mathcal{A}}(h(\delta_m), h(\delta_n))\,. && (h \circ \delta_M = \alpha)
\end{aligned}$$

(Inductive step) Without loss of generality, assume $|\,supp(\mu)| > 1$ (the proof follows similarly for $|\,supp(\nu)| > 1$). Then, there exists a nontrivial measurable partition $(N'; N')$ of $supp(\mu)$ such that $\mu(N_1), \mu(N_2) \neq \{0, 1\}$. Let $\omega \in \mathcal{C}(\mu, \nu)$ be minimal in Lemma 11.2 for $W_d^p(\mu, \nu)$ and $R = N_1 \times supp(\nu)$. Note that R is measurable (finite sets are always Borel measurable in the product space) and $\omega(R) = \mu(N_1) \notin \{0, 1\}$. Let $e = \omega(R)$. By the splitting lemma (Lemma 11.3), for any $\epsilon > 0$ such that $\int d^p \mathrm{d}\omega \leq W_d^p(\mu, \nu) + \varepsilon$, there exist $\mu_i, \nu_i \in \Pi[M]$ such that

$$\mu = \mu_1 +_e \mu_2 \quad \text{and} \quad \nu = \nu_1 +_e \nu_2\,,$$
$$W_d^p(\mu, \nu)^p + \epsilon \geq e W_d^p(\mu_1, \nu_1)^p + (1-e) W_d^p(\mu_2, \nu_2)^p\,.$$

Moreover, by the choice of R, we have that $supp(\mu_1) \subseteq N_1$, $supp(\mu_2) \subseteq N_2$, $supp(\nu_1) \subseteq supp(\nu)$, and $supp(\nu_2) = \emptyset$. Thus,

$$(\mu, \nu) \sqsubset (\mu_1, \nu_1) \qquad \text{and} \qquad (\mu, \nu) \sqsubset (\mu_2, \nu_2)\,.$$

Then the following holds:

$$\begin{aligned}
W_d^p(\mu, \nu)^p + \epsilon &\geq e W_d^p(\mu_1, \nu_1)^p + (1-e) W_d^p(\mu_2, \nu_2)^p && (\text{splitting lemma}) \\
&\geq e d^{\mathcal{A}}(h(\mu_1), h(\nu_1))^p + (1-e) d^{\mathcal{A}}(h(\mu_2), h(\nu_2))^p && (\text{hp. ind.}) \\
&\geq d^{\mathcal{A}}(h(\mu_1) +_e h(\mu_2), h(\nu_1) +_e h(\nu_2))^p && (\mathcal{A} \models (\text{IB})) \\
&= d^{\mathcal{A}}(h(\mu_1 +_e \mu_2), h(\nu_1 +_e \nu_2))^p && (h \text{ homo.}) \\
&= d^{\mathcal{A}}(h(\mu), h(\nu))^p && (\text{splitting lemma})
\end{aligned}$$

Since in the inequality above $\epsilon > 0$ is arbitrarily chosen, the proof is done. ∎

The next result follows directly by theorems 7.2 and 11.6.

Corollary 11.7 *The quantitative $\mathcal{B}$-algebras $\Pi[M]$ and $\mathbb{T}^d[M]$ are isomorphic with bijective isometry $h\colon \mathbb{T}^d[M] \longrightarrow \Pi[M]$ characterized, for $m \in M$ and $t, s \in \mathbb{T}M$ by*

$$h(m^{\cong}) = \delta_m\,, \qquad\qquad h((t +_e s)^{\cong}) = eh(t^{\cong}) + (1-e)h(s^{\cong})\,.$$

This means that the quantitative equational theory $\mathcal{U}^{IB}$, further extended with the axioms relative to the space (M, d), axiomatizes the p-Wasserstein metric induced by d; and for $p = 1$ it characterizes the Kantorovich metric.

11.3 The Continuous Case

We now focus on the class of the general Borel probability measures over a one-bounded complete separable metric space and prove that it forms a p-IB algebra. In this case we are not restricting to finitely-supported distributions. It turns out that this is the freely-generated algebra in the category of the p-IB algebras defined for complete separable *one-bounded* metric spaces.

Consider a complete separable metric space (M, d) with the metric taking values in $[0, 1]$. Let $\Delta[M]$ be the set of all Borel probability measures on M. Note that since (M, d) is complete and separable, all the measures in $\Delta[M]$ are Radon. We endow $\Delta[M]$ with the signature $\mathcal{B}$, where we define for arbitrary $\mu, \nu \in \Delta[M]$ and $r \in [0, 1]$,

$$\mu +_r \nu = r\mu + (1-r)\nu.$$

As shown previously, $\mathbb{T}^d[M] = (\mathbb{T}^d[M], \mathcal{B}, d^{\cong}_M)$ is a barycentric algebra isomorphic to $\Pi[M] = (\Pi[M], \mathcal{B}, W_d)$. However, we prove below that $(\mathbb{T}^d[M], d^{\cong}_M)$ is separable but it is not a complete metric space.

Consider the metric space $(\overline{\mathbb{T}^d[M]}, \overline{d^{\cong}_M})$ obtained by the completion of $(\mathbb{T}^d[M], d^{\cong}_M)$.

We need now to recall a series of definitions and results that relates the concept of weak topology and the p-Wasserstein distance.

Definition 11.8 *The p-**weak topology** on $\Delta[M]$ is the topology such that convergence of the sequence of measures ν_i to ν means that for all continuous real-valued functions f such that for arbitrary $m \in M$, $|f(m)| \leq C(1 + d(m_0, m)^p)$, for some $C \in \mathbb{R}_+$, and $m_0 \in M$,*

$$\int f \,\mathrm{d}\nu_i \rightarrow \int f \,\mathrm{d}\nu.$$

If (M, d) is a Polish space then it is known that p-Wasserstein W^p_d metrizes the p-weak topology on $\Delta[M]$ (see Theorem 6.9 and Corollary 6.13 in [Vil08]).

The following lemma is well known (see Theorem 6.18 in [Vil08]).

Proposition 11.9 *Let M be a Polish space and let $\{c_i\}_{i=1}^k$ be positive real numbers such that $\sum_{i=1}^k c_i = 1$. Let $\{m_i\}_{i=1}^k$ be points in M. Then measures of the form $\sum_{i=1}^k c_i\delta_{m_i}$ are p-weakly dense in $\Delta[M]$.*

Proof. Suppose that we are given a basic open neighbourhood of $\Delta[M]$

$$U = \left\{\mu : \left| \int_M f_i \, \mathrm{d}\nu - \int_M f_i \, \mathrm{d}\mu \right| < \varepsilon, i = 1, \ldots, k\right\},$$

where the f_i are bounded continuous functions, ν is a probability measure and $\varepsilon > 0$.

Fix an $\varepsilon > 0$. Now the functions f_i are measurable so there are simple functions g_i such that $\sup_x |f_i(x) - g_i(x)| < \varepsilon/2$ for each $i = 1, \ldots, k$. We partition M into disjoint Borel sets $A_j, j = 1, \ldots, l$ such that all the g_i are constant over each A_j.

Now choose a point m_j in each A_j and set $c_j = \nu(A_j)$. The measure $\mu := \sum_{j=1}^l c_j\delta_{m_j}$ is a convex combination of Dirac measures and has the property that for each A_j, $\mu(A_j) = \nu(A_j)$. Thus for each of the g_i

$$\int g_i \, \mathrm{d}\nu = \int g_i \, \mathrm{d}\mu.$$

Thus we have

$$\left|\int f_i \, \mathrm{d}\nu - \int f_i \, \mathrm{d}\mu\right| = \left|\int f_i \, \mathrm{d}\nu - \int g_i \, \mathrm{d}\nu + \int g_i \, \mathrm{d}\mu - \int f_i \, \mathrm{d}\mu\right| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves that μ is in U. Since U is an arbitrary basic open it shows that measures of the form μ are dense in the weak topology. ∎

Let $\overline{\mathbb{K}}_{\mathcal{B}}$ be the class of IB algebras with complete separable metric spaces. We prove that $\underline{\Delta[M]} = (\Delta[M], \mathcal{B}, W_d^p)$ is isomorphic, as a barycentric algebra, to $\overline{\mathbb{T}^d[M]} = (\overline{\mathbb{T}^d[M]}, \mathcal{B}, \overline{d_M^{\cong}})$.

Theorem 11.10 *If (M, d) is a complete separable metric space, then $\Delta[M] \in \overline{\mathbb{K}}_{\mathcal{B}}$. Moreover, $\Delta[M]$ is isomorphic to $\overline{\mathbb{T}^d[M]}$.*

Proof. Verifying the axioms of the barycentric algebras for $\Delta[M]$ is routine and follows closely the proof of Theorem 11.5. What we need to prove further is that $\Delta[M]$ is a complete separable metric space.

Let $D \subseteq M$ be a countable dense subset of M (its existence is guaranteed by the fact that (M, d) is a separable space). Now $\Pi[D]$ is of course not countable but we can take all distributions that assign only rational measures to points and get a countable set. We call this $\mathcal{P}[D]$ for short. We now show that it is dense in $\Delta[M]$.

Let $\rho \in \Delta[M]$. Since (M, d) is Polish, W_d^p metrizes the p-weak-topology on $\Delta[M]$, which is also a Polish space (Corollary 6.13 in [Vil08]). Moreover, $\Pi[M]$

is dense in $\Delta[M]$ with respect to this topology by Prop. 11.9. Hence, there exists a sequence $(\rho_i)_{i\in\mathbb{N}} \subseteq \Pi[M]$ of distributions with finite support on M that converges to ρ. Since D is dense in M and the rationals are dense in $[0,1]$, for any sequence $(\epsilon_i)_{i\in\mathbb{N}} \in [0,1]$ that converges to 0, we can find a sequence $(\rho'_i)_{i\in\mathbb{N}} \subseteq \mathcal{P}[D]$ such that $W_d^p(\rho_i, \rho'_i) < \epsilon_i$. Thus, $\{\rho_i \mid i \in \mathbb{N}\} \cup \{\rho'_i \mid i \in \mathbb{N}\}$ is a Cauchy sequence in $\Pi[M]$ and since $(\rho_i)_{i\in\mathbb{N}}$ converges to ρ and $\Pi[M]$ is complete, also $(\rho'_i)_{i\in\mathbb{N}}$ converges to ρ. And this proves that $\mathcal{P}[D]$ is dense in $\Delta[M]$.

In the previous section we have shown that $d_M^{\cong} = W_d^p$ on $\Pi[M]$, hence also on $\Pi[D]$. Since the completion of $\Pi[D]$ is unique and it gives us $(\Delta[M], d_M^{\cong})$, we obtain the isomorphism between $\overline{\mathbb{T}^d[M]}$ and $\Delta[M]$; hence, also the isomorphism of metric spaces. ∎

Next we show that $\overline{\mathbb{T}^d[M]}$ is the quantitative algebra in $\overline{\mathbb{K}}_{\mathcal{B}}$ freely generated from the complete separable metric space (M,d). Specifically, $\overline{\mathbb{T}^d[M]}$ has the universal mapping property for $(M,d) \in \mathbf{CSMet_1}$ (the category of complete separable one-bounded metric spaces with non-expansive maps) to the forgetful functor

$$U_{\mathbf{CSMet}} \colon \overline{\mathbb{K}}_{\mathcal{B}} \longrightarrow \mathbf{CSMet_1}\,.$$

This situation is described by the commutative diagram below (*cf.* Definition 3.5):

$$\begin{array}{ccc}
\text{in } \mathbf{CSMet_1} & \qquad & \text{in } \overline{\mathbb{K}}_{\mathcal{B}} \\
\begin{tikzcd}
(M,d) \arrow[r, "\eta_M"] \arrow[dr, "\alpha"'] & (\overline{\mathbb{T}^d[M]}, \overline{d_M^{\cong}}) \arrow[d, "h"] \\
& (A, d^{\mathcal{A}})
\end{tikzcd}
& &
\begin{tikzcd}
\overline{\mathbb{T}^d[M]} \arrow[d, dashed, "h"] \\
\mathcal{A}
\end{tikzcd}
\end{array}$$

Theorem 11.11 *$(\overline{\mathbb{T}^d[M]}, \eta_M)$ is a universal morphism from $(M,d) \in \mathbf{CSMet_1}$ to $U_{\mathbf{CSMet}}$.*

Proof. The proof of this theorem is essentially the same as that of Theorem 10.11.

As before, we can define two functors between the categories $\mathbf{SMet_1}$ of separable one-bounded metric spaces and $\mathbf{CSMet}$ of complete separable one-bounded metric spaces with non-expansive maps. These functors are $\mathbb{C} : \mathbf{SMet_1} \to \mathbf{CSMet_1}$ that maps a metric space to its completion and $\mathbb{I} : \mathbf{CSMet_1} \to \mathbf{SMet_1}$ that maps a complete metric space to itself. They define an adjunction $\mathbb{C} \dashv \mathbb{I}$.

$$\begin{tikzcd}
\mathbf{SMet_1} \arrow[r, bend left, "\mathbb{C}"] & \mathbf{CSMet_1} \arrow[l, bend left, "\mathbb{I}"]
\end{tikzcd}$$

The result now follows from Theorem 11.6 and from the universal property implicit in the adjunction.

■

11.4 Pointed Interpolative Barycentric Algebras

All the results presented in the previous section can be readily extended to the case of subprobability measures by introducing a new constant in the signature where the "missing mass" can reside. These are called *pointed* interpolative barycentric algebras.

Definition 11.12 (Pointed Interpolative Barycentric Algebra) *Let $\mathcal{B}$ be the barycentric signature, c a constant and $\mathcal{B}^+ = \mathcal{B} \cup \{c : 0\}$. A* pointed barycentric algebra *is a quantitative algebra $\mathcal{A} = (A, \mathcal{B}^+, \mathcal{R})$ that satisfies the axioms of interpolative barycentric algebras.*

We maintain the notations of the previous section and denote by $\mathcal{U}^{IB^+}$ the approximated equational theory induced by the barycentric axioms over the terms of the signature $\mathcal{B}^+$.

Let $\mathbb{K}(\mathcal{B}^+, \mathcal{U}^{IB})$ be the class of the pointed interpolative barycentric algebras.

The finitary case

Let (M, d) be a one-bounded metric space and let $|M| = sup\{d(m, n) \mid m, n \in M\}$ be its diameter. Assume, without loss of generality, that $M \cap \{c\} = \emptyset$ and let $M^+ = M \cup \{c\}$. We extend d to M^+ by assuming that

$$\text{for any} \quad m \in M, \quad d(m, c) = d(c, m) = |M|.$$

Obviously, (M^+, d) is a one-bounded metric space.

If we denote by $\Pi^-[M]$ and $\Pi[M]$ the classes of subprobabilistic distributions with finite support on M and probabilistic distributions with finite support on M respectively, observe that there exists a bijective map

$$\pi : \Pi[M^+] \longrightarrow \Pi^-[M]$$

given for $\mu \in \Pi[M^+]$ by $\pi(\mu)(m) = \mu(m)$ for $m \in M$.

The definition of π also guarantees that it defines a morphism of $\mathcal{B}$-barycentric algebras and the metric induced is $W_d^p : \Pi^-[M] \times \Pi^-[M] \longrightarrow [0, 1]$, the p-Wasserstein metric of d for some $p \geq 1$, on subprobability distributions with finite support.

Consider now the barycentric algebra $\mathbb{T}^d[M^+]$ constructed as in Section 7. Recall that, by Corollary 11.7 there exists an (unique) isomorphism of barycentric algebras $h : \mathbb{T}^d[M^+] \longrightarrow \Pi[M^+]$.

We use this result to prove the following theorem.

Theorem 11.13 *$\mathbb{T}^d[M^+]$ and $\Pi^-[M]$ are isomorphic barycentric algebras and the isomorphism is given by the map*

$$\pi \circ h : \mathbb{T}^d[M^+] \longrightarrow \Pi^-[M].$$

Observe that $(\pi \circ h)(c) = 0$, the null distribution. We can use the null distribution to interpret c in the algebra $\Pi^-[M]$ and thus, we regard it as a pointed barycentric algebra. This entails the following corollary of Theorem 11.13.

Corollary 11.14 *$\mathbb{T}^d[M^+]$ and $\Pi^-[M]$ are isomorphic pointed interpolative barycentric algebras and the unique isomorphism is given by the map*

$$\pi \circ h : \mathbb{T}^d[M^+] \longrightarrow \Pi^-[M].$$

Applying Theorem 11.5, we get further the following result.

Theorem 11.15 *If (M, d) is an one-bounded metric space, then $\Pi^-[M] \in \mathbb{K}(\mathcal{B}^+, \mathcal{U}^{IB})$ and*

$$\Pi^-[M] \models t =_\epsilon s \textit{ iff } W_d^p((\pi \circ h)(t), (\pi \circ h)(s)) \leq \epsilon.$$

The continuous case

As in the previous section, we now focus on *general* subprobability distributions over a one-bounded complete separable metric space and prove that it is a pointed interpolative barycentric algebra. By "general" we mean that we are not restricting to finitely supported distributions. It turns out that this is the initial algebra in the category of the pointed interpolative barycentric algebras defined for separable metric spaces.

Consider a one-bounded complete separable metric space (M, d). Let $\overline{\mathbb{K}}^+$ be the class of pointed interpolative barycentric algebras with complete separable metric spaces.

Let $\Delta[M]$ and $\Delta^-[M]$ be the set of all Borel probability distributions on M and of all Borel subprobability distributions on M respectively. Also in this case there exists a bijective map $\pi : \Delta[M^+] \longrightarrow \Delta^-[M]$ given for $\mu \in \Delta[M^+]$ by $\pi(\mu)(m) = \mu(m)$ for $m \in M$.

Applying Theorem 11.10, $\Delta[M^+] \in \overline{\mathbb{K}}^+$ and the metric induced by the barycentric axioms coincides with the p-Wasserstein metric W_d^p on $\Delta[M^+]$. Moreover, $\Delta[M^+]$ is isomorphic to $\overline{\mathbb{T}^d[M^+]}$.

Now, since $\Delta[M^+]$ is isomorphic to $\Delta^-[M]$ and $\mathbb{T}^d[M^+]$ is isomorphic to $\Pi[M^+]$ which is further isomorphic to $\Pi^-[M]$, we obtain directly the following theorem.

Theorem 11.16 *If (M, d) is an one-bounded complete separable metric space, then $\Delta^-[M] \in \overline{\mathbb{K}}^+$ and the metric induced by the barycentric axioms coincides with the p-Wasserstein metric W_d^p on $\Delta^-[M]$. Moreover, $\Delta^-[M]$ is isomorphic to $\overline{\mathbb{T}^d[M^+]}$.*

12 Related work

The closest related work is by van Breugel et al. [vBHMW07] and by Adamek et al. [AMM12] both of which were important precursors to the present work. The first paper really shows why the Hausdorff and Kantorovich metrics are canonical. The second one shows the finitary natures of these monads. In the paper by van Breugel et al. [vBHMW07] it was shown that the Kantorovich functor is left adjoint to a forgetful functor from a suitable algebraic category (mean-value algebras) to complete metric spaces. Similarly they show that a suitable Hausdorff functor can be treated in a similar way. Their results are intended to exhibit the power of an approach to solving recursive equations using the theory of accessible categories. Adamek et al. [AMM12] have studied the finitary versions of the same functors and have given equational presentations.

A fairly important difference with the present work is that we use the barycentric axioms rather than the mean value axioms. The major difference, however, is our use of quantitative equations that capture the idea of approximate equality.

The difference between the mean-value axiomatization and the barycentric axiomatization may seem unimportant but we feel that barycentric algebras are more fundamental. They allow *all* binary choices to be directly available; they are of course all definable from the mean-value *if you allow infinite terms but certainly not if you want everything to be finitary.* The barycentric algebras are the axioms for abstract convex spaces and arise widely in mathematics; see the historical remarks in [KP15]. Barycentric algebras work very well in other settings too. For example, if one takes the free pointed barycentric algebras in other categories like sets or cpos one gets the structures one expects: finite probability distributions for the case of sets and the valuation powerdomain for the case of continuous dcpos.

We do not see as yet how all this fits with the program being pursued by Bart Jacobs and his group at Nijmegen where they have a general notion of quantitative logic based on structures that they call an "effectus."[CJWW15]. There are many intriguing possibilities but we must defer a proper comparison until we have digested effectus theory more deeply. One of the motivating strands of that work was various dualities involving convex structures so there certainly should be connections.

13 Future work

There is clearly much more to do both in the general theory and in specific examples. A fundamental task is to understand how to combine effects just as in the non-quantitative case; many of the basic results [HPP06, HLPP07] apply. It should be possible to extend the results of Section 10.2 to metrics that take extended real values by suitable rescalings of the metric.

We are actively looking at Markov processes as an example; this could benefit from a many-sorted extension of the basic theory or could alternatively use recursive domain equations. As far as we know, an equational presentation of Markov processes does not exist. Other possible examples are general distributions coming from a suitable axiomatization of cones and also an axiomatization of Choquet capacities which are of interest in games.

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