A metric analogue of Stone duality for Markov processes

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I. Introduction

The Stone representation [Sto36] theorem is one of the recognized landmarks of mathematics. The Stone representation theorem [Sto36] states that every (abstract) boolean algebra is isomorphic to a boolean algebra of sets; in modern terminology one has an equivalence of categories between the category of boolean algebras and the (opposite of) the category of compact Hausdorff zero-dimensional spaces, or Stone spaces.

In this paper we develop exactly such a duality for continuous-time continuous-space transitions systems where transitions are governed by an exponentially-distributed waiting time, essentially a continuous-time Markov chain (CTMC) with a continuous space. The logical characterization of bisimulation for such systems was proved a few years ago [DGJP03] using much the same techniques as were used for labelled Markov processes [Pan09]. Recent work by the first two authors and Cardelli [CLM11a], [CLM11b] have established completeness theorems and finite model theorems for similar logics. Thus it seemed ripe to capture these logics algebraically and explore duality theory.

One of the critiques of logics and equivalences being used for the treatment of probabilistic systems is that boolean logic is not robust with respect to small perturbations of the real-valued system parameters. Accordingly, a theory of metrics [DGJP04] was developed and metric reasoning principles were advocated. In conjunction with our exploration of duality theory therefore we investigated the role of metrics and discovered a striking metric analogue of the duality theory. This paper describes both these theories. One can view the latter as the analogue of a completeness theorem for metric reasoning principles.

One of the points of departure of the present work from earlier work is the use of hemimetrics: analogues of pseudometrics that are not symmetric. This fits in well with the order structure of the boolean algebra. Nearly 25 years ago, Mike Smyth [Smy87] advocated the use of such structure to combine metric and domain theory ideas. The interplay between the hemimetric and the boolean algebra is somewhat delicate and had to be carefully examined for the duality to emerge. It is a pleasant feature that exactly these axioms relating the hemimetric and the boolean algebra are satisfied in our examples without any artificial fiddling.

We summarize the key results of the present work:

- a description of a new class of algebras that captures, in algebraic form, the probabilistic modal logics used for continuous Markov processes,
- a duality between these algebras and continuous Markov processes
- a (hemi)metrized version of the algebras and of the Markov processes and
- a metric analogue of the duality.

II. Definitions

Let M be a set and $d : M \times M \to \mathbb{R}$.

Definition 1. *We say that d is a* hemimetric *on M if for arbitrary $x, y \in M$,*

$$\text{(1):} \quad d(x,x) = 0$$
$$\text{(2):} \quad d(x,y) \leq d(x,z) + d(z,y)$$

We say that (M, d) is a hemimetric space.

Note that a hemimetric is not necessarily symmetric nor does $d(x,y) = 0$ imply that $x = y$. A symmetric hemimetric is called a *pseudometric*.

Definition 2. *For a hemimetric d on M we define the* Hausdorff hemimetric *d^H on the class of subsets of X by*

$$d^H(X,Y) = \sup_{x \in X} \inf_{y \in Y} d(x,y).$$

We also define the dual of the Hausdorff hemimetric d_H on the class of subsets of X by

$$d_H(X,Y) = \sup_{y \in Y} \inf_{x \in X} d(x,y).$$

Definition 3 (Continuous Markov processes)**.** *Given a measurable space (M, Σ), a* continuous Markov process *(CMK) is a tuple $\mathcal{M} = (M, \Sigma, \theta)$, where $\theta \in [\![M \to \Delta(M, \Sigma)]\!]$. M is the* support set *of $\mathcal{M}$ denoted by $supp(\mathcal{M})$. If $m \in M$, $(\mathcal{M}, m)$ is a* continuous Markov process *(CMP).*

Definition 4 (Aumann algebra)**.** *An* Aumann algebra *(AA) over the set $B \neq \emptyset$ is a structure $\mathcal{A} = (B, \top, \bot, \sim, \sqcup, \sqcap, \{F_r, G_r\}_{r \in \mathbb{Q}^+}, \sqsubseteq)$ where $\mathcal{B} = (B, \top, \bot, \sim, \sqcup, \sqcap, \sqsubseteq)$ is a meet-continuous boolean Algebra, for each $r \in \mathbb{Q}^+$, F_r, G_r :*

$B \to B$ *are monadic operations and the elements of* B *satisfy the axioms in Table I, for arbitrary* $a, b \in B$ *and* $r, s \in \mathbb{Q}^+$.

(AA1):	$\top \sqsubseteq F_0 a$
(AA2):	$F_{r+s} a \sqsubseteq \sim G_r a$, for $s > 0$
(AA3):	$\sim F_r a \sqsubseteq G_r a$
(AA4):	$(\sim F_r(a \sqcap b)) \sqcap (\sim F_s(a \sqcap \sim b)) \sqsubseteq \sim F_{r+s} a$
(AA5):	$(\sim G_r(a \sqcap b)) \sqcap (\sim G_s(a \sqcap \sim b)) \sqsubseteq \sim G_{r+s} a$
(AA6):	If $a \sqsubseteq b$ then $F_r a \sqsubseteq F_r b$
(AA7):	$\bigwedge\{F_r b \mid r < s\} = F_s b$
(AA8):	$\bigwedge\{G_r b \mid r > s\} = G_s b$
(AA9):	$\bigwedge\{F_r b \mid r > s\} = \bot$

TABLE I
AUMANN ALGEBRA

Definition 5 (Metrized Aumann algebra)**.** *A* metrized Aumann algebra *is a tuple* $(\mathcal{A}, \delta)$*, where* $\mathcal{A} = (B, \top, \bot, \sim, \sqcup, \sqcap, \{F_r, G_r\}_{r\in\mathbb{Q}^+}, \sqsubseteq)$ *is an Aumann algebra and* $\delta : B \times B \to [0, 1]$ *is a hemimetric on* B *satisfying, for arbitrary* $a, b \in B$*, and arbitrary filtered set* $A \subseteq B$ *for which there exists* $\bigwedge A' in B$*, the axioms in Table II.*

(H0):	if $\delta(a, b) = 0$, then $a \sqsubseteq b$
(H1):	$\delta(a, b) = \delta(a \sqcap (\sim b), b)$
(H2):	$\delta(b, \bigwedge A) = \inf_{a\in A} \delta(b, a)$
(H3):	$\delta(\bigwedge A, b) = \sup_{a\in A} \delta(a, b)$

TABLE II
HEMIMETRIC AXIOMS FOR METRIZED AA

III. RESULTS

We have a duality theorem between CMPs and Aumann Algebras.

Theorem 6 (Representation Theorem)**.** *(i) Any CMP* $\mathcal{M} = (M, \Sigma, \theta)$ *is bisimilar to* $\mathcal{M}(\mathcal{L}(\mathcal{M}))$ *and the bisimulation relation is given by the mapping* α *defined, for arbitrary* $m \in M$*, by*

$$m \mapsto \alpha(m) = \{\phi \in \mathcal{L}(\mathcal{M}) \mid \mathcal{M}, m \models \phi\}.$$

(ii) Any Aumann algebra $\mathcal{A} = (B, \top, \bot, \sim, \sqcup, \sqcap, \{F_r, G_r\}_{r\in\mathbb{Q}^+}, \sqsubseteq)$ *is isomorphic to* $\mathcal{L}(\mathcal{M}(\mathcal{A}))$ *and the isomorphism is given by the mapping* β *defined, for arbitrary* $a \in B$*, by*

$$a \mapsto \beta(a) = \bigwedge(\{\phi \in \mathcal{A}(\mathcal{L}(\mathcal{M}(\mathcal{A}))) \mid \forall u \in \mathcal{U}(B) \ s.\ t.\ \uparrow(a) \subseteq u, \mathcal{M}(\mathcal{A}), u \models \phi\}).$$

This extends to a duality between the hemi-metric spaces in the following sense.

Theorem 7 (The metric duality theorem)**.** *(i) Given a metrized CMP* $(\mathcal{M}, d)$ *with* $\mathcal{M} = (M, \Sigma, \theta)$*,* $\mathcal{M}$ *is bisimilar to* $\mathcal{M}(\mathcal{A}(\mathcal{L}(\mathcal{M})))$ *by the map* α *defined in the Representation Theorem and, in addition, for arbitrary* $m, n \in M$*,*

$$d(m, n) = (d^H)_H(\alpha(m), \alpha(n)).$$

(ii) Given a metrized AA $(\mathcal{A}, \delta)$ *with* $\mathcal{A} = (B, \top, \bot, \sim, \sqcup, \sqcap, \{F_r, G_r\}_{r\in\mathbb{Q}^+}, \sqsubseteq)$*,* $\mathcal{A}$ *is isomorphic to* $\mathcal{A}(\mathcal{L}(\mathcal{M}(\mathcal{A})))$ *by the map* β *defined in the Representation Theorem and, in addition, for arbitrary* $a, b \in B$

$$\delta(a, b) = (\delta^H)_H(\beta(a), \beta(b)).$$

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