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An introduction to Lagrangian and Hamiltonian mechanics

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These notes are dedicated to Dr. Frank Berkshire whose enthusiasm and knowledge inspired me as a student. The lecture notes herein, are largely based on the first half of Frank's Dynamics course that I attended as a third year undergraduate at Imperial College in the Autumn term of 1989.

Preface

Newtonian mechanics took the Apollo astronauts to the moon. It also took the voyager spacecraft to the far reaches of the solar system. However Newtonian mechanics is a consequence of a more general scheme. One that brought us quantum mechanics, and thus the digital age. Indeed it has pointed us beyond that as well. The scheme is Lagrangian and Hamiltonian mechanics. Its original prescription rested on two principles. First that we should try to express the state of the mechanical system using the minimum representation possible and which reflects the fact that the physics of the problem is coordinate-invariant. Second, a mechanical system tries to optimize its *action* from one split second to the next. These notes are intended as an elementary introduction into these ideas and the basic prescription of Lagrangian and Hamiltonian mechanics. The only physical principles we require the reader to know are: (i) Newton's three laws; (ii) that the kinetic energy of a particle is a half its mass times the magnitude of its velocity squared; and (iii) that work/energy is equal to the force applied times the distance moved in the direction of the force.

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1.1 Example problems

Many physical problems involve the minimization (or maximization) of a quantity that is expressed as an integral.

Example 1 (Euclidean geodesic). Consider the path that gives the shortest distance between two points in the plane, say (x_1, y_1) and (x_2, y_2). Suppose that the general curve joining these two points is given by $y = y(x)$. Then our goal is to find the function $y(x)$ that minimizes the arclength:

$$\begin{aligned} J(y) &= \int_{(x_1,y_1)}^{(x_2,y_2)} \mathrm{d}s \\ &= \int_{x_1}^{x_2} \sqrt{1 + (y_x)^2}\, \mathrm{d}x. \end{aligned}$$

Here we have used that for a curve $y = y(x)$, if we make a small increment in x, say Δx, and the corresponding change in y is Δy, then by Pythagoras' theorem the corresponding change in length along the curve is $\Delta s = +\sqrt{(\Delta x)^2 + (\Delta y)^2}$. Hence we see that

$$\Delta s = \frac{\Delta s}{\Delta x}\, \Delta x = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x.$$

Note further that here, and hereafter, we use $y_x = y_x(x)$ to denote the derivative of y, i.e. $y_x(x) = y'(x)$ for each x for which the derivative is defined.

Example 2 (Brachistochrome problem; John and James Bernoulli 1697). Suppose a particle/bead is allowed to slide freely along a wire under gravity (force $F = -gk$ where k is the unit upward vertical vector) from a point (x_1, y_1) to the origin $(0, 0)$. Find the curve $y = y(x)$ that minimizes the time of descent:

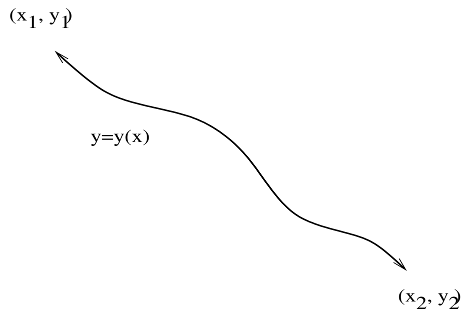

Fig. 1.1 In the Euclidean geodesic problem, the goal is to find the path with minimum total length between points (x_1, y_1) and (x_2, y_2).

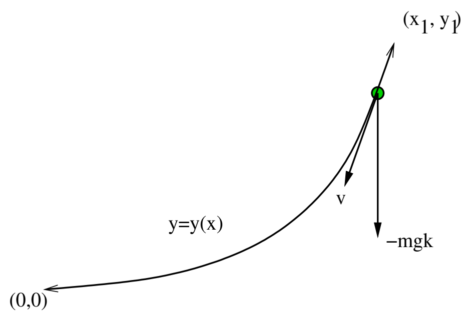

Fig. 1.2 In the Brachistochrome problem, a bead can slide freely under gravity along the wire. The goal is to find the shape of the wire that minimizes the time of descent of the bead.

$$
\begin{aligned}
J(y) &= \int_{(x_1,y_1)}^{(0,0)} \frac{1}{v}\,\mathrm{d}s \\
&= \int_{x_1}^{0} \frac{\sqrt{1+(y_x)^2}}{\sqrt{2g(y_1-y)}}\mathrm{d}x.
\end{aligned}
$$

Here we have used that the total energy, which is the sum of the kinetic and potential energies,

$$E = \tfrac{1}{2}mv^2 + mgy,$$

is constant. Assume the initial condition is $v = 0$ when $y = y_1$, i.e. the bead starts with zero velocity at the top end of the wire. Since its total energy is constant, its energy at any time t later, when its height is y and its velocity is v, is equal to its initial energy. Hence we have

$$\tfrac{1}{2}mv^2 + mgy = 0 + mgy_1 \qquad \Leftrightarrow \qquad v = +\sqrt{2g(y_1 - y)}.$$

1.2 Euler–Lagrange equation

We can see that the two examples above are special cases of a more general problem scenario.

Problem 1 (Classical variational problem). Suppose the given function F is twice continuously differentiable with respect to all of its arguments. Among all functions/paths $y = y(x)$, which are twice continuously differentiable on the interval $[a, b]$ with $y(a)$ and $y(b)$ specified, find the one which extremizes the *functional* defined by

$$J(y) := \int_a^b F(x, y, y_x)\,\mathrm{d}x.$$

Theorem 1 (Euler–Lagrange equation). *The function $u = u(x)$ that extremizes the functional J necessarily satisfies the Euler–Lagrange equation on $[a, b]$:*

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial u_x}\right) = 0.$$

Remark 1. Note for a given explicit function $F = F(x, y, y_x)$ for a given problem (such as the Euclidean geodesic and Brachistochrome problems above), we compute the partial derivatives $\partial F/\partial y$ and $\partial F/\partial y_x$ which will also be functions of x, y and y_x in general. Then using the chain rule to compute the term $(\mathrm{d}/\mathrm{d}x)(\partial F/\partial y_x)$, we see that the left hand side of the Euler–Lagrange equation will in general be a nonlinear function of x, y, y_x and y_{xx}. In other words the Euler–Lagrange equation represents a *nonlinear second order ordinary differential equation* for $y = y(x)$. This will be clearer when we consider explicit examples presently. The solution $y = y(x)$ of that ordinary differential equation which passes through $\big(a, y(a)\big)$ and $\big(b, y(b)\big)$ will be the function that extremizes J.

Proof. Consider the family of functions on $[a, b]$ given by

$$y^\epsilon(x) := u(x) + \epsilon\eta(x),$$

where the functions $\eta = \eta(x)$ are twice continuously differentiable and satisfy $\eta(a) = \eta(b) = 0$. Here ϵ is a small real parameter and the function $u = u(x)$ is our 'candidate' extremizing function. We set (see Evans [7, Section 3.3])

$$\varphi(\epsilon) := J(u + \epsilon\eta).$$

If the functional J has a local maximum or minimum at u, then u is a stationary function for J, and for all η we must have

$$\varphi'(0) = 0.$$

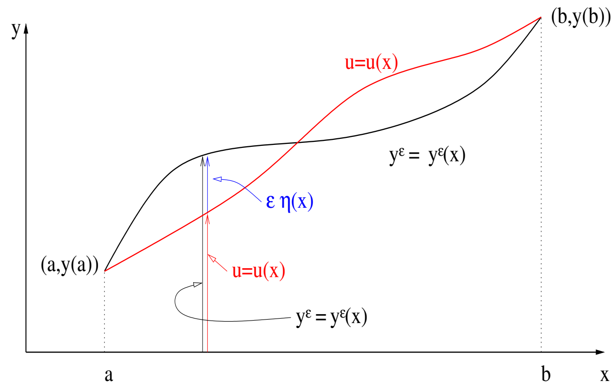

Fig. 1.3 We consider a family of functions $y^\epsilon := u + \epsilon\eta$ on $[a,b]$. The function $u = u(x)$ is the 'candidate' extremizing function and the functions $\epsilon\eta(x)$ represent perturbations from $u = u(x)$ which are parameterized by the real variable ϵ. We naturally assume $\eta(a) = \eta(b) = 0$.

To evaluate this condition for the integral functional J above, we first compute $\varphi'(\epsilon)$. By direct calculation with $y^\epsilon = u + \epsilon\eta$ and $y^\epsilon_x = u_x + \epsilon\eta_x$, we have

$$\begin{aligned}
\varphi'(\epsilon) &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} J(y^\epsilon) \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon} \int_a^b F\big(x, y^\epsilon(x), y^\epsilon_x(x)\big)\,\mathrm{d}x \\
&= \int_a^b \frac{\partial}{\partial\epsilon} F\big(x, y^\epsilon(x), y^\epsilon_x(x)\big)\,\mathrm{d}x \\
&= \int_a^b \left(\frac{\partial F}{\partial y^\epsilon}\frac{\partial y^\epsilon}{\partial\epsilon} + \frac{\partial F}{\partial y^\epsilon_x}\frac{\partial y^\epsilon_x}{\partial\epsilon}\right)\mathrm{d}x \\
&= \int_a^b \left(\frac{\partial F}{\partial y^\epsilon}\eta(x) + \frac{\partial F}{\partial y^\epsilon_x}\eta'(x)\right)\mathrm{d}x.
\end{aligned}$$

Note that we used the chain rule to write

$$\frac{\partial}{\partial\epsilon} F\big(x, y^\epsilon(x), y^\epsilon_x(x)\big) = \frac{\partial F}{\partial y^\epsilon}\frac{\partial y^\epsilon}{\partial\epsilon} + \frac{\partial F}{\partial y^\epsilon_x}\frac{\partial y^\epsilon_x}{\partial\epsilon}.$$

We use the integration by parts formula on the second term in the expression for $\varphi'(\epsilon)$ above to write it in the form

$$\int_a^b \left(\frac{\partial F}{\partial y^\epsilon_x}\right)\eta'(x)\,\mathrm{d}x = \left[\left(\frac{\partial F}{\partial y^\epsilon_x}\right)\eta(x)\right]_{x=a}^{x=b} - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y^\epsilon_x}\right)\eta(x)\,\mathrm{d}x.$$

Recall that $\eta(a) = \eta(b) = 0$, so the boundary term (first term on the right) vanishes in this last formula. Hence we see that

$$\varphi'(\epsilon) = \int_a^b \left(\frac{\partial F}{\partial y^\epsilon} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y_x^\epsilon} \right) \right) \eta(x) \, \mathrm{d}x.$$

If we now set $\epsilon = 0$, then the condition for u to be a critical point of J, which is $\varphi'(0) = 0$ for all η, is

$$\int_a^b \left(\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial u_x} \right) \right) \eta(x) \, \mathrm{d}x = 0,$$

for all η. Since this must hold for all functions $\eta = \eta(x)$, using Lemma 1 below, we can deduce that pointwise, i.e. for all $x \in [a, b]$, necessarily u must satisfy the Euler–Lagrange equation shown. □

Lemma 1 (Useful lemma). *If $\alpha(x)$ is continuous in $[a, b]$ and*

$$\int_a^b \alpha(x) \, \eta(x) \, \mathrm{d}x = 0$$

for all continuously differentiable functions $\eta(x)$ which satisfy $\eta(a) = \eta(b) = 0$, then $\alpha(x) \equiv 0$ in $[a, b]$.

Proof. Assume $\alpha(z) > 0$, say, at some point $a < z < b$. Then since α is continuous, we must have that $\alpha(x) > 0$ in some open neighbourhood of z, say in $a < \underline{z} < z < \overline{z} < b$. The choice

$$\eta(x) = \begin{cases} (x - \underline{z})^2 (\overline{z} - x)^2, & \text{for } x \in [\underline{z}, \overline{z}], \\ 0, & \text{otherwise}, \end{cases}$$

which is a continuously differentiable function, implies

$$\int_a^b \alpha(x) \, \eta(x) \, \mathrm{d}x = \int_{\underline{z}}^{\overline{z}} \alpha(x) \, (x - \underline{z})^2 (\overline{z} - x)^2 \, \mathrm{d}x > 0,$$

a contradiction. □

Remark 2. Some important theoretical and practical points to keep in mind are as follows.

1. The Euler–Lagrange equation is a *necessary* condition: if such a $u = u(x)$ exists that extremizes J, then u satisfies the Euler–Lagrange equation. Such a u is known as a *stationary function* of the *functional* J.
2. Note that the extremal solution u is independent of the coordinate system you choose to represent it (see Arnold [3, Page 59]). For example, in the Euclidean geodesic problem, we could have used polar coordinates (r, θ), instead of Cartesian coordinates (x, y), to express the total arclength J.

Formulating the Euler–Lagrange equations in these coordinates and then solving them will tell us that the extremizing solution is a straight line (only it will be expressed in polar coordinates).

3. Let $\mathbb{Y}$ denote a function space. In the context above $\mathbb{Y}$ was the space of twice continuously differentiable functions on $[a, b]$ which are fixed at $x = a$ and $x = b$. A *functional* is a real-valued map and here $J : \mathbb{Y} \to \mathbb{R}$.
4. We define the *first variation* $\delta J(u, \eta)$ of the functional J, at u in the direction η, to be $\delta J(u, \eta) := \varphi'(0)$.
5. Is u a maximum, minimum or saddle point for J? The physical context should hint towards what to expect. Higher order variations will give you the appropriate mathematical determination.
6. The functional J has a local minimum at u iff there is an open neighbourhood $U \subset \mathbb{Y}$ of u such that $J(y) \geq J(u)$ for all $y \in U$. The functional J has a local maximum at u when this inequality is reversed.
7. We generalize all these notions to multidimensions and systems presently.

Remark 3. An essential component in the proof of the Euler–Lagrange equation is the chain rule. We re-iterate here as it plays an important role throughout these notes. Suppose $f = f(u, v, w)$ is a function of three variables u, v and w. Suppose that $u = u(x, \epsilon)$, $v = v(x, \epsilon)$ and $w = w(x, \epsilon)$ are themselves functions of the variables x and ϵ. Then f is ultimately a function of x and ϵ as well and, for example, the chain states that

$$\frac{\partial f}{\partial \epsilon} \equiv \frac{\partial f}{\partial u}\,\frac{\partial u}{\partial \epsilon} + \frac{\partial f}{\partial v}\,\frac{\partial v}{\partial \epsilon} + \frac{\partial f}{\partial w}\,\frac{\partial w}{\partial \epsilon}.$$

The chain rule provides a mechanism to compute the partial derivative $\partial f / \partial \epsilon$ by breaking the calculation down to computing the partial derivatives on the right shown. Note that the partial derivative $\partial f / \partial u$ is computed keeping v and w fixed, while $\partial u / \partial \epsilon$ is computed keeping x fixed, and so forth. Note that the product rule simply corresponds to the special case $f = u\,v$, so that

$$\frac{\partial}{\partial \epsilon}(u\,v) = v\,\frac{\partial u}{\partial \epsilon} + u\,\frac{\partial v}{\partial \epsilon}.$$

Solution 1 (Euclidean geodesic). Recall, this variational problem concerns finding the shortest distance between the two points (x_1, y_1) and (x_2, y_2) in the plane. This is equivalent to minimizing the total arclength functional

$$J(y) = \int_{x_1}^{x_2} \sqrt{1 + (y_x)^2}\,\mathrm{d}x.$$

Hence in this case the *integrand* we denoted by $F = F(x, y, y_x)$ in the general theory above is

$$F(y_x) = \sqrt{1 + (y_x)^2}.$$

In particular, in this example, we note that $F = F(y_x)$ only. From the general theory outlined above, we know that the extremizing solution satisfies the

Euler–Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_x}\right) = 0.$$

Substituting the actual form for F we have in this case and using that $\partial F/\partial y = 0$ since $F = F(y_x)$ only, gives

$$\begin{aligned}
& -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial}{\partial y_x}\left(\left(1+(y_x)^2\right)^{\frac{1}{2}}\right)\right) = 0 \\
\Leftrightarrow \quad & \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y_x}{\left(1+(y_x)^2\right)^{\frac{1}{2}}}\right) = 0 \\
\Leftrightarrow \quad & \frac{y_{xx}}{\left(1+(y_x)^2\right)^{\frac{1}{2}}} - \frac{(y_x)^2 y_{xx}}{\left(1+(y_x)^2\right)^{\frac{3}{2}}} = 0 \\
\Leftrightarrow \quad & \frac{\left(1+(y_x)^2\right) y_{xx}}{\left(1+(y_x)^2\right)^{\frac{3}{2}}} - \frac{(y_x)^2 y_{xx}}{\left(1+(y_x)^2\right)^{\frac{3}{2}}} = 0 \\
\Leftrightarrow \quad & \frac{y_{xx}}{\left(1+(y_x)^2\right)^{\frac{3}{2}}} = 0 \\
\Leftrightarrow \quad & y_{xx} = 0.
\end{aligned}$$

Hence $y(x) = c_1 + c_2 x$ for some constants c_1 and c_2. Using the initial and starting point data we see that the solution is the straight line function

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1.$$

Note that this calculation might have been a bit shorter if we had recognised that this example corresponds to the third special case in the next section.

1.3 Alternative form and special cases

Lemma 2 (Alternative form). *The Euler–Lagrange equation given by*

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_x}\right) = 0$$

is equivalent to the equation

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(F - y_x \frac{\partial F}{\partial y_x}\right) = 0.$$

Proof. It is easiest to prove this result by starting with the second (alternative) form for the Euler–Lagrange equation and showing that it is equivalent to the first (original) form for the Euler–Lagrange equation above. The chain rule and product rules are required as follows. If $u = u(x)$, $v = v(x)$ and $w = w(x)$ are functions of x only and $F = F(u, v, w)$ then the chain rule tells us that

$$\frac{\mathrm{d}F}{\mathrm{d}x} \equiv \frac{\partial F}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial F}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\partial F}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}x}.$$

Then for example if $u(x) \equiv x$, $v(x) \equiv y(x)$ and $w(x) \equiv y_x(x)$ we have

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial y_x}\frac{\mathrm{d}y_x}{\mathrm{d}x}.$$

If $u = u(x)$ and $v = v(x)$ then the product rule tells us that

$$\frac{\mathrm{d}}{\mathrm{d}x}(u\,v) \equiv \frac{\mathrm{d}u}{\mathrm{d}x}\,v + u\,\frac{\mathrm{d}v}{\mathrm{d}x}.$$

Then if $u = y_x$ and $v = \partial F/\partial y_x$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(y_x \frac{\partial F}{\partial y_x}\Big) = y_{xx}\frac{\partial F}{\partial y_x} + y_x \frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{\partial F}{\partial y_x}\Big).$$

Using these applications of the chain and product rules in the alternative Euler–Lagrange equation generates, after cancellation, the original form. □

Corollary 1 (Special cases). *When the integrand F does not explicitly depend on one or more variables, then the Euler–Lagrange equations simplify considerably. We have the following three notable cases:*

1. *If $F = F(y, y_x)$ only, i.e. it does not* explicitly *depend on x, then the alternative form for the Euler–Lagrange equation implies*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F - y_x\frac{\partial F}{\partial y_x}\right) = 0 \qquad \Leftrightarrow \qquad F - y_x\frac{\partial F}{\partial y_x} = c,$$

   for some arbitrary constant c.
2. *If $F = F(x, y_x)$ only, i.e. it does not* explicitly *depend on y, then the Euler–Lagrange equation implies*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_x}\right) = 0 \qquad \Leftrightarrow \qquad \frac{\partial F}{\partial y_x} = c,$$

   for some arbitrary constant c.
3. *If $F = F(y_x)$ only, then the Euler–Lagrange equation implies $y_{xx} = 0$, i.e. $y = y(x)$ is a linear function of x and has the form*

$$y = c_1 + c_2 x,$$

for some constants c_1 and c_2.

Proof. We only need to prove Item 3 in the statement of the corollary. We use the chain rule—see the form given in the proof of Lemma 2. Replacing F by $\partial F/\partial y_x$ and setting $u(x) \equiv x$, $v(x) \equiv y(x)$ and $w(x) \equiv y_x(x)$, we see that the Euler–Lagrange equation implies

$$\begin{aligned} 0 &= \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_x}\right) \\ &= \frac{\partial F}{\partial y} - \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y_x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial}{\partial y_x}\left(\frac{\partial F}{\partial y_x}\right)\frac{\mathrm{d}y_x}{\mathrm{d}x}\right) \\ &= \frac{\partial F}{\partial y} - \left(\frac{\partial^2 F}{\partial x \partial y_x} + \frac{\partial^2 F}{\partial y \partial y_x}\, y_x + \frac{\partial^2 F}{\partial y_x \partial y_x}\, y_{xx}\right) \\ &= \frac{\partial F}{\partial y} - \left(\frac{\partial^2 F}{\partial y_x \partial x} + \frac{\partial^2 F}{\partial y_x \partial y}\, y_x + \frac{\partial^2 F}{\partial y_x \partial y_x}\, y_{xx}\right) \\ &= -y_{xx}\frac{\partial^2 F}{\partial y_x \partial y_x}. \end{aligned}$$

From the penultimate to ultimate line we used that $F = F(y_x)$ only. Hence assuming $\partial^2 F/\partial y_x \partial y_x \neq 0$ the result follows. □

Solution 2 (Brachistochrome problem). Recall, this variational problem concerns a particle/bead which can freely slide along a wire under the force of gravity. The wire is represented by a curve $y = y(x)$ from (x_1, y_1) to the origin $(0, 0)$. The goal is to find the shape of the wire, i.e. $y = y(x)$, which minimizes the time of descent of the bead, which is given by the functional

$$J(y) = \int_{x_1}^{0} \sqrt{\frac{1 + (y_x)^2}{2g(y_1 - y)}}\,\mathrm{d}x = \frac{1}{\sqrt{2g}} \int_{x_1}^{0} \sqrt{\frac{1 + (y_x)^2}{(y_1 - y)}}\,\mathrm{d}x.$$

Hence in this case, the integrand we denoted by $F = F(x, y, y_x)$ in the general theory above is

$$F(y, y_x) = \sqrt{\frac{1 + (y_x)^2}{(y_1 - y)}}.$$

From the general theory, we know that the extremizing solution satisfies the Euler–Lagrange equation. Note that the multiplicative constant factor $1/\sqrt{2g}$ should not affect the extremizing solution path; indeed it divides out of the Euler–Lagrange equations. Noting that the integrand F does not explicitly depend on x, then the alternative form for the Euler–Lagrange equation may be easier:

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(F - y_x\frac{\partial F}{\partial y_x}\right) = 0.$$

This implies that for some constant c, the Euler–Lagrange equation is

$$F - y_x \frac{\partial F}{\partial y_x} = c.$$

Now substituting the form for F into this gives

$$\begin{aligned}
& \frac{\big(1+(y_x)^2\big)^{\frac{1}{2}}}{(y_1-y)^{\frac{1}{2}}} - y_x \frac{\partial}{\partial y_x}\left(\frac{\big(1+(y_x)^2\big)^{\frac{1}{2}}}{(y_1-y)^{\frac{1}{2}}}\right) = c \\
\Leftrightarrow\quad & \frac{\big(1+(y_x)^2\big)^{\frac{1}{2}}}{(y_1-y)^{\frac{1}{2}}} - \frac{y_x}{(y_1-y)^{\frac{1}{2}}} \cdot \frac{\partial}{\partial y_x}\Big(\big(1+(y_x)^2\big)^{\frac{1}{2}}\Big) = c \\
\Leftrightarrow\quad & \frac{\big(1+(y_x)^2\big)^{\frac{1}{2}}}{(y_1-y)^{\frac{1}{2}}} - \frac{y_x}{(y_1-y)^{\frac{1}{2}}} \cdot \frac{\frac{1}{2}\cdot 2 \cdot y_x}{\big(1+(y_x)^2\big)^{\frac{1}{2}}} = c \\
\Leftrightarrow\quad & \frac{\big(1+(y_x)^2\big)^{\frac{1}{2}}}{(y_1-y)^{\frac{1}{2}}} - \frac{(y_x)^2}{(y_1-y)^{\frac{1}{2}}\big(1+(y_x)^2\big)^{\frac{1}{2}}} = c \\
\Leftrightarrow\quad & \frac{1+(y_x)^2}{(y_1-y)^{\frac{1}{2}}\big(1+(y_x)^2\big)^{\frac{1}{2}}} - \frac{(y_x)^2}{(y_1-y)^{\frac{1}{2}}\big(1+(y_x)^2\big)^{\frac{1}{2}}} = c \\
\Leftrightarrow\quad & \frac{1}{(y_1-y)\big(1+(y_x)^2\big)^{\frac{1}{2}}} = c \\
\Leftrightarrow\quad & \frac{1}{(y_1-y)\big(1+(y_x)^2\big)} = c^2.
\end{aligned}$$

We can now rearrange this equation so that

$$\begin{aligned}
& (y_x)^2 = \frac{1}{c^2(y_1-y)} - 1 \\
\Leftrightarrow\quad & (y_x)^2 = \frac{1-c^2y_1+c^2y}{c^2y_1-c^2y} \\
\Leftrightarrow\quad & (y_x)^2 = \frac{\frac{1}{c^2}-y_1+y}{y_1-y}.
\end{aligned}$$

If we set $a = y_1$ and $b = \frac{1}{c^2} - y_1$, then this equation becomes

$$y_x = \left(\frac{b+y}{a-y}\right)^{\frac{1}{2}}.$$

To find the solution to this ordinary differential equation we make the change of variable from $y = y(x)$ to the variable $\theta = \theta(x)$ where the functions y and θ are related as follows

$$y = \tfrac{1}{2}(a-b) - \tfrac{1}{2}(a+b)\cos\theta.$$

Using the chain rule this implies

$$y_x = \tfrac{1}{2}(a+b)\sin\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x}.$$

If we substitute these expressions for y and y_x into the ordinary differential equation above we get

$$\tfrac{1}{2}(a+b)\sin\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}x} = \left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{\frac{1}{2}}.$$

Now we use that $1/(\mathrm{d}\theta/\mathrm{d}x) = \mathrm{d}x/\mathrm{d}\theta$, and that $\sin\theta = \sqrt{1-\cos^2\theta}$, to get

$$\begin{aligned}
&& \frac{\mathrm{d}x}{\mathrm{d}\theta} &= \tfrac{1}{2}(a+b)\sin\theta \cdot \left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{\frac{1}{2}} \\
\Leftrightarrow && \frac{\mathrm{d}x}{\mathrm{d}\theta} &= \tfrac{1}{2}(a+b)\cdot(1-\cos^2\theta)^{\frac{1}{2}} \cdot \left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{\frac{1}{2}} \\
\Leftrightarrow && \frac{\mathrm{d}x}{\mathrm{d}\theta} &= \tfrac{1}{2}(a+b)\cdot(1+\cos\theta)^{\frac{1}{2}}(1-\cos\theta)^{\frac{1}{2}} \cdot \left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{\frac{1}{2}} \\
\Leftrightarrow && \frac{\mathrm{d}x}{\mathrm{d}\theta} &= \tfrac{1}{2}(a+b)(1+\cos\theta).
\end{aligned}$$

We can directly integrate this last equation to find x as a function of θ. In other words we can find the solution to the ordinary differential equation for $y = y(x)$ above in parametric form, which with some minor rearrangement, can expressed as (here d is an arbitrary constant of integration)

$$\begin{aligned}
x + d &= \tfrac{1}{2}(a+b)(\theta + \sin\theta), \\
y + b &= \tfrac{1}{2}(a+b)(1 - \cos\theta).
\end{aligned}$$

This is the parametric representation of a *cycloid*.

Remark 4. We note that in many example problems the integrand F has the form $F(y, y_x) = f(y)\left(1 + (y_x)^2\right)^{1/2}$ for some function $f = f(y)$. In this case the Euler–Lagrange equation, using the alternative form, implies that for an arbitrary constant c we have

$$\begin{aligned}
&& F - y_x \frac{\partial F}{\partial y_x} &= c \\
\Leftrightarrow && f(y)\left(1+(y_x)^2\right)^{1/2} - y_x\, f(y)\,\frac{y_x}{\left(1+(y_x)^2\right)^{1/2}} &= c \\
\Leftrightarrow && \frac{f(y)}{\left(1+(y_x)^2\right)^{1/2}} &= c.
\end{aligned}$$

1.4 Multivariable systems

We consider possible generalizations of functionals to be extremized. For more details see for example Keener [9, Chapter 5].

Problem 2 (Higher derivatives). Suppose we are asked to find the curve $y = y(x) \in \mathbb{R}$ that extremizes the functional

$$J(y) \coloneqq \int_a^b F(x, y, y_x, y_{xx})\,\mathrm{d}x,$$

subject to $y(a)$, $y(b)$, $y_x(a)$ and $y_x(b)$ being fixed. Here the functional quantity to be extremized depends on the curvature y_{xx} of the path. Necessarily the extremizing curve y satisfies the Euler–Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_x}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial y_{xx}}\right) = 0.$$

This is in general a nonlinear third order ordinary differential equation for $y = y(x)$. Note this follows by analogous arguments to those used in the proof for the classical variational problem above. Essentially, here we use the chain rule to expand the following term in the integral over x between a and b,

$$\begin{aligned}\frac{\partial}{\partial \epsilon} F\big(x, y^\epsilon(x), y_x^\epsilon(x), y_{xx}^\epsilon(x)\big) &= \frac{\partial F}{\partial y^\epsilon}\frac{\partial y^\epsilon}{\partial \epsilon} + \frac{\partial F}{\partial y_x^\epsilon}\frac{\partial y_x^\epsilon}{\partial \epsilon} + \frac{\partial F}{\partial y_{xx}^\epsilon}\frac{\partial y_{xx}^\epsilon}{\partial \epsilon} \\ &= \frac{\partial F}{\partial y^\epsilon}\,\eta + \frac{\partial F}{\partial y_x^\epsilon}\,\eta' + \frac{\partial F}{\partial y_{xx}^\epsilon}\,\eta''.\end{aligned}$$

We use integration by parts for the term $(\partial F/\partial y_x^\epsilon)\,\eta'$ as previously, and integration by parts twice for the term $(\partial F/\partial y_{xx}^\epsilon)\,\eta''$.

Question 1. Can you guess what the correct form for the Euler–Lagrange equation should be if $F = F(x, y, y_x, y_{xx}, y_{xxx})$ and so forth?

Problem 3 (Multiple dependent variables). Suppose we are asked to find the multidimensional curve $\boldsymbol{y} = \boldsymbol{y}(x) \in \mathbb{R}^N$ that extremizes the functional

$$J(\boldsymbol{y}) \coloneqq \int_a^b F(x, \boldsymbol{y}, \boldsymbol{y}_x)\,\mathrm{d}x,$$

subject to $\boldsymbol{y}(a)$ and $\boldsymbol{y}(b)$ being fixed. Note $x \in [a, b]$ but here $\boldsymbol{y} = \boldsymbol{y}(x)$ is a curve in N-dimensional space and is thus a vector so that (here $' \equiv \mathrm{d}/\mathrm{d}x$)

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{y}_x = \begin{pmatrix} y_1' \\ \vdots \\ y_N' \end{pmatrix}.$$

Necessarily the extremizing curve $\boldsymbol{y}$ satisfies a set of Euler–Lagrange equations, which are equivalent to a system of ordinary differential equations, given for $i = 1, \ldots, N$ by:

$$\frac{\partial F}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_i'}\right) = 0.$$

Problem 4 (Multiple independent variables). Suppose we are asked to find the field $y = y(\boldsymbol{x})$ that, for $\boldsymbol{x} \in \Omega \subseteq \mathbb{R}^n$, extremizes the functional

$$J(y) := \int_\Omega F(\boldsymbol{x}, y, \nabla y)\,\mathrm{d}\boldsymbol{x},$$

subject to y being fixed at the boundary $\partial\Omega$ of the domain Ω. Note here

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and $\nabla y \equiv \nabla_{\boldsymbol{x}} y$ is the gradient of y, i.e. it is the vector of partial derivatives of y with respect to each of the components x_i $(i = 1, \ldots, n)$ of $\boldsymbol{x}$, i.e.

$$\nabla y = \begin{pmatrix} \partial y/\partial x_1 \\ \vdots \\ \partial y/\partial x_n \end{pmatrix}.$$

Necessarily y satisfies an Euler–Lagrange equation which is a partial differential equation given by

$$\frac{\partial F}{\partial y} - \nabla \cdot \left(\nabla_{y_{\boldsymbol{x}}} F\right) = 0.$$

Here '$\nabla \cdot \; \equiv \nabla_{\boldsymbol{x}} \cdot$' is the usual divergence operator with respect to $\boldsymbol{x}$, and

$$\nabla_{y_{\boldsymbol{x}}} F := \begin{pmatrix} \partial F/\partial y_{x_1} \\ \vdots \\ \partial F/\partial y_{x_n} \end{pmatrix},$$

where to keep the formula readable, with ∇y the usual gradient of y, we have set $y_{\boldsymbol{x}} \equiv \nabla y$ so that $y_{x_i} = (\nabla y)_i = \partial y/\partial x_i$ for $i = 1, \ldots, n$.

*Example 3 (**Laplace's equation**).* The variational problem here is to find the field $\psi = \psi(x_1, x_2, x_3)$, for $\boldsymbol{x} = (x_1, x_2, x_3)^{\mathrm{T}} \in \Omega \subseteq \mathbb{R}^3$, that extremizes the mean-square gradient average

$$J(\psi) := \int_\Omega |\nabla\psi|^2\,\mathrm{d}\boldsymbol{x}.$$

In this case the integrand of the functional J is

$$F(\boldsymbol{x}, \psi, \nabla\psi) = |\nabla\psi|^2 \equiv (\psi_{x_1})^2 + (\psi_{x_2})^2 + (\psi_{x_3})^2.$$

Note that the integrand F depends on the partial derivatives of ψ only. Using the form for the Euler–Lagrange equation above we get

$$\begin{aligned}
& & -\nabla_{\boldsymbol{x}} \cdot \big(\nabla_{\psi_{\boldsymbol{x}}} F\big) &= 0 \\
\Leftrightarrow & & \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} \cdot \begin{pmatrix} \partial F/\partial \psi_{x_1} \\ \partial F/\partial \psi_{x_2} \\ \partial F/\partial \psi_{x_3} \end{pmatrix} &= 0 \\
\Leftrightarrow & & \frac{\partial}{\partial x_1}\big(2\psi_{x_1}\big) + \frac{\partial}{\partial x_2}\big(2\psi_{x_2}\big) + \frac{\partial}{\partial x_3}\big(2\psi_{x_3}\big) &= 0 \\
\Leftrightarrow & & \psi_{x_1 x_1} + \psi_{x_2 x_2} + \psi_{x_3 x_3} &= 0 \\
\Leftrightarrow & & \nabla^2 \psi &= 0.
\end{aligned}$$

This is *Laplace's equation* for ψ in the domain Ω; the solutions are called *harmonic functions*. Note that implicit in writing down the Euler–Lagrange partial differential equation above, we assumed that ψ was fixed at the boundary $\partial\Omega$, i.e. Dirichlet boundary conditions were specified.

*Example 4 (**Stretched vibrating string**).* Suppose a string is tied between the two fixed points $x = 0$ and $x = \ell$. Let $y = y(x,t)$ be the small displacement of the string at position $x \in [0,\ell]$ and time $t > 0$ from the equilibrium position $y = 0$. If μ is the uniform mass per unit length of the string which is stretched to a tension K, the kinetic and potential energy of the string are given by

$$T \coloneqq \tfrac{1}{2}\mu \int_0^\ell (y_t)^2 \, \mathrm{d}x, \qquad \text{and} \qquad V \coloneqq K\bigg(\int_0^\ell \big(1 + (y_x)^2\big)^{1/2} \, \mathrm{d}x - \ell\bigg)$$

respectively, where subscripts indicate partial derivatives and the effect of gravity is neglected. If the oscillations of the string are quite small, we can use the binomial expansion to approximate

$$\big(1 + (y_x)^2\big)^{1/2} \approx 1 + \tfrac{1}{2}(y_x)^2.$$

We thus approximate the expression for the potential energy V by

$$V \approx \frac{1}{2} K \int_0^\ell (y_x)^2 \, \mathrm{d}x.$$

We seek a solution $y = y(x,t)$ that extremizes the functional (this is the action functional as we see later)

$$\begin{aligned}J(y) &:= \int_{t_1}^{t_2} (T - V)\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} \int_0^{\ell} \tfrac{1}{2}\mu(y_t)^2 - \tfrac{1}{2}K(y_x)^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where t_1 and t_2 are two arbitrary times. In this case, with

$$\boldsymbol{x} = \begin{pmatrix} x \\ t \end{pmatrix} \qquad \text{and} \qquad \nabla y = \begin{pmatrix} \partial y/\partial x \\ \partial y/\partial t \end{pmatrix}$$

the integrand which in the general theory is denoted $F = F(\boldsymbol{x}, y, \nabla y)$ is

$$F(\nabla y) \equiv \tfrac{1}{2}\mu(y_t)^2 - \tfrac{1}{2}K(y_x)^2,$$

The Euler–Lagrange equation is thus

$$\begin{aligned} & -\nabla_{\boldsymbol{x}} \cdot \big(\nabla_{y_{\boldsymbol{x}}} F\big) = 0 \\ \Leftrightarrow \quad & \begin{pmatrix} \partial/\partial x \\ \partial/\partial t \end{pmatrix} \cdot \begin{pmatrix} \partial F/\partial y_x \\ \partial F/\partial y_t \end{pmatrix} = 0 \\ \Leftrightarrow \quad & \frac{\partial}{\partial x}\big(K y_x\big) - \frac{\partial}{\partial t}\big(\mu y_t\big) = 0 \\ \Leftrightarrow \quad & c^2\, y_{xx} - y_{tt} = 0, \end{aligned}$$

where $c^2 = K/\mu$. The partial differential equation $y_{tt} = c^2\, y_{xx}$ is known as the wave equation. It admits travelling wave solutions $y = y(x \pm ct)$ of speed $\pm c$.

1.5 Lagrange multipliers

For the moment, let us temporarily put aside variational calculus and consider a problem in standard multivariable calculus.

Problem 5 (Constrained optimization problem). Find the stationary points of the scalar function $f(\boldsymbol{x})$ where $\boldsymbol{x} = (x_1, \ldots, x_N)^{\mathrm{T}}$ subject to the constraints $g_k(\boldsymbol{x}) = 0$, where $k = 1, \ldots, m$, with $m < N$.

Note that the graph $y = f(\boldsymbol{x})$ of the function f represents a hyper-surface in $(N+1)$-dimensional space. The constraints are given implicitly; each one also represents a hyper-surface in $(N+1)$-dimensional space. In principle we could solve the system of m constraint equations, say for $x_1, \ldots, x_m$ in terms of the remaining variables $x_{m+1}, \ldots, x_N$. We could then substitute these into f, which would now be a function of $(x_{m+1}, \ldots, x_N)$ only. (We could solve the constraints for any subset of m variables x_i and substitute those in if we wished if this was easier, or avoided singularities, and so forth.) We would then proceed in the usual way to find the stationary points of f by considering the partial derivative of f with respect to all the remaining variables $x_{m+1}, \ldots, x_N$, setting those partial derivatives equal to zero, and then solving that system of equations. However solving the constraint equations may be very difficult, and the method of *Lagrange multipliers* provides an elegant alternative (see McCallum *et. al.* [14, Section 14.3]).

Example 5. Suppose we are given an explicit function $f = f(x, y, z)$ and the following two functions

$$g_1(x, y, z) = x^2 + y^2 + z^2 - 1,$$
$$g_2(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1,$$

where a, b and c are positive real constants with $a > 1$ and $b = c < 1$. We might imagine for example that f represents the temperature in a room or some such physical quantity. Assume the origin $x = 0$, $y = 0$, $z = 0$ is in the centre of the box-shaped room. Suppose we are asked, subject to the constraints $g_1 = 0$ and $g_2 = 0$, to find where in the room the temperature is highest. The constraint $g_1 = 0$ specifies that x, y and z are restricted/constrained to satisfy

$$x^2 + y^2 + z^2 = 1.$$

In other words our search for the highest temperature is restricted to the surface specified by this relation which is a sphere of radius one. We have an additional restriction $g_2 = 0$ however which specifies that x, y and z are restricted to satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

In other words our search for the highest temperature is restricted to the surface specified by this relation which is the surface of an ellipsoid—it is an ellipsoidal surface if revolution which is elongated along the x-axis. Our search for the highest temperature is thus restricted to the region of the room specified by both constraints $g_1 = 0$ and $g_2 = 0$. That region of the room is precisely where the two surfaces, the sphere and ellipsoid, intersect. Naturally the locus of the intersection of two surfaces in three-dimensional space are curves. In this case the curves are two identical concentric rings to the x-axis, equidistant to the origin. So our search boils down to finding where on these two rings the temperature f is highest.

At the algebraic level, we could solve the equation for the sphere to write z in terms of x and y as follows,

$$z^2 = 1 - x^2 - y^2.$$

We could take the square root to find z explicitly if required. We can solve the equation for the ellipsoid analogously to find z in terms of x and y as follows,

$$z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

Equating these two expressions for z^2 we can in principle find an expression for y in terms of x, i.e. an expression $y = y(x)$. We can substitute this into either of the expressions for z above to obtain z as a function of x only, i.e. an expression $z = z(x)$. If we then substitute these into $f = f(x, y, z)$ to obtain $f = f\big(x, y(x), z(x)\big)$ we have a function f of one variable only, namely x. We can in principle then proceed to find the global maximum of this function $f\big(x, y(x), z(x)\big)$ of x only using standard single variable analysis techniques.

The idea of the *method of Lagrange multipliers* is to convert the constrained optimization problem to an 'unconstrained' one as follows. Form the *Lagrangian function*

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) \coloneqq f(\boldsymbol{x}) + \sum_{k=1}^{m} \lambda_k \, g_k(\boldsymbol{x}),$$

where the parameter variables $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)^{\mathrm{T}}$ are known as the *Lagrange multipliers*. Note $\mathcal{L}$ is a function of both $\boldsymbol{x}$ and $\boldsymbol{\lambda}$, i.e. of $N + m$ variables. The partial derivatives of $\mathcal{L}$ with respect to all of its dependent variables are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_j} &= \frac{\partial f}{\partial x_j} + \sum_{k=1}^{m} \lambda_k \, \frac{\partial g_k}{\partial x_j}, \\
\frac{\partial \mathcal{L}}{\partial \lambda_k} &= g_k,
\end{aligned}$$

where $j = 1, \ldots, N$ and $k = 1, \ldots, m$. At the stationary points of $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda})$, necessarily all these partial derivatives must be zero, and we must solve the following 'unconstrained' problem:

$$\begin{aligned}
\frac{\partial f}{\partial x_j} + \sum_{k=1}^{m} \lambda_k \, \frac{\partial g_k}{\partial x_j} &= 0, \\
g_k &= 0,
\end{aligned}$$

where $j = 1, \ldots, N$ and $k = 1, \ldots, m$. Note we have $N+m$ equations in $N+m$ unknowns. Assuming that we can solve this system to find a stationary point $(\boldsymbol{x}_*, \boldsymbol{\lambda}_*)$ of $\mathcal{L}$ (there could be none, one, or more) then $\boldsymbol{x}_*$ is *also* a stationary point of the original *constrained* problem. Recall: what is important about the formulation of the Lagrangian function $\mathcal{L}$ we introduced above, is that the given constraints mean that (on the constraint manifold) we have $\mathcal{L} = f + 0$ and therefore the stationary points of f and $\mathcal{L}$ coincide.

*Remark 5 (**Geometric intuition**).* Suppose we wish to extremize (find a local maximum or minimum) the *objective function* $f(x, y)$ subject to the *constraint* $g(x, y) = 0$. We can think of this as follows. The graph $z = f(x, y)$

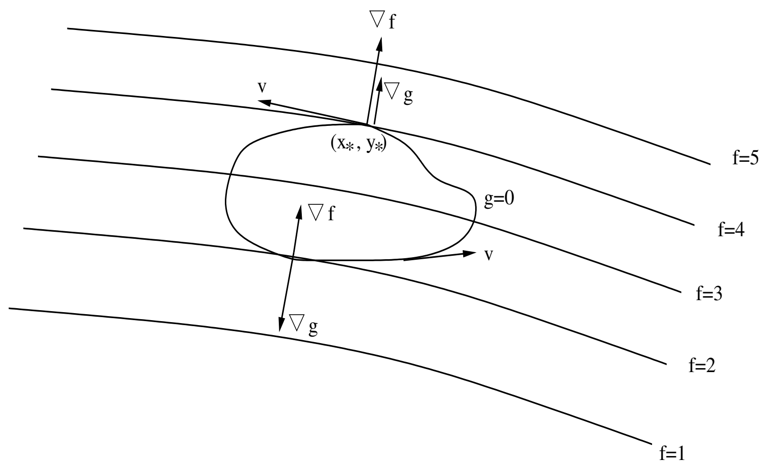

Fig. 1.4 At the constrained extremum ∇f and ∇g are parallel. (This is a rough reproduction of the figure on page 199 of McCallum *et. al.* [14])

represents a surface in three dimensional (x, y, z) space, while the constraint represents a curve in the x-y plane to which our movements are restricted.

Constrained extrema occur at points where the contours of f are tangent to the contours of g (and can also occur at the endpoints of the constraint). This can be seen as follows. At any point (x, y) in the plane ∇f points in the direction of maximum increase of f and thus perpendicular to the level contours of f. Suppose that the vector v is tangent to the constraining curve $g(x, y) = 0$. If the directional derivative $f_v = \nabla f \cdot v$ is positive at some point, then moving in the direction of v means that f increases (if the directional derivative is negative then f decreases in that direction). Thus at the point (x_*, y_*) where f has a constrained extremum we must have $\nabla f \cdot v = 0$ and so both ∇f and ∇g are perpendicular to v and therefore parallel. Hence for some scalar parameter λ (the Lagrange multiplier) we have at the constrained extremum:

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g = 0.$$

Notice that here we have three equations in three unknowns x, y, λ.

1.6 Constrained variational problems

A common optimization problem is to extremize a functional J with respect to paths y which are constrained in some way. We consider the following formulation.

Problem 6 (Constrained variational problem). Find the vector of functions $\boldsymbol{y} = \boldsymbol{y}(x)$ with N components

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix},$$

that extremize the functional

$$J(\boldsymbol{y}) \coloneqq \int_a^b F(x, \boldsymbol{y}, \boldsymbol{y}_x)\,\mathrm{d}x,$$

subject to the set of m constraints, for $k = 1, \ldots, m < N$:

$$G_k(x, \boldsymbol{y}) = 0.$$

To solve this constrained variational problem we generalize the method of Lagrange multipliers as follows. Note the m constraint equations above imply

$$\int_a^b \lambda_k(x)\, G_k(x, \boldsymbol{y})\,\mathrm{d}x = 0,$$

for each $k = 1, \ldots, m$. Note that the λ_k's are the Lagrange multipliers, which with the constraints expressed in this integral form, can in general be functions of x. We now form the equivalent of the Lagrangian function which here is the functional

$$\begin{aligned}\tilde{J}(\boldsymbol{y}, \boldsymbol{\lambda}) &\coloneqq \int_a^b F(x, \boldsymbol{y}, \boldsymbol{y}_x)\,\mathrm{d}x + \sum_{k=1}^{m} \int_a^b \lambda_k(x)\, G_k(x, \boldsymbol{y})\,\mathrm{d}x \\ &= \int_a^b \left(F(x, \boldsymbol{y}, \boldsymbol{y}_x) + \sum_{k=1}^{m} \lambda_k(x)\, G_k(x, \boldsymbol{y}) \right) \mathrm{d}x,\end{aligned}$$

where $\boldsymbol{\lambda} = \boldsymbol{\lambda}(x)$ is a vector of m Lagrange multiplier functions, i.e.

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}.$$

The integrand of the functional above is

$$\tilde{F}(x, \boldsymbol{y}, \boldsymbol{y}_x, \boldsymbol{\lambda}) \coloneqq F(x, \boldsymbol{y}, \boldsymbol{y}_x) + \sum_{k=1}^{m} \lambda_k(x)\, G_k(x, \boldsymbol{y}).$$

We now treat this as an unconstrained variational problem for the curves $\boldsymbol{y} = \boldsymbol{y}(x)$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}(x)$. From the classical variational problem if $(\boldsymbol{y}, \boldsymbol{\lambda})$

extremize $\tilde{J}$ then necessarily they must satisfy the Euler–Lagrange equations,

$$\begin{aligned}\frac{\partial \tilde{F}}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \tilde{F}}{\partial y_i'}\right) &= 0,\\ \frac{\partial \tilde{F}}{\partial \lambda_k} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \tilde{F}}{\partial \lambda_k'}\right) &= 0.\end{aligned}$$

Note that $\tilde{F} = \tilde{F}(x, \boldsymbol{y}, \boldsymbol{y}_x, \boldsymbol{\lambda})$ is independent of explicit $\boldsymbol{\lambda}'$ so $\partial \tilde{F}/\partial \lambda_k' = 0$. Further note that $\partial \tilde{F}/\partial \lambda_k = G_k$. Hence the Euler–Lagrange equations above simplify to

$$\begin{aligned}\frac{\partial \tilde{F}}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \tilde{F}}{\partial y_i'}\right) &= 0,\\ G_k(x, \boldsymbol{y}) &= 0,\end{aligned}$$

for $i = 1, \ldots, N$ and $k = 1, \ldots, m$. This is a system of *differential-algebraic equations*: the first set of relations are nonlinear second order ordinary differential equations, while the constraints are algebraic relations.

*Remark 6 (**Integral constraints**).* If the constraints are (already) in integral form so that we have

$$\int_a^b G_k(x, \boldsymbol{y})\,\mathrm{d}x = 0,$$

for $k = 1, \ldots, m$, then set

$$\begin{aligned}\tilde{J}(\boldsymbol{y}, \boldsymbol{\lambda}) &\coloneqq J(\boldsymbol{y}) + \sum_{k=1}^{m} \lambda_k \int_a^b G_k(x, \boldsymbol{y})\,\mathrm{d}x\\ &= \int_a^b \left(F(x, \boldsymbol{y}, \boldsymbol{y}_x) + \sum_{k=1}^{m} \lambda_k\, G_k(x, \boldsymbol{y})\right)\mathrm{d}x.\end{aligned}$$

We now have a variational problem with respect to the curve $\boldsymbol{y} = \boldsymbol{y}(x)$ but the constraints are classical in the sense that the Lagrangian multipliers $\boldsymbol{\lambda}$ here are simply variables (and not functions). Proceeding as above, with this hybrid variational constraint problem, with

$$\tilde{F}(x, \boldsymbol{y}, \boldsymbol{y}_x, \boldsymbol{\lambda}) \coloneqq F(x, \boldsymbol{y}, \boldsymbol{y}_x) + \sum_{k=1}^{m} \lambda_k\, G_k(x, \boldsymbol{y}),$$

we generate the Euler–Lagrange equations

$$\frac{\partial \tilde{F}}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \tilde{F}}{\partial y_i'}\right) = 0,$$

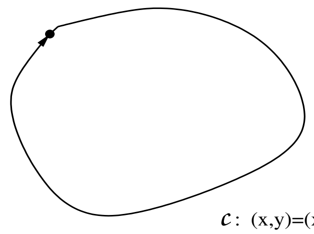

Fig. 1.5 For the isoperimetrical problem, the closed curve $\mathcal{C}$ has a fixed length ℓ, and the goal is to choose the shape that maximizes the area it encloses.

together with the integral constraint equations which result from taking the partial derivative of $\tilde{J}(\boldsymbol{y}, \boldsymbol{\lambda})$ with respect to all the components of $\boldsymbol{\lambda}$.

*Example 6 (**Dido's/isoperimetrical problem**).* The goal of this classical constrained variational problem is to find the shape of the closed curve, of a given fixed length ℓ, that encloses the maximum possible area. Suppose that the curve is given in parametric coordinates $\big(x(\tau), y(\tau)\big)$ where the parameter $\tau \in [0, 2\pi]$. By Stokes' theorem, the area enclosed by a closed contour $\mathcal{C}$ is

$$\tfrac{1}{2}\oint_{\mathcal{C}} x\,\mathrm{d}y - y\,\mathrm{d}x.$$

Hence our goal is to extremize the functional

$$J(x, y) := \tfrac{1}{2}\int_0^{2\pi} (x\dot{y} - y\dot{x})\,\mathrm{d}\tau,$$

subject to the constraint

$$\int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\,\mathrm{d}\tau = \ell.$$

The left hand side of this expression is the total arclength for a closed parametrically defined curve $(x, y) = \big(x(\tau), y(\tau)\big)$. This constraint can be expressed in standard integral form as follows. Since ℓ is fixed we have

$$\int_0^{2\pi} \tfrac{1}{2\pi}\ell\,\mathrm{d}\tau = \ell.$$

Hence the constraint can be expressed in the standard integral form

$$\int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2} - \tfrac{1}{2\pi}\ell\ \mathrm{d}\tau = 0.$$

To solve this variational constraint problem we use the method of Lagrange multipliers and form the functional

$$\tilde{J}(x,y,\lambda) := J(x,y) + \lambda\biggl(\int_0^{2\pi} \sqrt{\dot{x}^2+\dot{y}^2} - \tfrac{1}{2\pi}\ell \,\mathrm{d}\tau\biggr).$$

Substituting in the form for $J = J(x,y)$ above we find that

$$\tilde{J}(x,y,\lambda) = \tfrac{1}{2}\int_0^{2\pi} (x\dot{y} - y\dot{x})\,\mathrm{d}\tau + \lambda\biggl(\int_0^{2\pi} \sqrt{\dot{x}^2+\dot{y}^2} - \tfrac{1}{2\pi}\ell \,\mathrm{d}\tau\biggr).$$

Hence we can write

$$\tilde{J}(x,y,\lambda) = \int_0^{2\pi} \tilde{F}(x,y,\dot{x},\dot{y},\lambda)\,\mathrm{d}\tau,$$

where the integrand $\tilde{F}$ is given by

$$\tilde{F}(x,y,\dot{x},\dot{y},\lambda) = \tfrac{1}{2}(x\dot{y} - y\dot{x}) + \lambda(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}} - \lambda\tfrac{1}{2\pi}\ell.$$

Note there are two dependent variables x and y (here the parameter τ is the independent variable). By the theory above we know that the extremizing solution (x,y) necessarily satisfies an Euler–Lagrange system of equations, which are the pair of ordinary differential equations

$$\begin{aligned}
\frac{\partial \tilde{F}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}\tau}\biggl(\frac{\partial \tilde{F}}{\partial \dot{x}}\biggr) &= 0,\\
\frac{\partial \tilde{F}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}\tau}\biggl(\frac{\partial \tilde{F}}{\partial \dot{y}}\biggr) &= 0,
\end{aligned}$$

together with the integral constraint condition. Substituting the form for $\tilde{F}$ above, the pair of ordinary differential equations are

$$\begin{aligned}
\tfrac{1}{2}\dot{y} - \frac{\mathrm{d}}{\mathrm{d}\tau}\biggl(-\tfrac{1}{2}y + \frac{\lambda\dot{x}}{(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}}\biggr) &= 0,\\
-\tfrac{1}{2}\dot{x} - \frac{\mathrm{d}}{\mathrm{d}\tau}\biggl(\tfrac{1}{2}x + \frac{\lambda\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}}\biggr) &= 0.
\end{aligned}$$

This pair of equations is equivalent to the pair

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\biggl(y - \frac{\lambda\dot{x}}{(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}}\biggr) &= 0,\\
\frac{\mathrm{d}}{\mathrm{d}\tau}\biggl(x + \frac{\lambda\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}}}\biggr) &= 0.
\end{aligned}$$

Integrating both these equations with respect to τ we get

$$y - \frac{\lambda \dot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}} = c_2 \qquad \text{and} \qquad x + \frac{\lambda \dot{y}}{(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}} = c_1,$$

for arbitrary constants c_1 and c_2. Combining these last two equations reveals

$$(x - c_1)^2 + (y - c_2)^2 = \frac{\lambda^2 \dot{y}^2}{\dot{x}^2 + \dot{y}^2} + \frac{\lambda^2 \dot{x}^2}{\dot{x}^2 + \dot{y}^2} = \lambda^2.$$

Hence the solution curve is given by

$$(x - c_1)^2 + (y - c_2)^2 = \lambda^2,$$

which is the equation for a circle with radius λ and centre (c_1, c_2). The constraint condition implies $\lambda = \ell/2\pi$ and c_1 and c_2 can be determined from the initial or end points of the closed contour/path.

Remark 7. The isoperimetrical problem has quite a history. It was formulated in Virgil's poem the *Aeneid*, one account of the beginnings of Rome; see Wikipedia [18] or Montgomery [16]. Quoting from Wikipedia (Dido was also known as Elissa):

> Eventually Elissa and her followers arrived on the coast of North Africa where Elissa asked the local inhabitants for a small bit of land for a temporary refuge until she could continue her journeying, only as much land as could be encompassed by an oxhide. They agreed. Elissa cut the oxhide into fine strips so that she had enough to encircle an entire nearby hill, which was therefore afterwards named Byrsa "hide". (This event is commemorated in modern mathematics: The "isoperimetric problem" of enclosing the maximum area within a fixed boundary is often called the "Dido Problem" in modern calculus of variations.)

Dido found the solution—in her case a half-circle—and the semi-circular city of *Carthage* was founded.

*Example 7 (**Helmholtz's equation**).* This is a constrained variational version of the problem that generated the Laplace equation. The goal is to find the field $\psi = \psi(\boldsymbol{x})$ that extremizes the functional

$$J(\psi) := \int_\Omega |\nabla \psi|^2 \, \mathrm{d}\boldsymbol{x},$$

subject to the constraint

$$\int_\Omega \psi^2 \, \mathrm{d}\boldsymbol{x} = 1.$$

This constraint corresponds to saying that the total energy is bounded and in fact renormalized to unity. We assume zero boundary conditions, $\psi(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \partial\Omega \subseteq \mathbb{R}^n$. Using the method of Lagrange multipliers (for integral constraints) we form the functional

$$\tilde{J}(\psi, \lambda) := \int_\Omega |\nabla \psi|^2 \, \mathrm{d}\boldsymbol{x} + \lambda \left(\int_\Omega \psi^2 \, \mathrm{d}\boldsymbol{x} - 1 \right).$$

We can re-write this in the form

$$\tilde{J}(\psi,\lambda)=\int_{\Omega}|\nabla\psi|^2+\lambda\biggl(\psi^2-\frac{1}{|\Omega|}\biggr)\,\mathrm{d}\boldsymbol{x},$$

where $|\Omega|$ is the volume of the domain Ω. Hence the integrand in this case is

$$\tilde{F}(\boldsymbol{x},\psi,\nabla\psi,\lambda):=|\nabla\psi|^2+\lambda\biggl(\psi^2-\frac{1}{|\Omega|}\biggr).$$

The extremzing solution satisfies the Euler–Lagrange partial differential equation

$$\frac{\partial\tilde{F}}{\partial\psi}-\nabla\cdot\bigl(\nabla_{\psi_{\boldsymbol{x}}}\tilde{F}\bigr)=0,$$

together with the constraint equation. Note, directly computing, we have

$$\nabla_{\psi_{\boldsymbol{x}}}\tilde{F}=2\nabla\psi\qquad\text{and}\qquad\frac{\partial\tilde{F}}{\partial\psi}=2\lambda\psi.$$

Substituting these two results into the Euler–Lagrange partial differential equation we find

$$\begin{aligned}&& 2\lambda\psi-\nabla\cdot(2\nabla\psi)&=0\\ \Leftrightarrow&& \nabla^2\psi&=\lambda\psi.\end{aligned}$$

This is Helmholtz's equation on Ω. The Lagrange multiplier λ also represents an eigenvalue parameter.

1.7 Optimal linear-quadratic control

We can use calculus of variations techniques to derive the solution to an important problem in control theory. Suppose that a system state at any time $t\geqslant 0$ is recorded in the vector $\boldsymbol{q}=\boldsymbol{q}(t)$. Suppose further that the state evolves according to a *linear system* of differential equations and we can control the system via a set of *inputs* or *controls* $\boldsymbol{u}=\boldsymbol{u}(t)$, i.e. the system evolution is

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t}=A\boldsymbol{q}+B\boldsymbol{u}.$$

Here A is a matrix, which for convenience we will assume is constant, and $B=B(t)$ is another matrix mediating the controls.

Problem 7 (Optimal control problem). Starting in the state $\boldsymbol{q}(0)=\boldsymbol{q}_0$, bring this initial state to a final state $\boldsymbol{q}(T)$ at time $t=T>0$, as expediently as possible.

There are many criteria as to what constitutes "expediency". Here we will measure the cost on the system of our actions over $[0,T]$ by the *quadratic utility*

$$J(\boldsymbol{u}) \coloneqq \int_0^T \boldsymbol{q}^{\mathrm{T}}(t)C(t)\boldsymbol{q}(t) + \boldsymbol{u}^{\mathrm{T}}(t)D(t)\boldsymbol{u}(t)\,\mathrm{d}t + \boldsymbol{q}^{\mathrm{T}}(T)E\boldsymbol{q}(T).$$

Here $C = C(t)$, $D = D(t)$ and E are non-negative definite symmetric matrices. The final term represents a final state achievement cost. Note that we can in principle solve the system of linear differential equations above for the state $\boldsymbol{q} = \boldsymbol{q}(t)$ in terms of the control $\boldsymbol{u} = \boldsymbol{u}(t)$ so that $J = J(\boldsymbol{u})$ is a functional of $\boldsymbol{u}$ only (we see this presently).

Thus our goal is to find the control $\boldsymbol{u}_* = \boldsymbol{u}_*(t)$ that minimizes the cost $J = J(\boldsymbol{u})$ whilst respecting the constraint which is the linear evolution of the system state. We proceed as before. Suppose $\boldsymbol{u}_* = \boldsymbol{u}_*(t)$ exists. Consider perturbations to $\boldsymbol{u}_*$ on $[0,T]$ of the form

$$\boldsymbol{u} = \boldsymbol{u}_* + \epsilon\hat{\boldsymbol{u}}.$$

Changing the control/input changes the state $\boldsymbol{q} = \boldsymbol{q}(t)$ of the system so that

$$\boldsymbol{q} = \boldsymbol{q}_* + \epsilon\hat{\boldsymbol{q}}.$$

where we suppose here $\boldsymbol{q}_* = \boldsymbol{q}_*(t)$ to be the system evolution corresponding to the optimal control $\boldsymbol{u}_* = \boldsymbol{u}_*(t)$. Note that linear system perturbations $\epsilon\hat{\boldsymbol{q}}$ are linear in ϵ. Substituting these last two perturbation expressions into the differential system for the state evolution we find

$$\begin{aligned}
& & \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} &= A\boldsymbol{q} + B\boldsymbol{u} \\
&\Leftrightarrow & \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{q}_* + \epsilon\hat{\boldsymbol{q}}) &= A(\boldsymbol{q}_* + \epsilon\hat{\boldsymbol{q}}) + B(\boldsymbol{u}_* + \epsilon\hat{\boldsymbol{u}}) \\
&\Leftrightarrow & \frac{\mathrm{d}\boldsymbol{q}_*}{\mathrm{d}t} + \epsilon\frac{\mathrm{d}\hat{\boldsymbol{q}}}{\mathrm{d}t} &= A\boldsymbol{q}_* + \epsilon A\hat{\boldsymbol{q}} + B\boldsymbol{u}_* + \epsilon B\hat{\boldsymbol{u}} \\
&\Leftrightarrow & \frac{\mathrm{d}\hat{\boldsymbol{q}}}{\mathrm{d}t} &= A\hat{\boldsymbol{q}} + B\hat{\boldsymbol{u}},
\end{aligned}$$

where we have used that

$$\frac{\mathrm{d}\boldsymbol{q}_*}{\mathrm{d}t} = A\boldsymbol{q}_* + B\boldsymbol{u}_*.$$

The initial condition is $\hat{\boldsymbol{q}}(0) = 0$ as we assume $\boldsymbol{q}_*(0) = q_0$. We can solve this system of differential equations for $\hat{\boldsymbol{q}}$ in terms of $\hat{\boldsymbol{u}}$ by the variation of constants formula (using an integrating factor) as follows. We remark that we define the exponential of a square matrix through the exponential series. Hence for example for any real parameter t and square matrix A we set

$$\exp(tA) = I + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots.$$

With this in hand, if A is a constant matrix which is the case in our example here, we see by direct calculation that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\exp(tA)\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(I + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots\Big)\\ &= A + tA^2 + \frac{1}{2!}t^2A^3 + \cdots\\ &= A\Big(I + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots\Big)\\ &= A\,\exp(tA).\end{aligned}$$

By a similar calculation we also observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\exp(tA)\big) = \exp(tA)\,A.$$

Hence by direct calculation we see that

$$\begin{aligned} & & \frac{\mathrm{d}\hat{\boldsymbol{q}}}{\mathrm{d}t} &= A\hat{\boldsymbol{q}} + B\hat{\boldsymbol{u}}\\ \Leftrightarrow & & \frac{\mathrm{d}\hat{\boldsymbol{q}}}{\mathrm{d}t} - A\hat{\boldsymbol{q}} &= B\hat{\boldsymbol{u}}\\ \Leftrightarrow & & \exp(-At)\frac{\mathrm{d}\hat{\boldsymbol{q}}}{\mathrm{d}t} - \exp(-At)A\hat{\boldsymbol{q}} &= \exp(-At)B\hat{\boldsymbol{u}}\\ \Leftrightarrow & & \frac{\mathrm{d}}{\mathrm{d}t}\big(\exp(-At)\hat{\boldsymbol{q}}(t)\big) &= \exp(-At)B\hat{\boldsymbol{u}}(t)\\ \Leftrightarrow & & \exp(-At)\hat{\boldsymbol{q}}(t) &= \int_0^t \exp(-As)B(s)\hat{\boldsymbol{u}}(s)\,\mathrm{d}s\\ \Leftrightarrow & & \hat{\boldsymbol{q}}(t) &= \int_0^t \exp\big(A(t-s)\big)B(s)\hat{\boldsymbol{u}}(s)\,\mathrm{d}s.\end{aligned}$$

By the calculus of variations, if we set

$$\varphi(\epsilon) \coloneqq J(\boldsymbol{u}_* + \epsilon\hat{\boldsymbol{u}}),$$

then if the functional J has a minimum we have, for all $\hat{\boldsymbol{u}}$,

$$\varphi'(0) = 0.$$

Note that we can substitute our expression for $\hat{\boldsymbol{q}}(t)$ in terms of $\hat{\boldsymbol{u}}$ into $J(\boldsymbol{u}_* + \epsilon\hat{\boldsymbol{u}})$. Since $\boldsymbol{u} = \boldsymbol{u}_* + \epsilon\hat{\boldsymbol{u}}$ and $\boldsymbol{q} = \boldsymbol{q}_* + \epsilon\hat{\boldsymbol{q}}$ are linear in ϵ and the functional $J = J(\boldsymbol{u})$ is quadratic in $\boldsymbol{u}$ and $\boldsymbol{q}$, then $J(\boldsymbol{u}_* + \epsilon\hat{\boldsymbol{u}})$ must be quadratic in ϵ so that

$$\varphi(\epsilon) = \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2,$$

for some functionals $\varphi_0 = \varphi_0(\hat{\boldsymbol{u}})$, $\varphi_1 = \varphi_1(\hat{\boldsymbol{u}})$ and $\varphi_2 = \varphi_2(\hat{\boldsymbol{u}})$ independent of ϵ. Since $\varphi'(\epsilon) = \varphi_1 + 2\epsilon\varphi_2$, we see $\varphi'(0) = \varphi_1$. This term in $\varphi(\epsilon) = J(\boldsymbol{u}_* + \epsilon\hat{\boldsymbol{u}})$ by direct computation is thus

$$\varphi'(0) = 2\int_0^T \hat{\boldsymbol{q}}^{\mathrm{T}}(t)C(t)\boldsymbol{q}_*(t) + \hat{\boldsymbol{u}}^{\mathrm{T}}(t)D(t)\boldsymbol{u}_*(t)\,\mathrm{d}t + 2\hat{\boldsymbol{q}}^{\mathrm{T}}(T)E\boldsymbol{q}_*(T),$$

where we used that C, D and E are symmetric. We now substitute our expression for $\hat{\boldsymbol{q}}$ in terms of $\hat{\boldsymbol{u}}$ above into this formula for $\varphi'(0)$, this gives

$$\begin{aligned}
&\tfrac{1}{2}\varphi'(0) \\
&= \int_0^T \hat{\boldsymbol{q}}^{\mathrm{T}}(t)C(t)\boldsymbol{q}_*(t)\,\mathrm{d}t + \int_0^T \hat{\boldsymbol{u}}^{\mathrm{T}}(s)D(s)\boldsymbol{u}_*(s)\,\mathrm{d}s + \hat{\boldsymbol{q}}^{\mathrm{T}}(T)E\,\boldsymbol{q}_*(T) \\
&= \int_0^T\int_0^t \Big(\exp\big(A(t-s)\big)B(s)\hat{\boldsymbol{u}}(s)\Big)^{\mathrm{T}} C(t)\boldsymbol{q}_*(t) + \hat{\boldsymbol{u}}^{\mathrm{T}}(s)D(s)\boldsymbol{u}_*(s)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_0^T \Big(\exp\big(A(T-s)\big)B(s)\hat{\boldsymbol{u}}(s)\Big)^{\mathrm{T}}\,\mathrm{d}s\,E\,\boldsymbol{q}_*(T) \\
&= \int_0^T\int_s^T \Big(\exp\big(A(t-s)\big)B(s)\hat{\boldsymbol{u}}(s)\Big)^{\mathrm{T}} C(t)\boldsymbol{q}_*(t) + \hat{\boldsymbol{u}}^{\mathrm{T}}(s)D(s)\boldsymbol{u}_*(s)\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + \int_0^T \Big(\exp\big(A(T-s)\big)B(s)\hat{\boldsymbol{u}}(s)\Big)^{\mathrm{T}}\,\mathrm{d}s\,E\,\boldsymbol{q}_*(T),
\end{aligned}$$

where we have swapped the order of integration in the first term. Now we use the transpose of the product of two matrices is the product of their transposes in reverse order to get

$$\begin{aligned}
\tfrac{1}{2}\varphi'(0) &= \int_0^T \bigg(\hat{\boldsymbol{u}}^{\mathrm{T}}(s)B^{\mathrm{T}}(s)\int_s^T \exp\big(A^{\mathrm{T}}(t-s)\big)C(t)\boldsymbol{q}_*(t)\,\mathrm{d}t + \hat{\boldsymbol{u}}^{\mathrm{T}}(s)D(s)\boldsymbol{u}_*(s) \\
&\qquad + \hat{\boldsymbol{u}}(s)^{\mathrm{T}}B^{\mathrm{T}}(s)\exp\big(A^{\mathrm{T}}(T-s)\big)E\,\boldsymbol{q}_*(T)\bigg)\,\mathrm{d}s \\
&= \int_0^T \hat{\boldsymbol{u}}^{\mathrm{T}}(s)B^{\mathrm{T}}(s)\boldsymbol{p}(s) + \hat{\boldsymbol{u}}^{\mathrm{T}}(s)D(s)\boldsymbol{u}_*(s)\,\mathrm{d}s,
\end{aligned}$$

where for all $s \in [0,T]$ we set

$$\boldsymbol{p}(s) := \int_s^T \exp\big(A^{\mathrm{T}}(t-s)\big)C(t)\boldsymbol{q}_*(t)\,\mathrm{d}t + \exp\big(A^{\mathrm{T}}(T-s)\big)E\boldsymbol{q}_*(T).$$

We see the condition for a minimum, $\varphi'(0) = 0$, is equivalent to

$$\int_0^T \hat{\boldsymbol{u}}^{\mathrm{T}}(s)\big(B^{\mathrm{T}}(s)\boldsymbol{p}(s) + D(s)\boldsymbol{u}_*(s)\big)\,\mathrm{d}s = 0,$$

for all $\hat{\boldsymbol{u}}$. Hence a necessary condition for the minimum is that for all $t \in [0, T]$ we have

$$\boldsymbol{u}_*(t) = -D^{-1}(t)B^{\mathrm{T}}(t)\boldsymbol{p}(t).$$

Note that $\boldsymbol{p}$ depends solely on the optimal state $\boldsymbol{q}_*$. To elucidate this relationship further, note by definition $\boldsymbol{p}(T) = E\boldsymbol{q}_*(T)$ and differentiating $\boldsymbol{p}$ with respect to t we find

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -C\boldsymbol{q}_* - A^{\mathrm{T}}\boldsymbol{p}.$$

Using the expression for the optimal control $\boldsymbol{u}_*$ in terms of $\boldsymbol{p}(t)$ we derived, we see that $\boldsymbol{q}_*$ and $\boldsymbol{p}$ satisfy the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \boldsymbol{q}_* \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} A & -BD^{-1}(t)B^{\mathrm{T}} \\ -C(t) & -A^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} \boldsymbol{q}_* \\ \boldsymbol{p} \end{pmatrix}.$$

Define $S = S(t)$ to be the map $S\colon \boldsymbol{q}_* \mapsto \boldsymbol{p}$, i.e. so that $\boldsymbol{p}(t) = S(t)\boldsymbol{q}_*(t)$. Then

$$\boldsymbol{u}_*(t) = -D^{-1}(t)B^{\mathrm{T}}(t)S(t)\boldsymbol{q}_*(t),$$

and $S(t)$ we see characterizes the optimal current state feedback control, it tells how to choose the current optimal control $\boldsymbol{u}_*(t)$ in terms of the current state $\boldsymbol{q}_*(t)$. Finally we observe that since $\boldsymbol{p}(t) = S(t)\boldsymbol{q}_*(t)$, we have

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}t}\boldsymbol{q}_* + S\frac{\mathrm{d}\boldsymbol{q}_*}{\mathrm{d}t}.$$

Thus we see that

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t}\boldsymbol{q}_* &= \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} - S\frac{\mathrm{d}\boldsymbol{q}_*}{\mathrm{d}t} \\ &= (-C\boldsymbol{q}_* - A^{\mathrm{T}}S\boldsymbol{q}_*) - S(A\boldsymbol{q}_* - BD^{-1}A^{\mathrm{T}}S\boldsymbol{q}_*).\end{aligned}$$

Hence $S = S(t)$ satisfies the *Riccati equation*

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -C - A^{\mathrm{T}}S - SA - S\big(BD^{-1}A^{\mathrm{T}}\big)S.$$

Remark 8. We can easily generalize the argument above to the case when the coefficient matrix $A = A(t)$ is not constant. This is achieved by carefully replacing the flow map $\exp\big(A(t-s)\big)$ for the linear constant coefficient system $(\mathrm{d}/\mathrm{d}t)\hat{\boldsymbol{q}} = A\hat{\boldsymbol{q}}$, by the flow map for the corresponding linear nonautonomous system with $A = A(t)$, and carrying that through the rest of the computation.

Exercises

1.1. Euler–Lagrange alternative form
Show that the Euler–Lagrange equation is equivalent to

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(F - y_x \frac{\partial F}{\partial y_x}\right) = 0.$$

1.2. Soap film
A soap film is stretched between two rings of radius a which lie in parallel planes a distance $2x_0$ apart—the axis of symmetry of the two rings is coincident—see Figure 1.6.

(a) Explain why the surface area of the surface of revolution is given by

$$J(y) = 2\pi \int_{-x_0}^{x_0} y\sqrt{1 + (y_x)^2}\,\mathrm{d}x,$$

where radius of the surface of revolution is given by $y = y(x)$ for $x \in [-x_0, x_0]$.

(b) Show that extremizing the surface area $J(y)$ in part (a) leads to the following ordinary differential equation for $y = y(x)$:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = C^{-2}y^2 - 1$$

where C is an arbitrary constant.

(c) Use the substitution $y = C\cosh\theta$ and the identity $\cosh^2\theta - \sinh^2\theta = 1$ to show that the solution to the ordinary differential equation in part (b) is

$$y = C\cosh\big(C^{-1}(x + b)\big)$$

where b is another arbitrary constant. Explain why we can deduce that $b = 0$.

(d) Using the end-point conditions $y = a$ at $x = \pm x_0$, discuss the existence of solutions in relation to the ratio a/x_0.

1.3. Hanging rope
We wish to compute the shape $y = y(x)$ of a uniform heavy hanging rope that is supported at the points $(-a, 0)$ and $(a, 0)$. The rope hangs so as to minimize its total potential energy which is given by the functional

$$\int_{-a}^{+a} \rho g y\sqrt{1 + (y_x)^2}\,\mathrm{d}x,$$

where ρ is the mass density of the rope and g is the acceleration due to gravity. Suppose that the total length of the rope is fixed and given by ℓ, i.e. we have a constraint on the system given by

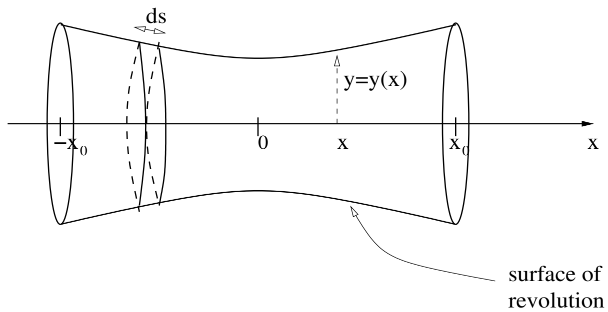

Fig. 1.6 Soap film stretched between two concentric rings. The radius of the surface of revolution is given by $y = y(x)$ for $x \in [-x_0, x_0]$.

$$\int_{-a}^{+a} \sqrt{1 + (y_x)^2}\,\mathrm{d}x = \ell.$$

(a) Use the method of Lagrange multipliers to show the Euler–Lagrange equations in this case is

$$(y_x)^2 = c^2(y + \lambda)^2 - 1,$$

where c is an arbitrary constant and λ the Lagrange multiplier. (*Hint:* you may find the alternative form for the Euler–Lagrange equation more useful here, and you may have to re-scale the Lagrange multiplier to get the precise form shown above.)

(b) Use the substitution $c(y + \lambda) = \cosh\theta$ to show that the solution to the ordinary differential equation in (a) is

$$y = c^{-1}\cosh\bigl(c(x + b)\bigr) - \lambda,$$

for some constant b. Since the problem is symmetric about the origin, what does this imply about b?

(c) Use the boundary conditions that $y = 0$ at $x = \pm a$ and the constraint condition, respectively, to show that

$$c\,\lambda = \cosh(ac),$$
$$c\,\ell = 2\sinh(ac).$$

Under what condition does the second equation have a real solution? What is the physical significance of this condition?

1.4. Spherical geodesic

The goal of this problem is to find the path on the surface of the sphere of radius r that minimizes the distance between two arbitrary points a and b

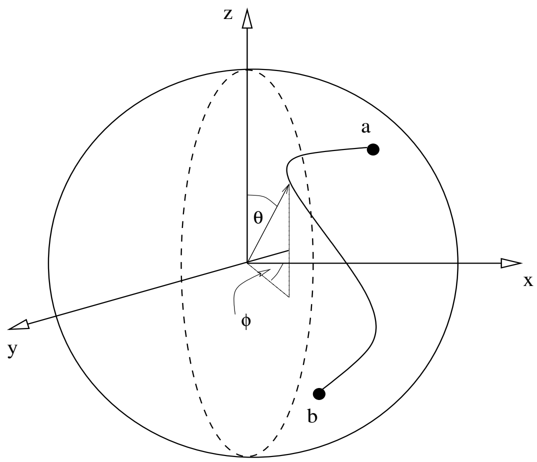

Fig. 1.7 The goal is to find the path on the surface of the sphere that minimizes the distance between two arbitrary points a and b that lie on the sphere. It is natural to use spherical polar coordinates (r, θ, ϕ) and to take the radial distance to be fixed, say $r = r_0$ with r_0 constant. The angles θ and ϕ are the latitude (measured from the north pole axis) and azimuthal angles, respectively. We can specify a path on the surface by a function $\phi = \phi(\theta)$.

that lie on the sphere. It is natural to use spherical polar coordinates and to take the radial distance to be fixed, say $r = r_0$ with r_0 constant. The relationship between spherical and cartesian coordinates is

$$x = r\,\sin\theta\,\cos\phi,$$
$$y = r\,\sin\theta\,\sin\phi,$$
$$z = r\,\cos\theta,$$

where θ and ϕ are the latitude (measured from the north pole axis) and azimuthal angles, respectively. See the setup in Figure 1.7. We thus wish to minimize the total arclength from the point a to the point b on the surface of the sphere, i.e. to minimize the total arclength functional

$$\int_a^b \mathrm{d}s,$$

where s measures arclength on the surface of the sphere.

(a) The position on the surface of the sphere can be specified by the azimuthal angle ϕ and the latitudinal angle θ as shown in Figure 1.7, i.e. we can specify a path on the surface by a function $\phi = \phi(\theta)$. Use spherical polar coordinates to show that we can express the total arclength functional in the form

$$J(\phi) := r_0 \int_{\theta_a}^{\theta_b} \sqrt{1 + \sin^2\theta\,(\phi')^2}\,\mathrm{d}\theta,$$

where $\phi' = \mathrm{d}\phi/\mathrm{d}\theta$. Here θ_a and θ_b are the latitude angles of points a and b on the sphere surface. *Hint:* you need to compute $\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2}$, though remember that $r = r_0$ is fixed.

(b) Write down the Euler–Lagrange equation that the path $\phi = \phi(\theta)$ that minimizes J must satisfy, and hence show that it is equivalent to the following first order ordinary differential equation:

$$\frac{\sin^2\theta \cdot \phi'}{\left(1 + \sin^2\theta \cdot (\phi')^2\right)^{\frac{1}{2}}} = c,$$

for some arbitrary constant c.

(c) Note that we can always orient our coordinate system so that $\theta_a = 0$. Using this initial condition and the Euler–Lagrange equation in part (b), deduce that the solution is an arc of a great circle (a great circle is cut out on the surface of a sphere by a plane that passes through the centre of the sphere).

(d) What is an everyday application of this knowledge?

1.5. Capillary surfaces

Consider a fluid of constant density ρ in a container, under the influence of a gravitational acceleration g. Our goal is to determine the shape of the free surface of the fluid. For simplicity we suppose that if x measures distance across the width of the container and y measures height from the flat bottom of the container, then the container is (long and) uniform in the third (horizontal) direction—see Figure 1.8 on the next page. Hence we assume the free surface is also uniform in the third (horizontal) direction and can be described by a single curve $y = y(x)$ for $-a \leqslant x \leqslant a$ as shown. The problem is to find the curve $y = y(x)$ that minimizes the energy functional

$$\sigma \int_{-a}^{a} \sqrt{1 + y_x^2}\,\mathrm{d}x - \tfrac{1}{2} \int_{-a}^{a} \rho g y^2\,\mathrm{d}x,$$

subject to the constraint

$$\int_{-a}^{a} y\,\mathrm{d}x = A.$$

Here σ is a constant representing the surface tension and A is a constant associated with the total volume of the fluid. The first term in the energy functional represents the deformation energy of the free surface while the second term represents the stored potential energy. We ignore the adhesion force at the wall for simplicity.

(a) By using the method of Lagrange multipliers, show that the free surface $y = y(x)$ necessarily satisfies the ordinary differential equation

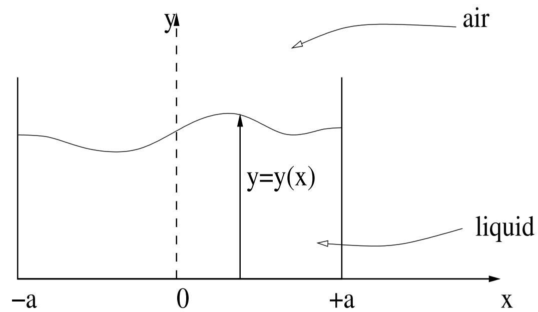

Fig. 1.8 A fluid of constant density lies in the container as shown. The set up is assumed to be uniform in the horizontal direction perpendicular to the x-axis. The goal is to determine the shape of the free surface $y = y(x)$.

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{1}{(b - cy + dy^2)^2} - 1,$$

where b is an arbitrary constant, c depends on the Lagrange multiplier and known constants and d depends on known constants. Please show explicitly how c is related to the Lagrange multiplier and constants σ, g and ρ, as well as how d is related to σ, g and ρ.

(b) Now assume that $d = 0$. Use the substitution

$$y = \frac{b}{c} - \frac{1}{c}\cos\theta$$

to show that there exist solutions to the corresponding ordinary differential equation in part (a) of the form

$$y = \frac{b}{c} - \frac{1}{c}\cos\big(\arcsin(cx + k)\big),$$

where k is an arbitrary constant.

(c) Explain why it would be reasonable to assume the constant $k = 0$ in our solution in part (b) above. Assuming this, use the constraint condition to derive a relationship between b and A. Comment on what implications this relationship has on the existence of solutions of the form in part (b).

1.6. Gravity train

The goal of this problem is to design a gravity train. The idea is to construct a tunnel, say between London and New York, through which a train falls under the force due to gravity only, which minimizes the time of travel between the two cities. We assume that the motion is drag-free, i.e. we ignore any air resistance, frictional or Coriolis forces. The set up is shown in Figure 1.9. Let the radius of the earth be R. We assume the tunnel path lies in the plane defined by the three points: the centre of the earth O, London A and New York B. It is natural to use plane polar coordinates r and θ, centred at O as shown in Figure 1.9. Hence the tunnel shape can be described by a curve $r = r(\theta)$. We assume that London is at angle $\theta = -\theta_0$ and New York is at angle $\theta = \theta_0$.

(a) The potential energy, due to the acceleration due to gravity which is oriented towards O, for a train of mass m which is at a distance r from O is given by

$$\frac{gmr^2}{2R},$$

where g is a constant (it's the acceleration due to gravity at the earth's surface). Assume that the train starts in London with zero velocity. Use that energy is conserved by the train to show that the velocity v of the train when it is at a distance r from O is given by

$$v = \left(\frac{g}{R}\right)^{1/2} (R^2 - r^2)^{1/2}.$$

(b) Take as given that in polar coordinates the arclength measure $\mathrm{d}s$ is given by

$$\mathrm{d}s = \big(r^2 + (r')^2\big)^{1/2}\,\mathrm{d}\theta,$$

where $r' = r'(\theta)$ is the derivative of $r = r(\theta)$ with respect to θ. Briefly explain why the total time the train will take to get from London to New York can be expressed in the form

$$\left(\frac{R}{g}\right)^{1/2} \int_{-\theta_0}^{\theta_0} \frac{\big(r^2 + (r')^2\big)^{1/2}}{(R^2 - r^2)^{1/2}}\,\mathrm{d}\theta.$$

(c) Show that the Euler–Lagrange equation for the path $r = r(\theta)$ that minimizes the total time integral given in part (b) above, is equivalent to the following first order ordinary differential equation:

$$(r')^2 = \frac{1}{c^2}\frac{r^4}{R^2 - r^2} - r^2,$$

for some arbitrary constant c. (*Hint:* Use the alternative form for the Euler–Lagrangian equation.)

(d) Using that $\mathrm{d}\theta/\mathrm{d}r = 1/r'(\theta)$, show that we can express the Euler–Lagrange equation in part (c) in the form

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)^2 = \frac{C^2}{R^2} \cdot \frac{R^2 - r^2}{r^2(r^2 - C^2)},$$

where the constant C is related to c through the relation $C^2 = c^2R^2/(1+c^2)$.

(e) By making the change of coordinates $r^2 = R^2 \cos^2 \psi + C^2 \sin^2 \psi$, and using the chain rule

$$\frac{\mathrm{d}\theta}{\mathrm{d}\psi} = \frac{\mathrm{d}\theta}{\mathrm{d}r} \cdot \frac{\mathrm{d}r}{\mathrm{d}\psi},$$

show that $\theta = \theta(\psi)$ satisfies the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}\psi} = \frac{C}{R} \cdot \frac{(R^2 - C^2) \tan^2 \psi}{R^2 + C^2 \tan^2 \psi}.$$

(*Hint:* Note that when taking the square root to compute $\mathrm{d}\theta/\mathrm{d}r$ from part (d) we ensure θ is an increasing function of ψ.)

Hence show that the optimal gravity train path is given parametrically by $\theta = \theta(\psi)$ and $r = r(\psi)$, where $\theta = \theta(\psi)$ given by

$$\theta = -\frac{C}{R}\psi + \arctan\left(\frac{C}{R} \tan \psi\right) - \theta_0.$$

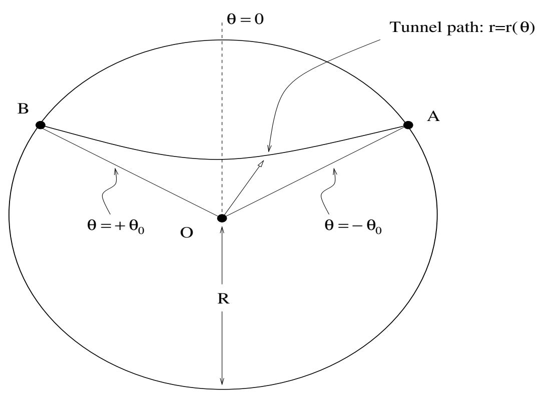

Fig. 1.9 The idea of a gravity train is to construct a tunnel, say between London and New York, which minimizes the time of travel between the two cities powered by the force due to gravity only.

Chapter 2
Lagrangian and Hamiltonian mechanics

2.1 Holonomic constraints and degrees of freedom

Consider a system of N particles in three dimensional space, each with position vector $\boldsymbol{r}_i(t)$ for $i = 1, \ldots, N$. Note that each $\boldsymbol{r}_i(t) \in \mathbb{R}^3$ is a 3-vector. We thus need $3N$ coordinates to specify the system, this is the *configuration space*. *Newton's 2nd law* tells us that the equation of motion for the ith particle is

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_i^{\text{ext}} + \boldsymbol{F}_i^{\text{con}},$$

for $i = 1, \ldots, N$. Here $\boldsymbol{p}_i = m_i \boldsymbol{v}_i$ is the linear momentum of the ith particle and $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i$ is its velocity. We decompose the total force on the ith particle into an external force $\boldsymbol{F}_i^{\text{ext}}$ and a *constraint* force $\boldsymbol{F}_i^{\text{con}}$. By external forces we imagine forces due to gravitational attraction or an electro-magnetic field, and so forth.

By a constraint on a particles we imagine that the particle's motion is limited in some rigid way. For example the particle/bead may be constrained to move along a wire or its motion is constrained to a given surface. If the system of N particles constitute a rigid body, then the distances between all the particles are rigidly fixed and we have the constraint

$$|\boldsymbol{r}_i(t) - \boldsymbol{r}_j(t)| = c_{ij},$$

for some constants c_{ij}, for all $i, j = 1, \ldots, N$. All of these are examples of *holonomic constraints*.

Definition 1 (Holonomic constraints). For a system of particles with positions given by $\boldsymbol{r}_i(t)$ for $i = 1, \ldots, N$, constraints that can be expressed in the form

$$g(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, t) = 0,$$

are said to be *holonomic*. Note they only involve the configuration coordinates.

We will *only* consider systems for which the constraints are holonomic. Systems with constraints that are non-holonomic are: gas molecules in a container (the constraint is only expressible as an inequality); or a sphere rolling on a rough surface without slipping (the constraint condition is one of matched velocities).

Let us suppose that for the N particles there are m holonomic constraints given by

$$g_k(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, t) = 0,$$

for $k = 1, \ldots, m$. The positions $\boldsymbol{r}_i(t)$ of all N particles are determined by $3N$ coordinates. However due to the constraints, the positions $\boldsymbol{r}_i(t)$ are not all independent. In principle, we can use the m holonomic constraints to eliminate m of the $3N$ coordinates and we would be left with $3N - m$ independent coordinates, i.e. the dimension of the configuration space is actually $3N - m$.

*Example 8 (**Two particles connected by a light rod**).* Suppose two particles can move freely in three-dimensional space, their position vectors at any time given by the vectors $\boldsymbol{r}_1 = \boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2 = \boldsymbol{r}_2(t)$, each with three components. Hence 6 pieces of information, the three components for each vector are required to specify the state of the system at any time t. The dimension of the configuration space is 6. Now suppose the two particles are connected by a light rigid rod of length ℓ. Thus for this system the vectors $\boldsymbol{r}_1 = \boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2 = \boldsymbol{r}_2(t)$ are restricted so that at any time t the constraint/condition

$$\big|\boldsymbol{r}_1(t) - \boldsymbol{r}_2(t)\big| = \ell$$

is satisfied. This constrain equation represents a single relation between the 6 configuration variables. In principle we can solve for any one of the configuration variables in terms of the other 5 configuration variables. Hence the system is constrained to evolve on a 5-dimensional submanifold of the 6-dimensional configuration space. The dimension of the configuration space is 5.

Definition 2 (Degrees of freedom). The dimension of the configuration space is called the *number of degrees of freedom*, see Arnold [3, Page 76].

Thus we can transform from the 'old' coordinates $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N$ to new *generalized coordinates* $q_1, \ldots, q_n$ where $n = 3N - m$:

$$\begin{aligned}
\boldsymbol{r}_1 &= \boldsymbol{r}_1(q_1, \ldots, q_n, t), \\
&\ \vdots \\
\boldsymbol{r}_N &= \boldsymbol{r}_N(q_1, \ldots, q_n, t).
\end{aligned}$$

2.2 D'Alembert's principle

We will restrict ourselves to systems for which the net work of the constraint forces is zero, i.e. we suppose

$$\sum_{i=1}^{N} \boldsymbol{F}_i^{\mathrm{con}} \cdot \mathrm{d}\boldsymbol{r}_i = 0,$$

for every small change $\mathrm{d}\boldsymbol{r}_i$ of the configuration of the system (for t fixed). In this section we follow the presentation given in Goldstein [12, Chapter 1] to where the reader is referred for more details. Recall that the work done by a particle is given by the force acting on the particle times the distance travelled in the direction of the force. So here for the ith particle, the constraint force applied is $\boldsymbol{F}_i^{\mathrm{con}}$ and suppose it undergoes a small displacement given by the vector $\mathrm{d}\boldsymbol{r}_i$. Since the dot product of two vectors gives the projection of one vector in the direction of the other, the dot product $\boldsymbol{F}_i^{\mathrm{con}} \cdot \mathrm{d}\boldsymbol{r}_i$ gives the work done by $\boldsymbol{F}_i^{\mathrm{con}}$ in the direction of the displacement $\mathrm{d}\boldsymbol{r}_i$.

If we combine the assumption that the net work of the constraint forces is zero with Newton's 2nd law

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_i^{\mathrm{ext}} + \boldsymbol{F}_i^{\mathrm{con}}$$

from the last section, we find

$$\begin{aligned}
& \sum_{i=1}^{N} \dot{\boldsymbol{p}}_i \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i=1}^{N} \big(\boldsymbol{F}_i^{\mathrm{ext}} + \boldsymbol{F}_i^{\mathrm{con}}\big) \cdot \mathrm{d}\boldsymbol{r}_i \\
\Leftrightarrow \quad & \sum_{i=1}^{N} \dot{\boldsymbol{p}}_i \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{F}_i^{\mathrm{ext}} \cdot \mathrm{d}\boldsymbol{r}_i + \sum_{i=1}^{N} \boldsymbol{F}_i^{\mathrm{con}} \cdot \mathrm{d}\boldsymbol{r}_i \\
\Leftrightarrow \quad & \sum_{i=1}^{N} \dot{\boldsymbol{p}}_i \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{F}_i^{\mathrm{ext}} \cdot \mathrm{d}\boldsymbol{r}_i .
\end{aligned}$$

In other words we have

$$\sum_{i=1}^{N} (\dot{\boldsymbol{p}}_i - \boldsymbol{F}_i^{\mathrm{ext}}) \cdot \mathrm{d}\boldsymbol{r}_i = 0,$$

for every small change $\mathrm{d}\boldsymbol{r}_i$. This represents *D'Alembert's principle*. Note in particular that no forces of constraint are present.

Remark 9. The assumption that the constraint force does no net work is quite general. It is true in particular for holonomic constraints. For example, for the case of a rigid body, the internal forces of constraint do no work as the distances $|\boldsymbol{r}_i - \boldsymbol{r}_j|$ between particles is fixed, then $\mathrm{d}(\boldsymbol{r}_i - \boldsymbol{r}_j)$ is perpendicular to $\boldsymbol{r}_i - \boldsymbol{r}_j$ and hence perpendicular to the force between them which is parallel to $\boldsymbol{r}_i - \boldsymbol{r}_j$. Similarly for the case of the bead on a wire or particle constrained to move on a surface—the normal reaction forces are perpendicular to $\mathrm{d}\boldsymbol{r}_i$.

In his *Mécanique Analytique [1788]*, Lagrange sought a "coordinate invariant expression for mass times acceleration", see Marsden and Ratiu [15, Page 231]. This lead to Lagrange's equations of motion. Consider the transformation to generalized coordinates

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, \ldots, q_n, t),$$

for $i = 1, \ldots, N$. If we consider a small increment in the displacements $\mathrm{d}\boldsymbol{r}_i$ then the corresponding increment in the work done by the external forces is

$$\sum_{i=1}^{N} \boldsymbol{F}_i^{\mathrm{ext}} \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i,j=1}^{N,n} \boldsymbol{F}_i^{\mathrm{ext}} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \mathrm{d}q_j = \sum_{j=1}^{n} Q_j \, \mathrm{d}q_j.$$

Here we have used the chain rule

$$\mathrm{d}\boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \mathrm{d}q_j,$$

and we set for $j = 1, \ldots, n$,

$$Q_j = \sum_{i=1}^{N} \boldsymbol{F}_i^{\mathrm{ext}} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}.$$

We think of the Q_j as *generalized forces*. We now assume the work done by these forces depends on the initial and final configurations only and not on the path between them. In other words we assume there exists a potential function $V = V(q_1, \ldots, q_n)$ such that

$$Q_j = -\frac{\partial V}{\partial q_j}$$

for $j = 1, \ldots, n$. Such forces are said to be *conservative*. We define the total *kinetic energy* to be

$$T := \sum_{i=1}^{N} \tfrac{1}{2} m_i |\boldsymbol{v}_i|^2,$$

and the *Lagrange function* or *Lagrangian* to be

$$L := T - V.$$

Theorem 2 (Lagrange's equations). *D'Alembert's principle, under the assumption the constraints are holonomic, is equivalent to the system of or-*

dinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0,$$

for $j = 1, \ldots, n$. *These are known as* Lagrange's equations *of motion.*

Proof. The change in kinetic energy mediated through the momentum—the first term in D'Alembert's principle—due to the increment in the displacements $\mathrm{d}\boldsymbol{r}_i$ is given by

$$\sum_{i=1}^{N} \dot{\boldsymbol{p}}_i \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i=1}^{N} m_i \dot{\boldsymbol{v}}_i \cdot \mathrm{d}\boldsymbol{r}_i = \sum_{i,j=1}^{N,n} m_i \dot{\boldsymbol{v}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}\mathrm{d}q_j.$$

From the product rule we know that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) &\equiv \dot{\boldsymbol{v}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} + \boldsymbol{v}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) \\ &\equiv \dot{\boldsymbol{v}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} + \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial q_j}.\end{aligned}$$

Also, by differentiating the transformation to generalized coordinates we see

$$\boldsymbol{v}_i \equiv \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j}\,\dot{q}_j \qquad \text{and} \qquad \frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_j} \equiv \frac{\partial \boldsymbol{r}_i}{\partial q_j}.$$

Using these last two identities we see that

$$\begin{aligned}\sum_{i=1}^{N} \dot{\boldsymbol{p}}_i \cdot \mathrm{d}\boldsymbol{r}_i &= \sum_{j=1}^{n}\left(\sum_{i=1}^{N} m_i \dot{\boldsymbol{v}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}\right)\mathrm{d}q_j \\ &= \sum_{j=1}^{n}\left(\sum_{i=1}^{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) - m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial q_j}\right)\right)\mathrm{d}q_j \\ &= \sum_{j=1}^{n}\left(\sum_{i=1}^{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_j}\right) - m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial q_j}\right)\right)\mathrm{d}q_j \\ &= \sum_{j=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}_j}\left(\sum_{i=1}^{N} \tfrac{1}{2} m_i |\boldsymbol{v}_i|^2\right)\right) - \frac{\partial}{\partial q_j}\left(\sum_{i=1}^{N} \tfrac{1}{2} m_i |\boldsymbol{v}_i|^2\right)\right)\mathrm{d}q_j.\end{aligned}$$

Hence we see that D'Alembert's principle is equivalent to

$$\sum_{j=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right)\mathrm{d}q_j = 0.$$

Since the q_j for $j = 1, \ldots, n$, where $n = 3N - m$, are all independent, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0,$$

for $j = 1, \ldots, n$. Using the definition for the generalized forces Q_j in terms of the potential function V gives the result. □

*Remark 10 (**Configuration space**).* As already noted, the n-dimensional subsurface of $3N$-dimensional space on which the solutions to Lagrange's equations lie is called the *configuration space*. It is parameterized by the n generalized coordinates $q_1, \ldots, q_n$.

*Remark 11 (**Non-conservative forces**).* If the system has forces that are not conservative it may still be possible to find a *generalized potential* function V such that

$$Q_j = -\frac{\partial V}{\partial q_j} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial V}{\partial \dot{q}_j}\right),$$

for $j = 1, \ldots, n$. From such potentials we can still deduce Lagrange's equations of motion. Examples of such generalized potentials are velocity dependent potentials due to electro-magnetic fields, for example the Lorentz force on a charged particle.

2.3 Hamilton's principle

We consider mechanical systems with holonomic constraints and all other forces conservative. Recall, we define the *Lagrange function* or *Lagrangian* to be

$$L = T - V,$$

where

$$T = \sum_{i=1}^{N} \tfrac{1}{2} m_i |\boldsymbol{v}_i|^2$$

is the total kinetic energy for the system, and V is its potential energy.

Definition 3 (Action). If the Lagrangian L is the difference of the kinetic and potential energies for a system, i.e. $L = T - V$, we define the *action* $A = A(\boldsymbol{q})$ from time t_1 to t_2, where $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$, to be the functional

$$A(\boldsymbol{q}) := \int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\,\mathrm{d}t.$$

Hamilton [1834] realized that Lagrange's equations of motion were equivalent to a variational principle (see Marsden and Ratiu [15, Page 231]).

Theorem 3 (Hamilton's principle of least action). *The correct path of motion of a mechanical system with holonomic constraints and conservative external forces, from time t_1 to t_2, is a stationary solution of the action. Indeed, the correct path of motion $\boldsymbol{q} = \boldsymbol{q}(t)$, with $\boldsymbol{q} = (q_1, \ldots, q_n)^T$, necessarily and sufficiently satisfies Lagrange's equations of motion for $j = 1, \ldots, n$:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0.$$

Quoting from Arnold [3, Page 60], it is Hamilton's form of the principle of least action "because in many cases the action of $\boldsymbol{q} = \boldsymbol{q}(t)$ is not only an extremal but also a minimum value of the action functional".

*Example 9 (**Simple harmonic motion**).* Consider a particle of mass m moving in a one dimensional Hookeian force field $-kx$, where k is a constant. The potential function $V = V(x)$ corresponding to this force field satisfies

$$\begin{aligned} & & -\frac{\partial V}{\partial x} &= -kx \\ \Leftrightarrow & & V(x) - V(0) &= \int_0^x k\xi \, \mathrm{d}\xi \\ \Leftrightarrow & & V(x) &= \tfrac{1}{2}x^2. \end{aligned}$$

The Lagrangian $L = T - V$ is thus given by

$$L(x, \dot{x}) = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2.$$

From Hamilton's principle the equations of motion are given by Lagrange's equations. Here, taking the generalized coordinate to be $q = x$, the single Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

Using the form for the Lagrangian above we find that

$$\frac{\partial L}{\partial x} = -k\,x \qquad \text{and} \qquad \frac{\partial L}{\partial \dot{x}} = m\,\dot{x},$$

and so Lagrange's equation of motion becomes

$$m\ddot{x} + kx = 0.$$

*Example 10 (**Kepler problem**).* Consider a particle of mass m moving in an inverse square law force field, $-\mu m/r^2$, such as a small planet or asteroid in the gravitational field of a star or larger planet. Hence the corresponding potential function satisfies

$$
\begin{aligned}
& & -\frac{\partial V}{\partial r} &= -\frac{\mu m}{r^2} \\
\Leftrightarrow & & V(\infty) - V(r) &= \int_r^\infty \frac{\mu m}{\rho^2}\,\mathrm{d}\rho \\
\Leftrightarrow & & V(r) &= -\frac{\mu m}{r}.
\end{aligned}
$$

The expression for the kinetic energy T is given in Cartesian coordinates by

$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2).$$

Making the change of coordinates from Cartesians $\big(x(t), y(t)\big)$ to plane polars $\big(r(t), \theta(t)\big)$ we have $x = r\cos\theta$ and $y = r\sin\theta$ and thus by the chain rule

$$
\begin{aligned}
\dot{x} &= \dot{r}\cos\theta + r(-\sin\theta)\,\dot{\theta}, \\
\dot{y} &= \dot{r}\sin\theta + r(\cos\theta)\,\dot{\theta}.
\end{aligned}
$$

Hence directly computing we find $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$ and so equivalently in plane polar coordinates we have

$$T = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2).$$

Hence the Lagrangian $L = T - V$ is thus given by

$$L(r, \dot{r}, \theta, \dot{\theta}) = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu m}{r}.$$

From Hamilton's principle the equations of motion are given by Lagrange's equations, which here, taking the generalized coordinates to be $q_1 = r$ and $q_2 = \theta$, are the pair of ordinary differential equations

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= 0, \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0.
\end{aligned}
$$

Using the form for the Lagrangian above we find

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\mu m}{r^2}, \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \text{and} \quad \frac{\partial L}{\partial \theta} = 0.$$

Substituting these expressions into Lagrange's equations above we find

$$
\begin{aligned}
m\ddot{r} - mr\dot{\theta}^2 + \frac{\mu m}{r^2} &= 0, \\
\frac{\mathrm{d}}{\mathrm{d}t}\big(mr^2\dot{\theta}\big) &= 0.
\end{aligned}
$$

*Remark 12 (**Non-uniqueness of the Lagrangian**).* Two Lagrangian's L_1 and L_2 that differ by the total time derivative of any function of $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$ and t generate the same equations of motion. In fact if

$$L_2(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = L_1(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \frac{\mathrm{d}}{\mathrm{d}t}\big(f(\boldsymbol{q}, t)\big),$$

then for $j = 1, \ldots, n$ direct calculation reveals that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_2}{\partial \dot{q}_j}\right) - \frac{\partial L_2}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_1}{\partial \dot{q}_j}\right) - \frac{\partial L_1}{\partial q_j}.$$

2.4 Constraints

Given a Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ for a system, suppose we realize the system has some constraints (so the q_j are not all independent). Suppose we have m *holonomic constraints* of the form

$$G_k(q_1, \ldots, q_n, t) = 0,$$

for $k = 1, \ldots, m < n$. We can now use the method of Lagrange multipliers with Hamilton's principle to deduce the equations of motion are given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} &= \sum_{k=1}^{m} \lambda_k(t) \frac{\partial G_k}{\partial q_j}, \\ G_k(q_1, \ldots, q_n, t) &= 0, \end{aligned}$$

for $j = 1, \ldots, n$ and $k = 1, \ldots, m$. We call the quantities on the right above

$$\sum_{k=1}^{m} \lambda_k(t) \frac{\partial G_k}{\partial q_j},$$

the *generalized forces of constraint.*

*Example 11 (**Simple pendulum**).* Consider the motion of a simple pendulum bob of mass m that swings at the end of a light rod of length a. The other end is attached so that the rod and bob can swing freely in a plane. If g is the acceleration due to gravity, then the Lagrangian $L = T - V$ is given by

$$L(r, \dot{r}, \theta, \dot{\theta}) = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta,$$

together with the constraint

$$r - a = 0.$$

We could just substitute $r = a$ into the Lagrangian, obtaining a system with one degree of freedom, and proceed from there. However, we will consider the system as one with two degrees of freedom, $q_1 = r$ and $q_2 = \theta$, together with a constraint $G(r) = 0$, where $G(r) = r - a$. Hamilton's principle and the method of Lagrange multipliers imply the system evolves according to the pair of ordinary differential equations together with the algebraic constraint given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= \lambda \frac{\partial G}{\partial r},\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= \lambda \frac{\partial G}{\partial \theta},\\ G &= 0.\end{aligned}$$

Substituting the form for the Lagrangian above, the two ordinary differential equations together with the algebraic constraint become

$$\begin{aligned}m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta &= \lambda,\\ \frac{\mathrm{d}}{\mathrm{d}t}\big(mr^2\dot{\theta}\big) + mgr\sin\theta &= 0,\\ r - a &= 0.\end{aligned}$$

Note that the constraint is of course $r = a$, which implies $\dot{r} = 0$. Using this, the system of differential algebraic equations thus reduces to

$$ma^2\ddot{\theta} + mga\sin\theta = 0,$$

which comes from the second equation above. The first equation tells us that the Lagrange multiplier is given by

$$\lambda(t) = -ma\dot{\theta}^2 - mg\cos\theta.$$

The Lagrange multiplier has a physical interpretation, it is the normal reaction force, which here is the tension in the rod.

*Remark 13 (**Non-holonomic constraints**).* Mechanical systems with some types of non-holonomic constraints can also be treated, in particular constraints of the form

$$\sum_{j=1}^{n} A(\boldsymbol{q}, t)_{kj}\dot{q}_j + b_k(\boldsymbol{q}, t) = 0,$$

for $k = 1, \ldots, m$, where $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$. Note the assumption is that these equations are not integrable, in particular not exact, otherwise the constraints would be holonomic.

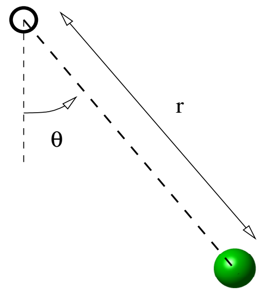

Fig. 2.1 The mechanical problem for the simple pendulum, can be thought of as a particle of mass m moving in a vertical plane, that is constrained to always be a distance a from a fixed point. In polar coordinates, the position of the mass is (r, θ) and the constraint is $r = a$.

2.5 Hamiltonian mechanics

We consider mechanical systems that are holonomic and and conservative (or for which the applied forces have a generalized potential). For such a system we can construct a Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$, where $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$, which is the difference of the total kinetic T and potential V energies. These mechanical systems evolve according to the n Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0,$$

for $j = 1, \ldots, n$. These are each second order ordinary differential equations and so the system is determined for all time once $2n$ initial conditions $\big(\boldsymbol{q}(t_0), \dot{\boldsymbol{q}}(t_0)\big)$ are specified (or n conditions at two different times). The state of the system is represented by a point $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$ in *configuration space*.

Definition 4 (Generalized momenta). We define the *generalized* momenta for a Lagrangian mechanical system for $j = 1, \ldots, n$ to be

$$p_j = \frac{\partial L}{\partial \dot{q}_j}.$$

Note that we have $p_j = p_j(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ in general, where $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$ and $\dot{\boldsymbol{q}} = (\dot{q}_1, \ldots, \dot{q}_n)^{\mathrm{T}}$.

In terms of the generalized momenta, Lagrange's equations become

$$\dot{p}_j = \frac{\partial L}{\partial q_j},$$

for $j = 1, \ldots, n$. Further, *in principle*, we can solve the relations above which define the generalized momenta, to find functional expressions for the $\dot{q}_j$ in terms of q_i, p_i and t, i.e. we can solve the relations defining the generalized momenta to find $\dot{q}_j = \dot{q}_j(\boldsymbol{q}, \boldsymbol{p}, t)$ where $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$ and $\boldsymbol{p} = (p_1, \ldots, p_n)^{\mathrm{T}}$.

Definition 5 (Hamiltonian). We define the *Hamiltonian function* as the *Legendre transform* of the Lagrangian function, i.e. we define it to be

$$H(\boldsymbol{q}, \boldsymbol{p}, t) := \dot{\boldsymbol{q}} \cdot \boldsymbol{p} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t),$$

where $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$ and $\boldsymbol{p} = (p_1, \ldots, p_n)^{\mathrm{T}}$ and we suppose $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{p}, t)$.

Note that in this definition we used the notation for the dot product

$$\dot{\boldsymbol{q}} \cdot \boldsymbol{p} = \sum_{j=1}^{n} \dot{q}_j \, p_j .$$

Remark 14. The *Legendre transform* is nicely explained in Arnold [3, Page 61] and Evans [7, Page 121]. In practice, as far as solving example problems herein, it simply involves the task of solving the equations for the generalized momenta above, to find $\dot{q}_j$ in terms of q_i, p_i and t.

From Lagrange's equation of motion we can deduce Hamilton's equations of motion, using the definitions for the generalized momenta and Hamiltonian,

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \qquad \text{and} \qquad H = \sum_{j=1}^{n} \dot{q}_j \, p_j - L.$$

Theorem 4 (Hamilton's equations of motion). *Lagrange's equations of motion imply Hamilton's canonical equations, for $i = 1, \ldots, n$ we have,*

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}. \end{aligned}$$

These consist of $2n$ first order equations of motion.

Proof. Using the definition of the Hamiltonian in terms of the Lagrangian and noting $\dot{q}_j = \dot{q}_j(\boldsymbol{q}, \boldsymbol{p}, t)$ for $j = 1, \ldots, n$, we use: (i) the chain and product rules; (ii) that $\partial p_j / \partial p_i = 0$ if $i \neq j$ while $\partial p_i / \partial p_i = 1$ and (iii) the definition of the generalized momenta. Directly computing using this sequence of results gives

$$\begin{aligned}\frac{\partial H}{\partial p_i} &= \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial p_i} p_j + \sum_{j=1}^{n} \dot{q}_j \frac{\partial p_j}{\partial p_i} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial p_i} \\ &= \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial p_i} p_j + \dot{q}_i - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial p_i} \\ &= \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial p_i} p_j + \dot{q}_i - \sum_{j=1}^{n} p_j \frac{\partial \dot{q}_j}{\partial p_i} \\ &= \dot{q}_i,\end{aligned}$$

for $i = 1, \ldots, n$. Again using the definition of the Hamiltonian as above, we use: (i) the chain and product rules; (ii) that $\partial q_j / \partial q_i = 0$ if $i \neq j$ while $\partial q_i / \partial q_i = 1$; (iii) the definition of the generalized momenta and (iv) Lagrange's equations of motion in the form $\dot{p}_j = \partial L / \partial q_j$. Directly computing using this sequence of results gives

$$\begin{aligned}\frac{\partial H}{\partial q_i} &= \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial q_i} p_j - \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial q_i} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial q_i} \\ &= \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial q_i} p_j - \frac{\partial L}{\partial q_i} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial q_i} \\ &= \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial q_i} p_j - \frac{\partial L}{\partial q_i} - \sum_{j=1}^{n} p_j \frac{\partial \dot{q}_j}{\partial q_i} \\ &= -\frac{\partial L}{\partial q_i} \\ &= -\dot{p}_i,\end{aligned}$$

for $i = 1, \ldots, n$. Collecting these relations together, we see that Lagrange's equations of motion imply Hamilton's canonical equations as shown. □

Two further observations are also useful. First, if the Lagrangian $L = L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is independent of explicit t, then when we solve the equations that define the generalized momenta we find $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{p})$. Hence we see that

$$H = \dot{\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{p}) \cdot \boldsymbol{p} - L\big(\boldsymbol{q}, \dot{\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{p})\big),$$

i.e. the Hamiltonian $H = H(\boldsymbol{q}, \boldsymbol{p})$ is also independent of t explicitly. Second, in general, using the chain rule and Hamilton's equations we see that

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} \\ &= \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial t}.\end{aligned}$$

Hence we have

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}.$$

Hence if H does *not* explicitly depend on t then

$$H \text{ is a } \begin{cases}\text{constant of the motion,} \\ \text{conserved quantity,} \\ \text{integral of the motion.}\end{cases}$$

Hence the absence of explicit t dependence in the Hamiltonian H could serve as a more general definition of a conservative system, though in general H may not be the total energy. However for *simple mechanical systems* for which the kinetic energy $T = T(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is a homogeneous quadratic function in $\dot{\boldsymbol{q}}$, and the potential $V = V(\boldsymbol{q})$, then the Hamiltonian H *will be* the total energy. To see this, suppose

$$T = \sum_{i,j=1}^{n} c_{ij}(q)\,\dot{q}_i\dot{q}_j,$$

i.e. a homogeneous quadratic function in $\dot{\boldsymbol{q}}$; necessarily $c_{ij} = c_{ji}$. Then we have

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{j=1}^{n} c_{kj}(q)\,\dot{q}_j + \sum_{i=1}^{n} c_{ik}(q)\,\dot{q}_i = 2\sum_{i=1}^{n} c_{ik}(q)\,\dot{q}_i$$

which implies

$$\sum_{k=1}^{n} \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = 2T.$$

Thus the Hamiltonian $H = 2T - (T - V) = T + V$, i.e. the total energy.

2.6 Hamiltonian formulation: summary

To construct Hamilton's canonical equations for a mechanical system proceed as follows:

1. Choose your generalized coordinates $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$ and construct

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T - V.$$

2. Define and compute the generalized momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i},$$

for $i = 1, \ldots, n$. Solve these relations to find $\dot{q}_i = \dot{q}_i(\boldsymbol{q}, \boldsymbol{p}, t)$.

3. Construct and compute the Hamiltonian function

$$H = \sum_{j=1}^{n} \dot{q}_j \, p_j - L,$$

4. Write down Hamilton's equations of motion

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}, \end{aligned}$$

for $i = 1, \ldots, n$, and evaluate the partial derivatives of the Hamiltonian on the right.

*Example 12 (**Simple harmonic oscillator**).* The Lagrangian for the simple harmonic oscillator, which consists of a mass m moving in a quadratic potential field with characteristic coefficient k, is

$$L(x, \dot{x}) = \tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2.$$

The corresponding generalized momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

which is the usual momentum. This implies $\dot{x} = p/m$ and so the Hamiltonian is given by

$$\begin{aligned} H(x, p) &= \dot{x} \, p - L(x, \dot{x}) \\ &= \frac{p}{m} \, p - \left(\tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2 \right) \\ &= \frac{p^2}{m} - \left(\tfrac{1}{2} m \left(\frac{p}{m} \right)^2 - \tfrac{1}{2} k x^2 \right) \\ &= \tfrac{1}{2} \frac{p^2}{m} + \tfrac{1}{2} k x^2. \end{aligned}$$

Note this last expression is the sum of the kinetic and potential energies and so H is the total energy. Hamilton's equations of motion are thus given by

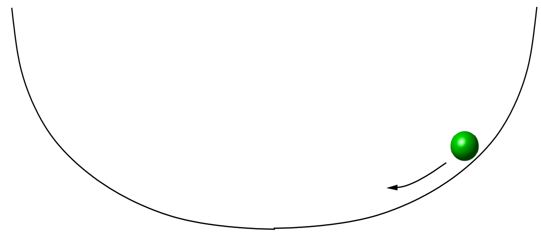

Fig. 2.2 The mechanical problem for the simple harmonic oscillator consists of a particle moving in a quadratic potential field. As shown, we can think of a ball of mass m sliding freely back and forth in a vertical plane, without energy loss, inside a parabolic shaped bowl. The horizontal position $x(t)$ is its displacement.

$$\begin{aligned}\dot{x} &= \partial H/\partial p, \\ \dot{p} &= -\partial H/\partial x,\end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned}\dot{x} &= p/m, \\ \dot{p} &= -kx.\end{aligned}$$

Note that combining these two equations, we get the usual equation for a harmonic oscillator: $m\ddot{x} = -kx$.

*Example 13 (**Kepler problem**).* Recall the Kepler problem for a mass m moving in an inverse-square central force field with characteristic coefficient μ. The Lagrangian $L = T - V$ is

$$L(r, \dot{r}, \theta, \dot{\theta}) = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu m}{r}.$$

Hence the generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \text{and} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}.$$

These imply $\dot{r} = p_r/m$ and $\dot{\theta} = p_\theta/mr^2$ and so the Hamiltonian is given by

$$\begin{aligned} H(r, \theta, p_r, p_\theta) &= \dot{r}\, p_r + \dot{\theta} p_\theta - L(r, \dot{r}, \theta, \dot{\theta}) \\ &= \frac{1}{m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \left(\tfrac{1}{2}m\left(\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2 r^4}\right) + \frac{\mu m}{r}\right) \\ &= \tfrac{1}{2}\left(\frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2}\right) - \frac{\mu m}{r}, \end{aligned}$$

which in this case is also the total energy. Hamilton's equations of motion are

$$\begin{aligned}\dot{r} &= \partial H/\partial p_r, \\ \dot{\theta} &= \partial H/\partial p_\theta, \\ \dot{p_r} &= -\partial H/\partial r, \\ \dot{p_\theta} &= -\partial H/\partial \theta,\end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned}\dot{r} &= p_r/m, \\ \dot{\theta} &= p_\theta/mr^2, \\ \dot{p_r} &= p_\theta^2/mr^3 - \mu m/r^2, \\ \dot{p_\theta} &= 0.\end{aligned}$$

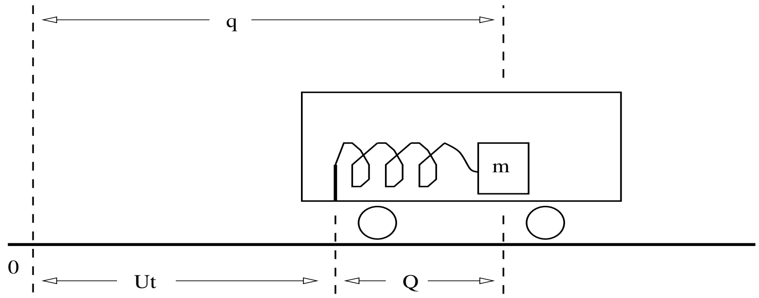

Fig. 2.3 Mass-spring system on a massless cart.

Note that $\dot{p_\theta} = 0$, i.e. we have that p_θ is constant for the motion. This property corresponds to the *conservation of angular momentum.*

Remark 15. The Lagrangian $L = T - V$ may change its functional form if we use different variables $(\boldsymbol{Q}, \dot{\boldsymbol{Q}})$ instead of $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, but its magnitude will not change. *However*, the functional form and magnitude of the Hamiltonian both depend on the generalized coordinates chosen. In particular, the Hamiltonian H may be conserved for one set of coordinates, but not for another.

*Example 14 (**Harmonic oscillator on a moving platform**).* Consider a mass-spring system, mass m and spring stiffness k, contained within a massless cart which is translating horizontally with a fixed velocity U—see Figure 2.3. The constant velocity U of the cart is maintained by an external agency. The Lagrangian $L = T - V$ for this system is

$$L(q, \dot{q}, t) = \tfrac{1}{2} m\dot{q}^2 - \tfrac{1}{2} k(q - Ut)^2.$$

The resulting equation of motion of the mass is

$$m\,\ddot{q} = -k(q - Ut).$$

If we set $Q \coloneqq q - Ut$, then the equation of motion is

$$m\,\ddot{Q} = -kQ.$$

Let us now consider the Hamiltonian formulation using two different sets of coordinates. First, using the generalized coordinate q, the corresponding generalized momentum is $p = m\dot{q} \Leftrightarrow \dot{q} = p/m$ and the Hamiltonian is

$$\begin{aligned} H(q,p,t) &= \dot{q}p - L(q,\dot{q},t) \\ &= \frac{p^2}{m} - \left(\tfrac{1}{2}m\frac{p^2}{m^2} - \tfrac{1}{2}k(q-Ut)^2\right) \\ &= \tfrac{1}{2}\frac{p^2}{m} + \tfrac{1}{2}k(q-Ut)^2. \end{aligned}$$

Here the Hamiltonian H *is* the total energy, but it is *not* conserved.

Second, using the generalized coordinate Q, if substitute $q = Q + Ut$ and $\dot{q} = \dot{Q} + U$ into the Lagrangian $L = L(q, \dot{q}, t)$ above we obtain the Lagrangian $\tilde{L} = \tilde{L}(Q, \dot{Q}, t)$ given by

$$\tilde{L}(Q,\dot{Q}) = \tfrac{1}{2}m\dot{Q}^2 + m\dot{Q}U + \tfrac{1}{2}mU^2 - \tfrac{1}{2}kQ^2.$$

Here, the generalized momentum is $P = m\dot{Q} + mU \Leftrightarrow \dot{Q} = (P - mU)/m$ and the Hamiltonian is

$$\begin{aligned} \tilde{H}(Q,P) &= \dot{Q}P - L(Q,\dot{Q}) \\ &= \frac{(P-mU)}{m}P \\ &\quad - \left(\tfrac{1}{2}m\frac{(P-mU)^2}{m^2} + m\frac{(P-mU)}{m}U + \tfrac{1}{2}mU^2 - \tfrac{1}{2}kQ^2\right) \\ &= \tfrac{1}{2}\frac{(P-mU)^2}{m} + \tfrac{1}{2}kQ^2 - \tfrac{1}{2}mU^2. \end{aligned}$$

Note that $\tilde{H}$ does not explicitly depend on t. Hence $\tilde{H}$ *is* conserved, but it is *not* the total energy.

2.7 Symmetries, conservation laws and cyclic coordinates

We have already seen that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t}.$$

Hence if the Hamiltonian does not depend explicitly on t, then it is a *constant* or *integral* of the motion; sometimes called Jacobi's integral. It *may* be the total energy. Further from the definition of the generalized momenta $p_i = \partial L/\partial \dot{q}_i$, Lagrange's equations, and Hamilton's equations for the generalized momenta, we have

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = \frac{\partial L}{\partial q_i}.$$

From these relations we can see that if q_i is explicitly absent from the Lagrangian L, then it is explicitly absent from the Hamiltonian H, and

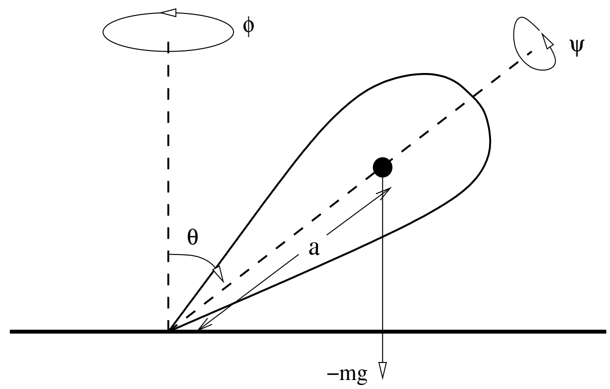

Fig. 2.4 Simple axisymmetric top: this consists of a body of mass m that spins around its axis of symmetry moving under gravity. It has a fixed point on its axis of symmetry—here this is the pivot at the narrow pointed end that touches the ground. The centre of mass is a distance a from the fixed point. The configuration of the top is given in terms of the Euler angles (θ, ϕ, ψ) shown. Typically the top also precesses around the vertical axis $\theta = 0$.

$$\dot{p}_i = 0.$$

Hence p_i is a conserved quantity, i.e. constant of the motion. Such a q_i is called *cyclic* or *ignorable*. Note that for such coordinates q_i, the transformation

$$\begin{aligned} t &\to t + \Delta t, \\ q_i &\to q_i + \Delta q_i, \end{aligned}$$

leave the Lagrangian/Hamiltonian unchanged. This invariance signifies a fundamental *symmetry* of the system.

*Example 15 (**Kepler problem**).* Recall, the Lagrangian $L = T - V$ for the Kepler problem is

$$L(r, \dot{r}, \theta, \dot{\theta}) = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu m}{r},$$

and the Hamiltonian is

$$H(r, \theta, p_r, p_\theta) = \tfrac{1}{2} m\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\mu m}{r}.$$

Note that L, H are independent of t and therefore H is conserved; and here it is the total energy. Further, L, H are independent of θ and therefore $p_\theta = mr^2\dot{\theta}$ is conserved also. We have thus just established two integrals of the motion, namely H and p_θ.

*Example 16 (**Axisymmetric top**).* Consider a simple axisymmetric top of mass m with a fixed point on its axis of symmetry; see Figure 2.4. Suppose the centre of mass is a distance a from the fixed point, and the principle moments of inertia are $A = B \neq C$. We assume there are no torques about the symmetry or vertical axes. The configuration of the top is given in terms of the Euler angles (θ, ϕ, ψ) as shown in Figure 2.4. The Lagrangian $L = T - V$ is

$$L = \tfrac{1}{2}A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \tfrac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mga\cos\theta.$$

The generalized momenta are

$$\begin{aligned} p_\theta &= A\dot{\theta}, \\ p_\phi &= A\dot{\phi}\sin^2\theta + C\cos\theta(\dot{\psi} + \dot{\phi}\cos\theta), \\ p_\psi &= C(\dot{\psi} + \dot{\phi}\cos\theta). \end{aligned}$$

Using these, the Hamiltonian is given by

$$H = \tfrac{1}{2}\frac{p_\theta^2}{A} + \tfrac{1}{2}\frac{(p_\phi - p_\psi\cos\theta)^2}{A\sin^2\theta} + \tfrac{1}{2}\frac{p_\psi^2}{C} + mga\cos\theta.$$

We see that L, H are both independent of t, ψ and ϕ. Hence H, p_ψ and p_ϕ are conserved. Respectively, H, p_ψ and p_ϕ represent the total energy (as the kinetic energy is a homogeneous quadratic function of the generalized velocities $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$), the angular momentum about the symmetry axis and the angular momentum about the vertical.

*Remark 16 (**Noether's Theorem**).* To accept only those symmetries which leave the Lagrangian unchanged is needlessly restrictive. When searching for conservation laws (integrals of the motion), we can in general consider transformations that leave the *action integral* 'invariant enough' so that we get the same equations of motion. This is the idea underlying (Emmy) Noether's theorem; see Arnold [3, Page 88].

2.8 Poisson brackets

Definition 6 (Poisson brackets). For two functions $u = u(\boldsymbol{q}, \boldsymbol{p})$ and $v = v(\boldsymbol{q}, \boldsymbol{p})$, where $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$ and $\boldsymbol{p} = (p_1, \ldots, p_n)^{\mathrm{T}}$, we define their Poisson bracket to be

$$[u, v] := \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} \right).$$

The Poisson bracket satisfies some properties that can be checked directly. For example, for any function $u = u(\boldsymbol{q}, \boldsymbol{p})$ and constant c we have $[u, c] = 0$. We also observe that if $v = v(\boldsymbol{q}, \boldsymbol{p})$ and $w = w(\boldsymbol{q}, \boldsymbol{p})$ then $[uv, w] = u[v, w] + v[u, w]$. Three crucial properties are summarized in the following lemma.

Lemma 3 (Lie algebra properties). *The bracket satisfies the following properties for all functions* $u = u(\boldsymbol{q}, \boldsymbol{p})$, $v = v(\boldsymbol{q}, \boldsymbol{p})$ *and* $w = w(\boldsymbol{q}, \boldsymbol{p})$ *and scalars* λ *and* μ*:*

1. *Skew-symmetry:* $[v, u] = -[u, v]$*;*
2. *Bilinearity:* $[\lambda u + \mu v, w] = \lambda[u, w] + \mu[v, w]$*;*
3. *Jacobi's identity:* $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$.

These three properties define a non-associative algebra known as a Lie algebra*.*

Two further simple examples of Lie algebras are: the vector space of vectors equipped with the wedge or vector product $[\boldsymbol{u}, \boldsymbol{v}] = \boldsymbol{u} \wedge \boldsymbol{v}$ and the vector space of matrices equipped with the matrix commutator product $[A, B] = AB - BA$.

Corollary 2 (Properties from the definition). *Using that all the coordinates* $(q_1, \ldots, q_n)$ *and* $(p_1, \ldots, p_n)$ *are independent, we immediately deduce by direct substitution into the definition, the following results for any* $u = u(\boldsymbol{q}, \boldsymbol{p})$ *and all* $i, j = 1, \ldots, n$*:*

$$\frac{\partial u}{\partial q_i} = [u, p_i], \quad \frac{\partial u}{\partial p_i} = -[u, q_i], \quad [q_i, q_j] = 0, \quad [p_i, p_j] = 0 \quad \text{and} \quad [q_i, p_j] = \delta_{ij}.$$

The Kronecker delta δ_{ij} *is zero when* $i \neq j$ *and unity when* $i = j$*.*

Corollary 3 (Poisson bracket for canonical variables). *If the variables* $\boldsymbol{q} = (q_1, \ldots, q_n)^T$ *and* $\boldsymbol{p} = (p_1, \ldots, p_n)^T$ *are canonical Hamilton variables, i.e. they satisfy Hamilton's equations for some Hamiltonian* H*, then for any function* $f = f(\boldsymbol{q}, \boldsymbol{p}, t)$ *we have*

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + [f, H].$$

Proof. Using Hamilton's equations of motion (in vector notation) we know

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \nabla_{\boldsymbol{p}} H \qquad \text{and} \qquad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\nabla_{\boldsymbol{q}} H.$$

Thus by the chain rule and then Hamilton's equations we find

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} \cdot \nabla_{\boldsymbol{q}} f + \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \cdot \nabla_{\boldsymbol{p}} f \\ &= \frac{\partial f}{\partial t} + \nabla_{\boldsymbol{p}} H \cdot \nabla_{\boldsymbol{q}} f + (-\nabla_{\boldsymbol{q}} H) \cdot \nabla_{\boldsymbol{p}} f \\ &= \frac{\partial f}{\partial t} + [f, H], \end{aligned}$$

by definition using the commutative property of the scalar (dot) product. □

All the properties above are essential for the next two results of this section.

First, remarkably the the Poisson bracket is *invariant under canonical transformations*. By this we mean the following. Suppose we make a transformation of the canonical coordinates $(\boldsymbol{q}, \boldsymbol{p})$, satisfying Hamilton's equations with respect to a Hamiltonian H, to new canonical coordinates $(\boldsymbol{Q}, \boldsymbol{P})$, satisfying Hamilton's equations with respect to a Hamiltonian K, where

$$\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{q}, \boldsymbol{p}) \qquad \text{and} \qquad \boldsymbol{P} = \boldsymbol{P}(\boldsymbol{q}, \boldsymbol{p}).$$

For two functions $u = u(\boldsymbol{q}, \boldsymbol{p})$ and $v = v(\boldsymbol{q}, \boldsymbol{p})$ define $U = U(\boldsymbol{Q}, \boldsymbol{P})$ and $V = V(\boldsymbol{Q}, \boldsymbol{P})$ by the identities

$$U(\boldsymbol{Q}, \boldsymbol{P}) = u(\boldsymbol{q}, \boldsymbol{p}) \qquad \text{and} \qquad V(\boldsymbol{Q}, \boldsymbol{P}) = v(\boldsymbol{q}, \boldsymbol{p}).$$

Then we have the following result—whose proof we leave as an exercise in the chain rule! (*Hint:* start with $[u, v]_{\boldsymbol{q},\boldsymbol{p}}$ and immediately substitute the definitions for U and V; also see Arnold [3, Page 216].)

Lemma 4 (Invariance under canonical transformations). *The Poisson bracket is invariant under a canonical transformations, i.e. we have*

$$[U, V]_{\boldsymbol{Q},\boldsymbol{P}} = [u, v]_{\boldsymbol{q},\boldsymbol{p}}.$$

Second, we can use Poisson Brackets to (potentially) generate further *constants of the motion*. Using Corollary 3 on the Poisson bracket of canonical variables we see Hamilton's equations of motion are equivalent to the system of equations

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = [\boldsymbol{q}, H] \qquad \text{and} \qquad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = [\boldsymbol{p}, H].$$

Here, by $[\boldsymbol{q}, H]$ we mean the vector with components $[q_i, H]$, and observe we have simply substituted the components of $\boldsymbol{q}$ and $\boldsymbol{p}$ for f in Corollary 3. If $u = u(\boldsymbol{q}, \boldsymbol{p}, t)$ is a constant of the motion then

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0,$$

and by Corollary 3 we see that

$$\frac{\partial u}{\partial t} = [H, u].$$

If $u = u(\boldsymbol{q}, \boldsymbol{p})$ only and does not depend explicitly on t then it must must Poisson commute with the Hamiltonian H, i.e. it must satisfy

$$[H, u] = 0.$$

Now suppose we have two constants of the motion $u = u(\boldsymbol{q}, \boldsymbol{p})$ and $v = v(\boldsymbol{q}, \boldsymbol{p})$, so that $[H, u] = 0$ and $[H, v] = 0$. Then by Jacobi's identity we find

$$[H,[u,v]] = -[u,[v,H]] - [v,[H,u]] = 0.$$

In other words it appears as though $[u,v]$ is another constant of the motion. (This result also extends to the case when u and v explicitly depend on t.) Indeed, in principle, we can generate a sequence of constants of the motion u, v, $[u,v]$, $[u,[u,v]]$, ... Sometimes we generate new constants of the motion by this procedure, i.e. we get new information, but often we generate a constant of the motion we already know about.

*Example 17 (**Kepler problem**).* Consider the Kepler problem of a particle of mass m in a three-dimensional central force field with Hamiltonian

$$H = \tfrac{1}{2}\frac{p_1^2 + p_2^2 + p_3^2}{m} + V(r)$$

where $V = V(r)$ is the potential with $r = \sqrt{q_1^2 + q_2^2 + q_3^2}$. Known constants of the motion are $u = q_2p_3 - q_3p_2$ and $v = q_3p_1 - q_1p_3$ and their Poisson bracket $[u,v] = q_1p_2 - q_2p_1$ is another constant of the motion (not too surprisingly in this case).

Exercises

2.1. Motion of relativistic particles
A particle with position $\boldsymbol{x}(t) \in \mathbb{R}^3$ at time t prescribes a path that minimizes the functional

$$\int_{t_0}^{t_1} \left(-m_0c^2\sqrt{1 - \frac{|\dot{\boldsymbol{x}}|^2}{c^2}} - U(\boldsymbol{x}) \right) \mathrm{d}t,$$

subject to $\boldsymbol{x}(t_0) = \boldsymbol{a}$ and $\boldsymbol{x}(t_1) = \boldsymbol{b}$. Show the equation of evolution of the particle is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \right) = -\nabla U, \qquad \text{where} \qquad m = \frac{m_0}{\sqrt{1 - \frac{|\dot{\boldsymbol{x}}|^2}{c^2}}}.$$

This is the equation of motion of a relativistic particle in an inertial system, under the influence of the force $-\nabla U$. Note the *relativistic mass* m of the particle depends on its velocity. This mass goes to infinity if the particle approaches the velocity of light c.

2.2. Central force field
A particle of mass m moves under the central force $F = -\mathrm{d}V(r)/\mathrm{d}r$ in the spherical coordinate system such that

$$x = r\cos\phi\sin\theta,$$
$$y = r\sin\phi\sin\theta,$$
$$z = r\cos\theta.$$

The total kinetic energy of the system $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ in spherical polar coordinates is $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$. Hence the Lagrangian is given by

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - V(r).$$

From this Lagrangian, show that: (a) the quantity $mr^2\dot{\phi}\sin^2\theta$ is a constant of the motion (call this h); (b) the two remaining equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = \frac{h^2}{m^2r^2}\cot\theta\,\mathrm{cosec}^2\theta,$$
$$\ddot{r} = r\dot{\theta}^2 + \frac{h^2}{m^2r^3}\mathrm{cosec}^2\theta - \frac{1}{m}\frac{\mathrm{d}V}{\mathrm{d}r}.$$

2.3. Spherical pendulum

An inextensible string of length ℓ is fixed at one end and has a bob of mass m attached to the other. This bob swings freely under gravity, forming a spherical pendulum. The Lagrangian for this system is given by

$$L(\theta,\phi,\dot{\theta},\dot{\phi}) = \tfrac{1}{2}m\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + mg\ell\cos\theta,$$

where (θ,ϕ) are spherical angle coordinates centred at the fixed end of the pendulum: θ measures the angle to the vertical downward direction, and ϕ represents the azimuthal angle. In addition, g represents the acceleration due to gravity.

(a) Briefly explain why the Lagrangian for a spherical pendulum has the form shown.

(b) Identify a conserved quantity.

(c) Write down the pair of Euler–Lagrange equations for this system and use them to show, first that

$$m\ell^2\sin^2\theta\cdot\dot{\phi} = \text{constant} = K,$$

and second, the equation of motion for θ is given by

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta + \left(\frac{K}{m\ell^2}\right)^2\cot\theta\,\mathrm{cosec}^2\theta.$$

(d) There is a solution to the spherical pendulum for which the bob traces out a horizontal circle—i.e. the bob rotates maintaining a fixed angle θ_0 to the vertical. Show that this is only possible for values of θ_0 satisfying

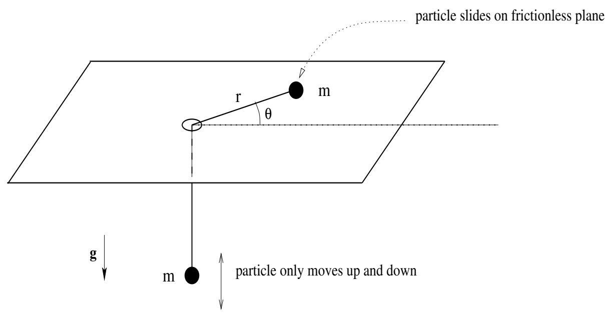

Fig. 2.5 Horizontal Atwood's machine: a string of length ℓ, with a mass m at each end, passes through a hole in a horizontal frictionless plane. One mass moves horizontally on the plane, the other hangs vertically downwards and only moves up and down

$$\left(\frac{g}{\ell}\right)\left(\frac{m\ell^2}{K}\right)^2 \sin^4\theta_0 = \cos\theta_0.$$

Does a solution $0 < \theta_0 < \pi$ exist?

2.4. Pendulum with moving frictionless support

A pendulum system consists of a light rod, of length ℓ, with a mass M connected at one end that can slide freely along the x-axis, and a mass m at the other end that swings freely in the vertical plane containing the x-axis—see Figure 2.6 on the next page. Let $\mu(t)$ represent the position of the mass M along the x-axis, and $\theta(t)$ the angle the rod makes with the vertical, at time t.

(a) Show that the Lagrangian for this system is

$$L(\mu,\theta,\dot{\mu},\dot{\theta}) = \tfrac{1}{2}M\dot{\mu}^2 + \tfrac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{\mu}^2 + 2\ell\dot{\mu}\dot{\theta}\cos\theta) + mg\ell\cos\theta.$$

(b) Derive explicit expressions for the generalized momenta p_μ and p_θ corresponding to the coordinates μ and θ, respectively.

(c) Explain why the Hamiltonian (no need to derive it) and p_μ are constants of the motion.

(d) Assume $p_\mu = 0$ (this corresponds to assuming that the centre of mass of the system is not uniformly translating in the x-direction) and show that

$$(M+m)\mu = -m\ell\sin\theta + A,$$

where A is an arbitrary constant.

(e) Using the result from part (d), write down Lagrange's equation of motion for the angle $\theta = \theta(t)$.

(f) Using the result for $\mu = \mu(t)$ in part (d) above, show that the position of the mass m at time t in Cartesian x and y coordinates is given by

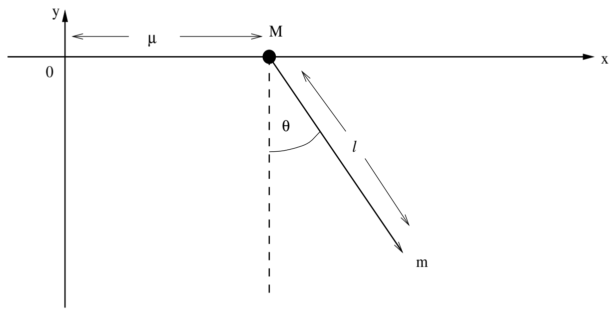

Fig. 2.6 Pendulum with moving frictionless support.

$$x = \left(\frac{M\ell}{M+m}\right)\sin\theta + \frac{A}{M+m},$$
$$y = -\ell\cos\theta.$$

What is the shape of this curve with respect to the x and y coordinates?

2.5. Horizontal Atwood machine

A string of length ℓ has a mass m at each end, and passes through a hole in a horizontal frictionless plane. One mass moves horizontally on the plane, the other hangs vertically downwards—see Figure 2.5. The Lagrangian for this system has the form

$$L(r,\theta,\dot{r},\dot{\theta}) = \tfrac{1}{2}m(2\dot{r}^2 + r^2\dot{\theta}^2) + mg(\ell - r),$$

where (r,θ) are the plane polar coordinates of the mass that moves on the plane.

(a) Show that the generalized momenta p_r and p_θ corresponding to the coordinates r and θ, respectively, are given by

$$p_r = 2m\dot{r} \qquad \text{and} \qquad p_\theta = mr^2\dot{\theta}.$$

(b) Using the results from part (a), show that the Hamiltonian for this system is given by

$$H(r,\theta,p_r,p_\theta) = \tfrac{1}{4}\frac{p_r^2}{m} + \tfrac{1}{2}\frac{p_\theta^2}{mr^2} - mg(\ell - r).$$

(c) Explain why the Hamiltonian H and generalized momentum p_θ are constants of the motion. Explain why the Hamiltonian equals the total energy of the system.

(d) In light of the information in part (c) above, we can express the Hamiltonian in the form

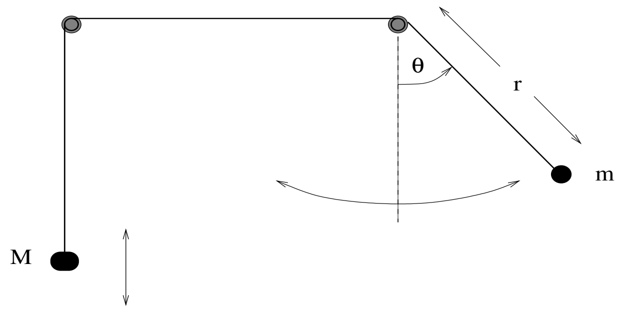

Fig. 2.7 Swinging Atwood's machine: a string of length ℓ, with a mass M at one end and a mass m at the other, is stretched over two pulleys. The mass M hangs vertically downwards; it only moves up and down. The mass m is free to swing in a vertical plane.

$$H(r, \theta, p_r, p_\theta) = \tfrac{1}{4}\frac{p_r^2}{m} + V(r),$$

where

$$V(r) = \tfrac{1}{2}\frac{p_\theta^2}{mr^2} - mg(\ell - r).$$

In other words, we can now think of the system as a particle moving in a potential given by $V(r)$.

Sketch V as a function of r. Describe qualitatively the different dynamics for the particle you might expect to see. (*Hint:* Use that the Hamiltonian, which is the total energy, is a constant of the motion.)

2.6. Swinging Atwood machine

The *swinging Atwood machine* is a mechanism that resembles a simple Atwood machine except that one of the masses is allowed to swing in a two-dimensional plane—see Figure 2.7. A string of length ℓ, with a mass M at one end and a mass m at the other, is stretched over two frictionless pulleys as shown in Figure 2.7. The mass M hangs vertically downwards; it only moves up and down. The mass m on the other hand is free to swing in a vertical plane as shown. The Lagrangian for this system has the form

$$L(r, \theta, \dot{r}, \dot{\theta}) = \tfrac{1}{2}(M + m)\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - gr(M - m\cos\theta),$$

where (r, θ) are the plane polar coordinates of the mass m that can swing in the vertical plane. Here g is the acceleration due to gravity.

(a) Show that the generalized momenta p_r and p_θ corresponding to the coordinates r and θ, respectively, are given by

$$p_r = (M + m)\dot{r} \qquad \text{and} \qquad p_\theta = mr^2\dot{\theta}.$$

(b) Using the results from part (a), show that the Hamiltonian for this system is given by

$$H(r,\theta,p_r,p_\theta) = \tfrac{1}{2}\frac{p_r^2}{(M+m)} + \tfrac{1}{2}\frac{p_\theta^2}{mr^2} + gr(M - m\cos\theta).$$

(c) Explain why the Hamiltonian H is a constant of the motion. Is the Hamiltonian H equal to the total energy?

(d) By either using Lagrange's equations of motion, or, using Hamilton's equations of motion, show that the swinging Atwood's machine evolves according to a pair of second order ordinary differential equations

$$\begin{aligned} \ddot{r} &= \frac{1}{M+m}\big(mr\dot{\theta}^2 - g(M - m\cos\theta)\big), \\ r\,\ddot{\theta} &= -2\dot{r}\dot{\theta} - g\sin\theta. \end{aligned}$$

2.7. Pendulum with moving support

A pendulum system consists of a light rod, of length ℓ, with a mass m connected at one end which is free to move in a vertical plane—see Figure 2.8. The other end of the rod is forced to move vertically so that its displacement at any time t, is given by $y = -h(t)$ where $h = h(t)$ is a given function. If θ is the angle the rod makes with the downward verticle direction as shown in Figure 2.8, the position of the mass m in Cartesian coordinates $x = x(t)$ and $y = y(t)$ at time t is given by

$$\begin{aligned} x &= \ell\sin\theta, \\ y &= -h - \ell\cos\theta. \end{aligned}$$

(a) Explain why the Lagrangian for this system is given by

$$L(\theta,\dot{\theta}) = \tfrac{1}{2}m\big(\ell^2\dot{\theta}^2 - 2\ell\sin\theta\,\dot{h}\dot{\theta} + \dot{h}^2\big) + mg(h + \ell\cos\theta).$$

(b) Is the Hamiltonian for this system conserved? Is it the total energy?

(c) Derive an explicit expression for the generalized momentum p corresponding to the coordinate θ.

(d) Derive the Hamiltonian for this system.

(e) Using Lagrange's equations of motion or Hamilton's equations of motion, show that the ordinary differential equation governing the evolution of the mass m is given by

$$\ddot{\theta} = \frac{1}{\ell}\sin\theta\big(\ddot{h} - g\big).$$

What do the terms on the right of this evolution equation correspond to?

2.8. Bead on a rotating wire

Consider a particle bead constrained to move on a circular wire as shown in Figure 2.9 (on the next page). The bead can slide freely along the wire. The bead is assumed to have mass m. The forces that act on it are that

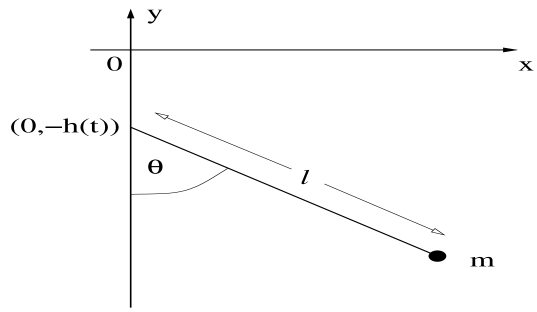

Fig. 2.8 Pendulum moving in a fixed vertical plane with support forced to move vertically so that its displacement at any time t, is given by $y = -h(t)$ where $h = h(t)$ is a given function.

due to gravity and the constraining forces that keep it on the wire. The wire itself rotates about a vertical axis. The position of the bead can be given in spherical polar coordinates (ℓ, θ, ϕ) where ℓ is the radial distance to the origin, this is *fixed*, θ is the angle to the vertical downward direction—see Figure 2.9 (on the next page). Also, ϕ is the azimuthal angle which measures how far round from the x-axis the particle bead is. You are given that the total kinetic energy for the bead is

$$T = \tfrac{1}{2}m(\ell^2\dot{\theta}^2 + \ell^2 \sin^2\theta\,\dot{\phi}^2),$$

while its potential energy is given by

$$V = -mg\ell\cos\theta,$$

where g is the acceleration due to gravity.

(a) Write down the Lagrangian $L(\theta, \phi, \dot{\theta}, \dot{\phi})$ for the bead-wire system.

(b) Looking at the form for the Lagrangian, explain briefly why:

(i) the quantity $p_\phi := m\ell^2 \sin^2\theta\,\dot{\phi}$ is a constant of the motion;

(ii) the corresponding Hamiltonian will be conserved;

(iii) the corresponding Hamiltonian will be the total energy.

(c) Write down Lagrange's equations of motion for the bead-wire system—you should have two.

(d) Now taking this problem in a different direction, assume the wire is rotating at a constant angular velocity Ω, i.e. we have $\dot{\phi} \equiv \Omega$. Use this fact in your equation of motion for θ from part (c) to show that

$$\ddot{\theta} = \sin\theta\cos\theta \cdot \Omega^2 - \frac{g}{\ell}\sin\theta.$$

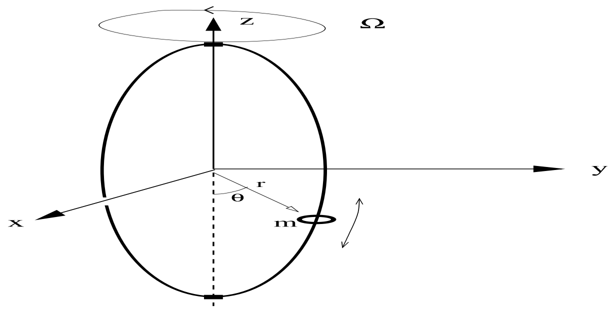

Fig. 2.9 The particle bead is constrained to move on the circular wire. The circular wire itself rotates with a fixed angular speed of Ω radians per second about a fixed vertical axis.

(e) What happens to the equation of motion in part (d) when $\Omega = 0$? What do the equations of motion represent in that case? And what is the significance of g/ℓ?

(f) Now suppose $\Omega \neq 0$. Is there a solution to the equations of motion in part (d), where the wire rotates with the bead at a non-zero fixed angle, say $\theta = \theta_0$? Explain your reasoning.

2.9. Falling stick

A uniform stick of mass m and length ℓ falls from an inclined position on a frictionless plane as shown in Figure 2.10 on the next page. Note that throughout its motion the lower end of the stick is always in contact with plane, this is the point P shown in Figure 2.10 which moves as the stick falls. Assume that the stick falls in a vertical plane and the angle the stick makes with the frictionless plane is $\theta = \theta(t)$.

(a) Explain briefly why the centre of mass of the stick has Cartesian coordinates $x = x(t)$ and $y = y(t)$ where

$$y = \frac{\ell}{2}\sin\theta.$$

(b) Explain briefly why the total kinetic energy T for the system is given by

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \frac{\ell}{4}\cos^2\theta\,\dot{\theta}^2\right) + \tfrac{1}{2}C\,\dot{\theta}^2,$$

where C is the moment of inertia of the stick, and why the total potential energy V for the system is given by

$$V = mg\frac{\ell}{2}\sin\theta.$$

(c) What is the form of the Lagrangian for this system? Without deriving the Hamiltonian explain why it is conserved for this system. Is the Hamiltonian the total energy of the system? Are there any other constants of the motion?

(d) Write down Lagrange's equations for motion for $x = x(t)$ and $\theta = \theta(t)$. What does the Lagrange equation for $x = x(t)$ tell us? Show that Lagrange's equation of motion for $\theta = \theta(t)$ can be written in the form

$$\Big(m\frac{\ell^2}{4}\cos^2\theta + C\Big)\ddot{\theta} - m\frac{\ell^2}{4}\sin\theta\,\cos\theta\,\dot{\theta}^2 + mg\frac{\ell}{2}\cos\theta = 0.$$

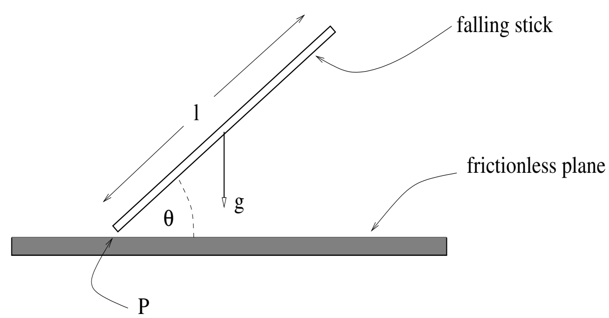

Fig. 2.10 A uniform stick of mass m and length ℓ falls from an inclined position on a frictionless plane. Throughout its motion the lower end of the stick is always in contact with plane, this is the point P shown which moves as the stick falls. We assume the stick falls in a vertical plane and the angle the stick makes with the frictionless plane is $\theta = \theta(t)$ as shown.

2.10. Particle in a cone

A cone of semi-angle α has its axis vertical and vertex downwards, as in Figure 2.11. A point mass m slides without friction on the inside of the cone under the influence of gravity which acts along the negative z direction. The Lagrangian for the particle is

$$L(r, \theta, \dot{r}, \dot{\theta}) = \tfrac{1}{2}m\left(r^2\dot{\theta}^2 + \frac{\dot{r}^2}{\sin^2\alpha}\right) - \frac{mgr}{\tan\alpha},$$

where (r, θ) are plane polar coordinates as shown in Figure 2.11.

(a) Show that the generalized momenta p_r and p_θ corresponding to the coordinates r and θ, respectively, are given by

$$p_r = \frac{m\dot{r}}{\sin^2\alpha} \qquad \text{and} \qquad p_\theta = mr^2\dot{\theta}.$$

(b) Show that the Hamiltonian for this system is given by

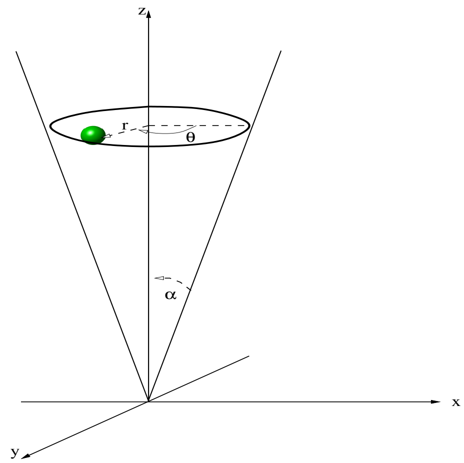

Fig. 2.11 Particle sliding without friction inside a cone of semi-angle α, axis vertical and vertex downwards.

$$H(r,\theta,p_r,p_\theta) = \tfrac{1}{2}\frac{\sin^2\alpha}{m}\,p_r^2 + \tfrac{1}{2}\frac{p_\theta^2}{mr^2} + \frac{mgr}{\tan\alpha}.$$

(c) Explain why the Hamiltonian H and generalized momentum p_θ are constants of the motion.

(d) In light of the information in part (c) above, we can express the Hamiltonian in the form

$$H(r,\theta,p_r,p_\theta) = \tfrac{1}{2}\frac{\sin^2\alpha}{m}\,p_r^2 + V(r),$$

where

$$V(r) = \tfrac{1}{2}\frac{p_\theta^2}{mr^2} + \frac{mgr}{\tan\alpha}.$$

In other words, we can now think of the system as a particle moving in a potential given by $V(r)$.

Sketch V as a function of r. Describe qualitatively the different dynamics for the particle you might expect to see.

2.11. Spherical pendulum revisited

An inextensible string of length ℓ is fixed at one end and has a bob of mass m attached to the other. This bob swings freely under gravity, forming a spherical pendulum. Recall the form of the Lagrangian from the exercise above. Write down the Hamiltonian for this system, and identify a constant, J, of the motion (distinct from the Hamiltonian, which is also conserved). If θ is the angle the string makes with the vertical, show that the Hamiltonian can be written in the form

$$H = \tfrac{1}{2}\frac{p_\theta^2}{m\ell^2} + U(\theta).$$

where θ and p_θ are the canonical coordinates and $U(\theta)$ is the effective potential. Sketch $U(\theta)$ showing that it has a local minimum at θ_0, where θ_0 satisfies,

$$\left(\frac{J}{m\ell}\right)\cos\theta_0 = g\ell\sin^4\theta_0.$$

Briefly describe the possible behaviour of this system.

2.12. Axisymmetric top

The *axisymmetric top* is the spinning top that was once a standard child's toy—see Figure 2.4 in the notes. It consists of a body of mass m that spins around its axis of symmetry moving under gravity. It has a fixed point on its axis of symmetry—in Figure 2.4 it is pivoted at the narrow pointed end that touches the ground. Suppose the centre of mass is a distance a from the fixed point, and the constant principle moments of inertia are $A = B \neq C$. We assume there are no torques about the symmetry or vertical axes. The configuration of the top is given in terms of the Euler angles (θ, ϕ, ψ) as shown in Figure 2.4. Besides spinning about its axis of symmetry, the top typically precesses about the vertical axis $\theta = 0$. The Lagrangian for an axisymmetric top has the form

$$L = \tfrac{1}{2}A(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \tfrac{1}{2}C(\dot\psi + \dot\phi\cos\theta)^2 - mga\cos\theta.$$

Here g is the acceleration due to gravity.

(a) Show that the generalized momenta p_θ, p_ϕ and p_ψ corresponding to the coordinates θ, ϕ and ψ, respectively, are given by

$$\begin{aligned}
p_\theta &= A\dot\theta,\\
p_\phi &= A\dot\phi\sin^2\theta + C\cos\theta(\dot\psi + \dot\phi\cos\theta),\\
p_\psi &= C(\dot\psi + \dot\phi\cos\theta).
\end{aligned}$$

By looking at the form for the Lagrangian L, explain why p_ϕ and p_ψ are constants of the motion. Further, explain why we know that the Hamiltonian H is a constant of the motion as well. Is the Hamiltonian H equal to the total energy?

(b) The Hamiltonian for this system can be expressed in the form (*note*, you may take this as given):

$$H = \tfrac{1}{2}\frac{p_\theta^2}{A} + U(\theta),$$

where

$$U(\theta) = \tfrac{1}{2}\frac{(p_\phi - p_\psi\cos\theta)^2}{A\sin^2\theta} + \tfrac{1}{2}\frac{p_\psi^2}{C} + mga\cos\theta.$$

Using the substitution $z = \cos\theta$ in the potential U, and recalling from part (a) above that p_ϕ, p_ψ and H are constants of the motion, show that the motion of the axisymmetric top must at all times satisfy the constraint $F(z) = 0$ where

$$F(z) \coloneqq (p_\phi - p_\psi z)^2 + \big((A/C)p_\psi^2 + 2Amgaz + p_\theta^2 - 2AH\big)(1 - z^2).$$

(c) Note that the function $F = F(z)$ in part (b) above is a cubic polynomial in z and $-1 \leq z \leq +1$. Assume for the moment that p_θ is fixed (i.e. temporarily ignore that $p_\theta = p_\theta(t)$ evolves, and along with $\theta = \theta(t)$, describes the motion of the axisymmetric top) and thus consider $F = F(z)$ as simply a function of z. Show that $F(-1) > 0$ and $F(+1) > 0$. Using this result, explain why, between $z = -1$ and $z = +1$, $F(z)$ may have *at most* two roots.

(d) Returning to the dynamics of the axisymmetric top, note that $U(\theta) \leq H$ and that $U(\theta) = H$ if and only if $p_\theta = 0$, i.e. when $\dot{\theta} = 0$. Assuming that, when $p_\theta = 0$, the function $F = F(z)$ has exactly two roots $z_1 = \cos\theta_1$ and $z_2 = \cos\theta_2$, describe the possible motion of the top in this case.

Chapter 3
Geodesic flow

3.1 Geodesic equations

We have already seen the simple example of the curve that minimizes the distance between two points in Euclidean space—unsurprisingly a straight line. What if the two points lie on a surface or manifold? Here we wish to determine the characterizing properties of curves that minimize the distance between two such points. To begin to answer this question, we need to set out the essential components we require. Indeed, the concepts we have discussed so far and our examples above have hinted at the need to consider the notion of a manifold and its concomitant components. We assume in this section the reader is familiar with the notions of Hausdorff topology, manifolds, tangent spaces and tangent bundles. A comprehensive introduction to these can be found in Abraham and Marsden [1, Chapter 1] or Marsden and Ratiu [15]. Thorough treatments of the material in this section can for example be found in Abraham and Marsden [1], Marsden and Ratiu [15] and Jost [8].

We now introduce a mechanism to measure the "distance" between two points on a manifold. To achieve this we need to be able to measure the length of, and angles between, tangent vectors—see for example Jost [8, Section 1.4] or Tao [17]. The mechanism for this in Euclidean space is the scalar product between vectors.

Definition 7 (Riemannian metric and manifold). On a smooth manifold $\mathcal{M}$ we assign, to any point $\boldsymbol{q} \in \mathcal{M}$ and pair of vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathrm{T}_{\boldsymbol{q}}\mathcal{M}$, an inner product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{g(\boldsymbol{q})}.$$

This assignment is assumed to be smooth with respect to the base point $\boldsymbol{q} \in \mathcal{M}$, and $g = g(\boldsymbol{q})$ is known as the *Riemannian metric*. The length of a tangent vector $\boldsymbol{u} \in \mathrm{T}_{\boldsymbol{q}}\mathcal{M}$ is then

$$\|\boldsymbol{u}\|_{g(\boldsymbol{q})} := \langle \boldsymbol{u}, \boldsymbol{u} \rangle_{g(\boldsymbol{q})}^{1/2}.$$

A *Riemannian manifold* is a smooth manifold $\mathcal{M}$ equipped with a Riemannian metric.

Remark 17. Every smooth manifold can be equipped with a Riemannian metric; see Jost [8, Theorem 1.4.1].

In local coordinates with $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}} \in \mathcal{M}$, $\boldsymbol{u} = (u_1, \ldots, u_n)^{\mathrm{T}} \in \mathrm{T}_{\boldsymbol{q}}\mathcal{M}$ and $\boldsymbol{v} = (v_1, \ldots, v_n)^{\mathrm{T}} \in \mathrm{T}_{\boldsymbol{q}}\mathcal{M}$, the Riemannian metric $g = g(\boldsymbol{q})$ is a real symmetric invertible positive definite matrix and the inner product above is

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{g(\boldsymbol{q})} := \sum_{i,j=1}^{n} g_{ij}(\boldsymbol{q}) u_i v_j.$$

We are interested in minimizing the length of a smooth curve on $\mathcal{M}$, however as we will see, computations based on the energy of a curve are simpler.

Definition 8 (Length and energy of a curve). Let $\boldsymbol{q}\colon [a,b] \to \mathcal{M}$ be a smooth curve. Then we define the *length* of this curve by

$$\ell(\boldsymbol{q}) := \int_a^b \big\| \dot{\boldsymbol{q}}(t) \big\|_{g(\boldsymbol{q}(t))} \, \mathrm{d}t.$$

We define the *energy* of the curve to be

$$E(\boldsymbol{q}) := \tfrac{1}{2} \int_a^b \big\| \dot{\boldsymbol{q}}(t) \big\|^2_{g(\boldsymbol{q}(t))} \, \mathrm{d}t.$$

Remark 18. The energy $E(\boldsymbol{q})$ is the *action* associated with the curve $\boldsymbol{q} = \boldsymbol{q}(t)$ on $[a,b]$.

*Remark 19 (**Distance function**).* The *distance* between two points $\boldsymbol{q}_a, \boldsymbol{q}_b \in \mathcal{M}$, assuming $\boldsymbol{q}(a) = \boldsymbol{q}_a$ and $\boldsymbol{q}(b) = \boldsymbol{q}_b$, can be defined as $\mathrm{d}(a,b) := \inf_{\boldsymbol{q}} \big\{ \ell(\boldsymbol{q}) \big\}$. This distance function satisfies the usual axioms (positive definiteness, symmetry in its arguments and the triangle inequality); see Jost [8, pp. 15–16].

*Example 18 (**Length on sphere surface**).* Recall the spherical geodesic Problem 1.4 in the Exercises at the end of Chapter 1. Here we parameterized the curve using $t \in [a,b]$ and the latitude $\theta = \theta(t)$ and azimuthal $\phi = \phi(t)$ angles prescribe the curve on the sphere surface. Hence we have $q_1 = \theta$, $q_2 = \phi$ and $\dot{q}_1 = \dot{\theta}$, $\dot{q}_2 = \dot{\phi}$. Using the form for the arclength in Problem 1.4 we see

$$\begin{aligned} \big\| \dot{\boldsymbol{q}}(t) \big\|_{g(\boldsymbol{q}(t))} &= \Bigg(\sum_{i,j=1}^{2} g_{ij}(\boldsymbol{q}) \dot{q}_i \dot{q}_j \Bigg)^{\frac{1}{2}} \\ &= \big(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \big)^{\frac{1}{2}} \end{aligned}$$

and so

$$g(\boldsymbol{q}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

By the Cauchy–Schwarz inequality we know that

$$\int_a^b \left\|\dot{\boldsymbol{q}}(t)\right\|_{g(\boldsymbol{q}(t))} \mathrm{d}t \leqslant |b-a|^{\frac{1}{2}} \left(\int_a^b \left\|\dot{\boldsymbol{q}}(t)\right\|^2_{g(\boldsymbol{q}(t))} \mathrm{d}t \right)^{\frac{1}{2}}.$$

Using the definitions of $\ell(\boldsymbol{q})$ and $E(\boldsymbol{q})$ this implies

$$\big(\ell(\boldsymbol{q})\big)^2 \leqslant 2\,|b-a|\,E(\boldsymbol{q}).$$

We have *equality* in this last statement, i.e. we have $\big(\ell(\boldsymbol{q})\big)^2 = 2\,|b-a|\,E(\boldsymbol{q})$, if and only if $\dot{\boldsymbol{q}}$ is constant. The length $\ell(\boldsymbol{q})$ of a smooth curve $\boldsymbol{q} = \boldsymbol{q}(t)$ is invariant to reparameterization; see for example Jost [8, p. 17]. Hence we can always parameterize a curve so as to arrange for $\dot{\boldsymbol{q}}$ to be constant (this is also known as parameterization proportional to arclength). After such a parameterization, minimizing the energy is equivalent to minimizing the length, and this is how we proceed henceforth.

We are now in a position to derive the *geodesic equations* that characterize the curves that minimize the length/energy between two arbitrary points on a smooth manifold $\mathcal{M}$. For more background and further reading see Abraham and Marsden [1, p, 224–5], Marsden and Ratiu [15, Section 7.5], Montgomery [16, pp. 6–10] and Tao [17]. Our goal is thus to minimize the total energy of a path $\boldsymbol{q} = \boldsymbol{q}(t)$ where $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$. The total energy of a path $\boldsymbol{q} = \boldsymbol{q}(t)$, between $t = a$ and $t = b$, as we have seen, can be defined in terms of a Lagrangian function $L = L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ as follows

$$E(\boldsymbol{q}) := \int_{t_1}^{t_2} L\big(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)\big)\,\mathrm{d}t.$$

The path $\boldsymbol{q} = \boldsymbol{q}(t)$ that minimizes the total energy $E = E(\boldsymbol{q})$ necessarily satisfies the Euler–Lagrange equations. Here these take the form of Lagrange's equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0,$$

for each $j = 1, \ldots, n$. In the following, we use g^{ij} to denote the inverse matrix of the symmetric positive definite matrix g_{ij} (where $i, j = 1, \ldots, n$) so that

$$\sum_{k=1}^{n} g^{ik} g_{kj} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta function that is 1 when $i = j$ and 0 otherwise.

Lemma 5 (Geodesic equations). *Lagrange's equations of motion for the Lagrangian*

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) := \tfrac{1}{2} \sum_{i,k=1}^{n} g_{ik}(\boldsymbol{q})\, \dot{q}_i \dot{q}_k,$$

are given in local coordinates by the system of ordinary differential equations

$$\ddot{q}_i + \sum_{j,k=1}^{n} \Gamma^i_{jk}\, \dot{q}_j \dot{q}_k = 0,$$

where the quantities Γ^i_{jk} *are known as the* Christoffel symbols *and for* $i, j, k = 1, \ldots, n$ *are given by*

$$\Gamma^i_{jk} := \sum_{\ell=1}^{n} \tfrac{1}{2} g^{i\ell} \left(\frac{\partial g_{\ell j}}{\partial q_k} + \frac{\partial g_{k\ell}}{\partial q_j} - \frac{\partial g_{jk}}{\partial q_\ell} \right).$$

Proof. We complete the proof in three steps as follows.

Step 1. For the Lagrangian $L = L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ defined in the statement of the lemma, using the product and chain rules, we find that

$$\frac{\partial L}{\partial q_j} = \tfrac{1}{2} \sum_{i,k=1}^{n} \frac{\partial g_{ik}}{\partial q_j}\, \dot{q}_i \dot{q}_k$$

and

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}_j} &= \tfrac{1}{2} \sum_{k=1}^{n} g_{jk}\, \dot{q}_k + \tfrac{1}{2} \sum_{i=1}^{n} g_{ij}\, \dot{q}_i \\ &= \tfrac{1}{2} \sum_{k=1}^{n} g_{jk}\, \dot{q}_k + \tfrac{1}{2} \sum_{k=1}^{n} g_{kj}\, \dot{q}_k \\ &= \sum_{k=1}^{n} g_{jk}\, \dot{q}_k, \end{aligned}$$

where in the last step we utilized the symmetry of g. Using this last expression we see

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_j} \right) &= \sum_{k=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t} \big(g_{jk}\, \dot{q}_k \big) \\ &= \sum_{k=1}^{n} \frac{\mathrm{d}g_{jk}}{\mathrm{d}t}\, \dot{q}_k + \sum_{k=1}^{n} \big(g_{jk}\, \ddot{q}_k \big) \\ &= \sum_{k,\ell=1}^{n} \frac{\partial g_{jk}}{\mathrm{d}q_\ell}\, \dot{q}_\ell \dot{q}_k + \sum_{k=1}^{n} \big(g_{jk}\, \ddot{q}_k \big). \end{aligned}$$

Substituting the expressions above into Lagrange's equations of motion, using that the summation indices i and k can be relabelled and that g is symmetric, we find

$$\sum_{k=1}^{n} g_{jk}\ddot{q}_k + \sum_{k,\ell=1}^{n}\Big(\frac{\partial g_{jk}}{\partial q_\ell} - \tfrac{1}{2}\frac{\partial g_{k\ell}}{\partial q_j}\Big)\dot{q}_k\dot{q}_\ell = 0,$$

for each $j = 1, \ldots, n$.

Step 2. We multiply the last equations from Step 1 by g^{ij} and sum over the index j. This gives

$$\begin{aligned}
&\sum_{j,k=1}^{n} g^{ij}g_{jk}\ddot{q}_k + \sum_{j,k,\ell=1}^{n} g^{ij}\Big(\frac{\partial g_{jk}}{\partial q_\ell} - \tfrac{1}{2}\frac{\partial g_{k\ell}}{\partial q_j}\Big)\dot{q}_k\dot{q}_\ell = 0\\
\Leftrightarrow\quad &\ddot{q}_i + \sum_{j,k,\ell=1}^{n} g^{ij}\Big(\frac{\partial g_{jk}}{\partial q_\ell} - \tfrac{1}{2}\frac{\partial g_{k\ell}}{\partial q_j}\Big)\dot{q}_k\dot{q}_\ell = 0\\
\Leftrightarrow\quad &\ddot{q}_i + \sum_{j,k,\ell=1}^{n} g^{i\ell}\Big(\frac{\partial g_{\ell j}}{\partial q_k} - \tfrac{1}{2}\frac{\partial g_{jk}}{\partial q_\ell}\Big)\dot{q}_j\dot{q}_k = 0,
\end{aligned}$$

where in the last step we relabelled summation indices.

Step 3. By using the symmetry of g and performing some further relabelling of summation indices, we see

$$\begin{aligned}
\sum_{j,k,\ell=1}^{n} g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial q_k} - \tfrac{1}{2}\frac{\partial g_{jk}}{\partial q_\ell}\right)\dot{q}_j\dot{q}_k &= \sum_{j,k,\ell=1}^{n} \tfrac{1}{2}g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial q_k} + \frac{\partial g_{\ell j}}{\partial q_k} - \frac{\partial g_{jk}}{\partial q_\ell}\right)\dot{q}_j\dot{q}_k\\
&= \sum_{j,k,\ell=1}^{n} \tfrac{1}{2}g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial q_k} + \frac{\partial g_{k\ell}}{\partial q_j} - \frac{\partial g_{jk}}{\partial q_\ell}\right)\dot{q}_j\dot{q}_k\\
&= \sum_{j,k=1}^{n} \Gamma^i_{ij}\,\dot{q}_j\dot{q}_k,
\end{aligned}$$

where the Γ^i_{ij} are those given in the statement of the lemma. □

Remark 20. On a smooth Riemannian manifold there is unique torsion free *connection* ∇ or *covariant derivative* defined on the tangent bundle known as the *Levi–Civita* connection. A curve $\boldsymbol{q}\colon [a,b] \to \mathcal{M}$ is *autoparallel* or *geodesic* with respect to ∇ if

$$\nabla_{\dot{\boldsymbol{q}}}\,\dot{\boldsymbol{q}} \equiv \boldsymbol{0},$$

i.e. the tangent field of $\boldsymbol{q} = \boldsymbol{q}(t)$ is parallel along $\boldsymbol{q} = \boldsymbol{q}(t)$; see Jost [8, Definition 3.1.6]. In local coordinates, the coordinates of $\nabla_{\dot{\boldsymbol{q}}}\,\dot{\boldsymbol{q}}$ are given by

$$(\nabla_{\dot{\boldsymbol{q}}}\,\dot{\boldsymbol{q}})_i = \ddot{q}_i + \sum_{j,k=1}^{n} \Gamma^i_{jk}\dot{q}_j\dot{q}_k,$$

for $i = 1, \dots, n$, where the quantities Γ^i_{jk} are equivalent to the Christoffel symbols above.

Now suppose that $\mathcal{M}$ is an embedded submanifold in a higher dimensional Euclidean space $\mathbb{R}^N$. Then $\mathcal{M}$ is a Riemannian manifold. (Nash's Theorem says that any Riemannian manifold can be embedded in a higher dimensional Euclidean space $\mathbb{R}^N$.) For an embedded submanifold $\mathcal{M}$, its Riemannian metric is that naturally induced from the embedding Euclidean space using the map from intrinsic to extrinsic coordinates. For example, in our discussion of D'Alembert's principle, we saw that the dynamics of the system evolved in the Euclidean space $\mathbb{R}^{3N}$, parameterized by the extrinsic coordinates $\boldsymbol{r}_1$, $\dots$, $\boldsymbol{r}_N$, though subject to $m < 3N$ constraints. The dynamics thus evolves on the submanifold $\mathcal{M}$ prescribed by the constraints. The dimension of $\mathcal{M}$ is $n = 3N - m$ and so locally $\mathcal{M}$ can be parameterized by the coordinates $(q_1, \dots, q_n)$. Further, by making the change of coordinates

$$\begin{aligned} \boldsymbol{r}_1 &= \boldsymbol{r}_1(q_1, \dots, q_n), \\ &\vdots \\ \boldsymbol{r}_N &= \boldsymbol{r}_N(q_1, \dots, q_n), \end{aligned}$$

we demonstrated that D'Alembert's principle is equivalent to Lagrange's equations of motion. Indeed the geodesic equations we derived above were Lagrange's equations of motion for the Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\|\dot{\boldsymbol{q}}\|^2_{g(\boldsymbol{q})}$. Thus, starting with Lagrange's equations of motion for this Lagrangian, we can trace the direction of implication back through to D'Alembert's principle and thus deduce that

$$\ddot{\boldsymbol{q}} \perp \mathrm{T}_{\boldsymbol{q}}\mathcal{M}.$$

As we argued in the lead up to D'Alembert's principle, this makes sense on a intuitive and physical level. The system is free to evolve on the manifold of constraint and, assuming no external forces, the forces of constraint must act normally to the manifold (normal to the tangent plane at every point $\boldsymbol{q} \in \mathcal{M}$) in order to maintain the evolution on the manifold. In other words, as Tao [17] eminently puts it,

> "... the normal quantity $\ddot{q}$ then corresponds to the *centripetal* force necessary to keep the particle lying in $\mathcal{M}$ (otherwise it would fly off along a tangent line to $\mathcal{M}$, as per Newtons first law)."

3.2 Euler equations for incompressible flow

We formally show here how the Euler equations for incompressible flow represent a geodesic submanifold flow. The argument is an extension of that above for finite dimensional geodesic submanifold flow to an infinite dimensional context and is originally due to Arnold. Our arguments essentially combine those in Daneri and Figalli [6] and Tao [17] and we cite Chorin and Marsden [5] and Malham [13] for further background material on incompressible fluid mechanics.

The Euler equations for incompressible flow in a bounded Lipschitz domain $\mathcal{D} \subseteq \mathbb{R}^d$ and on an arbitrary finite time interval $[0,T]$ are prescribed as follows. The velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},t)$ for $(\boldsymbol{x},t) \in \mathcal{D} \times [0,T]$ evolves according to the system of nonlinear partial differential equations

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= -\nabla p, \\ \nabla \cdot \boldsymbol{u} &= 0. \end{aligned}$$

Here the field $p = p(\boldsymbol{x},t)$ is the pressure field. We need to augment this system of equations with the zero flux boundary condition for all $(\boldsymbol{x},t) \in \partial\mathcal{D} \times [0,T]$, i.e. we have $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, where $\boldsymbol{n}$ is the unit outer normal to $\partial\mathcal{D}$. We also need to prescribe initial data $\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x})$ for some given function $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x})$ on $\mathcal{D}$.

From the Lagrangian viewpoint we can describe the flow through the "swarm" of particle trajectories prescribed by integral curves of the local velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},t)$. Let $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{x},t) \in \mathbb{R}^d$ represent the position at time $t \geqslant 0$ of the particle that started at position $\boldsymbol{x} \in \mathcal{D}$ at time $t = 0$. The flow $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{x},t)$ satisfies the system of equations

$$\begin{aligned} \dot{\boldsymbol{q}}(\boldsymbol{x},t) &= \boldsymbol{u}\big(\boldsymbol{q}(\boldsymbol{x},t),t\big), \\ \boldsymbol{q}(\boldsymbol{x},0) &= \boldsymbol{x}, \end{aligned}$$

for all $(\boldsymbol{x},t) \in \mathcal{D} \times [0,T]$. A straightforward though lengthy calculation (see Chorin and Marsden [5] or Malham [13]) reveals that the Jacobian of the flow $\det \nabla\boldsymbol{q} = \det \nabla_{\boldsymbol{x}}\, \boldsymbol{q}(\boldsymbol{x},t)$ satisfies the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\det \nabla \boldsymbol{q}\big) = \big(\nabla \cdot \boldsymbol{u}(\boldsymbol{q},t)\big)\, \det \nabla \boldsymbol{q}.$$

Since we assume the flow is incompressible so $\nabla \cdot \boldsymbol{u} = 0$, and $\boldsymbol{q}(\boldsymbol{x},0) = \boldsymbol{x}$, we deduce for all $t \in [0,T]$ we have $\det \nabla\boldsymbol{q} \equiv 1$. This means that for all $t \in [0,T]$, $\boldsymbol{q}(\,\cdot\,,t) \in \mathrm{SDiff}(\mathcal{D})$, the group of measure preserving diffeomorphisms on $\mathcal{D}$. If we now differentiate the evolution equations for $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{x},t)$ above with respect to time t and compare this to the Euler equations, we obtain the following equivalent formulation for the Euler equations of incompressible flow in terms of $\boldsymbol{q}_t(\boldsymbol{x}) \coloneqq \boldsymbol{q}(\boldsymbol{x},t)$ for all $t \in [0,T]$:

$$\ddot{\boldsymbol{q}}_t = -\nabla p(\boldsymbol{q}_t, t),$$

with the constraint

$$\boldsymbol{q}_t \in \mathrm{SDiff}(\mathcal{D}).$$

We now demonstrate formally how these equations can be realized as a geodesic flow on SDiff($\mathcal{D}$) equipped with the $\mathrm{L}^2(\mathcal{D};\mathbb{R}^d)$ metric induced from the inner product

$$\langle \boldsymbol{v}, \boldsymbol{\omega} \rangle_{\mathrm{L}^2(\mathcal{D};\mathbb{R}^d)} := \int_{\mathcal{D}} \boldsymbol{v} \cdot \boldsymbol{\omega} \, \mathrm{d}\boldsymbol{x}.$$

We consider SDiff($\mathcal{D}$) as an infinite dimensional submanifold of Diff($\mathcal{D}$), the group of diffeomorphisms on $\mathcal{D}$. By analogy with the finite dimensional case, the curve $\boldsymbol{q} \colon t \mapsto \boldsymbol{q}_t$ on SDiff($\mathcal{D}$) is geodesic if

$$\ddot{\boldsymbol{q}} \perp \mathrm{T}_{\boldsymbol{q}}\mathrm{SDiff}(\mathcal{D}).$$

Remark 21. Suppose the flow is not constrained and evolves on Diff($\mathcal{D}$). Then a geodesic would be characterized by $\ddot{\boldsymbol{q}} = 0$, i.e. Newton's Second Law, as the metric is flat. For the intrinsic velocity field (see below) we would have $\partial \boldsymbol{u}/\partial t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = 0$; see Tao [17].

Our goal now is to characterize the vector space orthogonal to $\mathrm{T}_{\boldsymbol{q}}\mathrm{SDiff}(\mathcal{D})$. Using the definition of the flow above, we define the intrinsic velocity field $\boldsymbol{u}_t(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}, t)$ by $\boldsymbol{u}_t := \dot{\boldsymbol{q}}_t \circ \boldsymbol{q}_t^{-1}$. Using the result for the Jacobian $\det \nabla \boldsymbol{q}$ above we note that the inner product $\langle \cdot\,, \cdot \rangle$ above is invariant to transformations $\boldsymbol{x} \mapsto \boldsymbol{q}_t(\boldsymbol{x}) \in \mathrm{SDiff}(\mathcal{D})$. Further, we can thus characterize

$$\mathrm{T}_{\boldsymbol{q}}\mathrm{SDiff}(\mathcal{D}) = \big\{ \boldsymbol{u}(\boldsymbol{q}_t, t) \colon\; \nabla \cdot \boldsymbol{u}(\boldsymbol{q}_t, t) = 0 \text{ and } \boldsymbol{u} \cdot \boldsymbol{n}(\boldsymbol{q}_t, t) = 0 \text{ on } \partial\mathcal{D} \big\}.$$

Equivalently, since it is the Lie algebra of SDiff($\mathcal{D}$) which is the vector space of all smooth divergence-free vector fields parallel to the boundary, we have

$$\mathrm{T}_{\boldsymbol{q}}\mathrm{SDiff}(\mathcal{D}) = \big\{ \boldsymbol{u}_t \colon\; \nabla \cdot \boldsymbol{u}_t = 0 \text{ and } \boldsymbol{u}_t \cdot \boldsymbol{n} = 0 \text{ on } \partial\mathcal{D} \big\}.$$

The Helmholtz–Hodge Decomposition Theorem tells us that any vector $\boldsymbol{\omega} \in \mathrm{L}^2(\mathcal{D}, \mathbb{R}^d)$ can be orthogonally decomposed into a divergence-free vector $\boldsymbol{u}$ parallel to $\partial\mathcal{D}$ and a solenoidal vector $\boldsymbol{p}$. In other words we can write

$$\boldsymbol{\omega} = \boldsymbol{u} + \boldsymbol{p},$$

where $\nabla \cdot \boldsymbol{u} = 0$ and $\boldsymbol{p} = \nabla p$ for some scalar field p. See for example Chorin and Marsden [5, p. 37]. In terms of tangent spaces, we have

$$\mathrm{T}_{\boldsymbol{q}}\mathrm{Diff}(\mathcal{D}) = \mathrm{T}_{\boldsymbol{q}}\mathrm{SDiff}(\mathcal{D}) \oplus \big(\mathrm{T}_{\boldsymbol{q}}\mathrm{SDiff}(\mathcal{D})\big)^{\perp}$$

where

$$\big(\mathrm{T}_{\boldsymbol{q}}\mathrm{SDiff}(\mathcal{D})\big)^{\perp} = \big\{ \boldsymbol{p} \in \mathrm{L}^2(\mathcal{D}, \mathbb{R}^d) \colon \boldsymbol{p} = \nabla p \big\}.$$

Hence we deduce that $\ddot{\boldsymbol{q}}_t = \nabla p(\boldsymbol{q}_t, t)$ for $\boldsymbol{q}_t \in \mathrm{SDiff}(\mathcal{D})$, which are the Euler equations for incompressible flow.

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